BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 1, March 2005

EXACT LAWS FOR RANDOMLY SELECTED ORDER STATISTICS

BY

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Abstract. Let $\{X, X_{nj}, 1 \leq j \leq m, n \geq 1\}$ be i.i.d. random variables with a generalized Pareto distribution where $EX = \infty$. We randomly select one of our order statistics from $\{X_{n(1)}, \ldots, X_{n(m)}\}$ with a predetermined set of probabilities. Calling that new random variable Y_n we explore whether or not we can obtain constants a_n and b_N so that $\sum_{n=1}^{N} a_n Y_n / b_N$ converges in some sense to a nonzero constant, thus creating an Exact Law of Large Numbers.

1. Introduction. We first observe i.i.d. random variables $\{X_{n1}, \ldots, X_{nm}\}$ with the common distribution function

$$F_X(x) = \begin{cases} 1 - (\frac{\gamma e}{\lambda})^{\lambda} \frac{(\lg x)^{\lambda}}{x^{\gamma}} & \text{if } x \geq e^{\lambda/\gamma} \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma > 0$ and $\lambda \geq 0$. Then we order our random variables so that $X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(m)}$. Thus our order statistics have the density

$$f_{X_{n(k)}}(x) \quad = \quad \frac{m!}{(k-1)!(m-k)!} \left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)} \left[1 - \left(\frac{\gamma e}{\lambda}\right)^{\lambda} \frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{k-1}$$

Received by the editors August 7, 2003 and in revised form November 27, 2003.

AMS (2000) subject classification: Primary 60F15, Secondary 60F05.

Key words and phrases: Strong law of large numbers, weak law of large numbers, exact laws, slow variation.

$$\times \frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma \lg x - \lambda] I(x \geq e^{\lambda/\gamma})}{x^{\gamma(m-k+1)+1}}$$

where $k = 1, \ldots, m$. Next, we set the probability that we select $X_{n(k)}$ as $p_k = P\{Y_n = X_{n(k)}\}$, $k = 1, \ldots, m$. It is important to know which is the largest $X_{n(k)}$ we are allowed to choose. We may not be interested in $X_{n(m)}$ and in that case we set $p_m = 0$. Hence we need to identify which p_k's are nonzero from our larger order statistics. Thus we define $\nu = \max\{k : p_k > 0\}$, which tells us that $p_\nu > 0$ while $p_{\nu+1} = 0$. So, in reality we are only looking at the set $\{X_{n(1)}, \ldots, X_{n(\nu)}\}$.

An important observation is that the condition $\gamma(m-\nu+1) = 1$ implies that $0 < \gamma \leq 1$. Therefore, since $\lambda \geq 0$ we have $EX = \infty$. The conditon $\gamma(m - \nu + 1) > 1$ is of no interest since that would imply that all the order statistics in our collection would have a finite mean and thus the usual Strong Law of Large Numbers would hold. The situation of $\gamma(m-\nu+1) < 1$ presents a different type of problem. It is known that if we are only looking at one order statistic in that case, then no Exact Laws of Large Numbers can be established, including Exact Weak Laws, see Adler (2004).

We define Exact Laws of Large Numbers as a law of large numbers in which the limit is a nonzero constant. This type of result also goes by the name of the fair games problem. The idea is to make a gambling game fair to both the house and the gambler. We consider the sum of random variables as the winnings after N plays of a game. Likewise the normalizing constant should be looked upon as the amount the gambler has paid after those N games. If the ratio converges to a nonzero constant, then we can make the game fair to everyone by multiplying our norming sequence (also known as the cumulative entrance fee) by that very same constant in order to make the limit equal to one. Hence, neither the house nor the gambler has an unfair advantage and thus both parties should be interested in playing the game. The most famous of all the fair games problems is the St. Petersburg game.

Initially these type of limit theorems were called Exact Sequences since $\{b_N, N \geq 1\}$ in the limit was equal to our partial sums. The idea for that name came from Rogozin's (1968) paper where he only found Exact Upper Sequences. Rogozin could only find upper limits since he was dealing with i.i.d. random variables. There is no way to find an Exact Strong Law for i.i.d. random variables, whenver $EX = 0$ or $E|X| = \infty$, as one can see via the theorems in this and other papers on this subject. That is why we need to examine weighted sums of i.i.d. random variables. However, the name Exact Sequences was already in use, so the name Exact Laws was created. The best source for Exact Laws is Adler (2000). For a multidimensional version one would use Adler and Qi (2004). There are also complete convergence results for the larger order statistics from a triangular array, see Adler (2002).

The theorems in this paper show how to select the proper subsets of order statistics in order to obtain an Exact Law of Large Numbers, where at least one (the largest order statistic in our collection) doesn't have a finite mean. They also show which weights and norms are permissible.

As usual we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. We use the constant C to denote a generic real number that is not necessarily the same in each appearance. Note that we use it as both an lower and upper bound in our proofs. It is used to show that a limit is bounded away from zero and from infinity in these respectively different cases. Also we use Theorem 1 from page 281 of Feller (1971) throughout the paper when dealing with slowly varying functions. This theorem allows us to basically pull these slowly varying functions out of our integrals and summations.

2. Exact strong law of large numbers. In this section we show how to obtain Exact Strong Laws. We need to choose the proper weights and norming sequences in order to have a fair game between the house and the player. Thus we want to select a_n and b_n so that

$$\frac{\sum_{n=1}^{N} a_n Y_n}{b_N} \to 1 \qquad \text{almost surely}$$

as N goes to infinity. Note that we do not need to have one as our limit, any nonzero constant will do.

Theorem 1. *If $\gamma(m-\nu+1)=1$ and $\beta>0$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{[\lg n]^{\beta-\lambda/\gamma-2}}{n} Y_n}{[\lg N]^\beta} = \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)\beta(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \quad \textit{almost surely.}$$

Proof. Let $a_n = [\lg n]^{\beta-\lambda/\gamma-2}/n$, $b_n = [\lg n]^\beta$ and $c_n = b_n/a_n = n[\lg n]^{\lambda/\gamma+2}$. We partition our sum into three terms

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N} a_n Y_n &= \frac{1}{b_N}\sum_{n=1}^{N} a_n [Y_n I(Y_n \le c_n) - EY_n I(Y_n \le c_n)] \\ &\quad + \frac{1}{b_N}\sum_{n=1}^{N} a_n Y_n I(Y_n > c_n) + \frac{1}{b_N}\sum_{n=1}^{N} a_n E Y_n I(Y_n \le c_n).\end{aligned}$$

The first term vanishes almost surely via the Khintchine-Kolmogorov convergence theorem and Kronecker's lemma, see Chow and Teicher (1997), since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{c_n^2} EY_n^2 I(Y_n \le c_n) &= \sum_{k=1}^{\nu} p_k \sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_{n(k)}^2 I(X_{n(k)} \le c_n) \\ &< C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda(m-k+1)} x^2\, dx}{x^{\gamma(m-k+1)+1}} \\ &= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda(m-v+1)+\lambda(\nu-k)} dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)-1}} \\ &= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\, dx}{x^{\gamma(\nu-k)}}\end{aligned}$$

$$= C\sum_{k=1}^{\nu-1}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}dx}{x^{\gamma(\nu-k)}}$$
$$+C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{c_n}(\lg x)^{\lambda/\gamma}\,dx.$$

It is important to observe the case of $k=\nu$ very closely. This term is bounded above by

$$C\sum_{n=1}^{\infty}\frac{1}{c_n^2}c_n[\lg c_n]^{\lambda/\gamma} = C\sum_{n=1}^{\infty}\frac{[\lg c_n]^{\lambda/\gamma}}{c_n} < C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma}}{n[\lg n]^{\lambda/\gamma+2}}$$
$$= C\sum_{n=1}^{\infty}\frac{1}{n[\lg n]^2} < \infty.$$

We need not be as careful with the other terms in our sum. They converge very easily, since whenever $k<\nu$ it follows that

$$\sum_{k=1}^{\nu-1}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\,dx}{x^{\gamma(\nu-k)}} < \sum_{k=1}^{\nu-1}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-1)}\,dx}{x^{\gamma}}$$
$$< C\sum_{n=1}^{\infty}\left(\frac{1}{c_n^2}\right)\left(\frac{[\lg c_n]^{\lambda/\gamma+\lambda(\nu-1)}}{c_n^{\gamma-1}}\right) = C\sum_{n=1}^{\infty}\frac{[\lg c_n]^{\lambda/\gamma+\lambda(\nu-1)}}{c_n^{\gamma+1}}$$
$$< C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda(\nu-1)}}{[n(\lg n)^{\lambda/\gamma+2}]^{\gamma+1}} = C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda(\nu-1)}}{n^{\gamma+1}[\lg n]^{\lambda/\gamma+\lambda+2\gamma+2}}$$
$$= C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda(\nu-2)-2\gamma-2}}{n^{\gamma+1}} < \infty$$

since $\gamma>0$. Thus our first term does vanish with probability one.

As for our second term we use the Borel-Cantelli lemma since

$$\sum_{n=1}^{\infty}P\{Y_n>c_n\} < C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\int_{c_n}^{\infty}\frac{(\lg x)^{\lambda(m-k+1)}\,dx}{x^{\gamma(m-k+1)+1}}$$
$$= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\int_{c_n}^{\infty}\frac{(\lg x)^{\lambda(m-\nu+1)+\lambda(\nu-k)}\,dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)+1}}$$
$$= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\int_{c_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\,dx}{x^{\gamma(\nu-k)+2}} < C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{[\lg c_n]^{\lambda/\gamma+\lambda(\nu-k)}}{c_n^{\gamma(\nu-k)+1}}$$

$$
\begin{aligned}
&< \; C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda(\nu-k)}}{[n(\lg n)^{\lambda/\gamma+2}]^{\gamma(\nu-k)+1}} \\
&= \; C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda(\nu-k)}}{n^{\gamma(\nu-k)+1}[\lg n]^{\lambda/\gamma+\lambda(\nu-k)+2\gamma(\nu-k)+2}} \\
&= \; C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{n^{\gamma(\nu-k)+1}[\lg n]^{2\gamma(\nu-k)+2}} < C\sum_{n=1}^{\infty}\frac{1}{n[\lg n]^2} < \infty.
\end{aligned}
$$

Thus the third and last term will give us our desired limit. Our truncated first moment is

$$
\begin{aligned}
EY_nI(Y_n \le c_n) \; &= \; \sum_{k=1}^{\nu}\frac{p_k m!}{(m-k)!(k-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)} \\
&\times \int_{e^{\lambda/\gamma}}^{c_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{k-1}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-k+1)}}.
\end{aligned}
$$

Now, when $k < \nu$ we have

$$
\begin{aligned}
&\sum_{k=1}^{\nu-1}\frac{p_k m!}{(m-k)!(k-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)}\int_{e^{\lambda/\gamma}}^{c_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{k-1} \\
&\qquad\times\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-k+1)}} \\
&< \; C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda(m-k+1)}\,dx}{x^{\gamma(m-k+1)}} = C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda(m-\nu+1)+\lambda(\nu-k)}\,dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)}} \\
&= \; C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\,dx}{x^{\gamma(\nu-k)+1}} < C\int_{e^{\lambda/\gamma}}^{c_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-1)}\,dx}{x^{\gamma+1}} = O(1)
\end{aligned}
$$

since $\gamma > 0$. Hence we may attack the case of $k = \nu$

$$
\begin{aligned}
&\frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-\nu+1)} \\
&\qquad\times\int_{e^{\lambda/\gamma}}^{c_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{\nu-1}\frac{(\lg x)^{\lambda(m-\nu+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-\nu+1)}} \\
&= \; \frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}(-1)^j\left(\frac{\gamma e}{\lambda}\right)^{\lambda j}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda j+\lambda/\gamma-1}[\gamma \lg x - \lambda]\, dx}{x^{\gamma j+1}} \\
\sim\quad & \frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda/\gamma-1}[\gamma \lg x - \lambda]\, dx}{x} \\
\sim\quad & \frac{p_\nu \gamma m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda/\gamma}\, dx}{x} \\
\sim\quad & \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} [\lg c_n]^{\lambda/\gamma+1} \\
\sim\quad & \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} [\lg n]^{\lambda/\gamma+1}
\end{aligned}
$$

because

$$
\begin{aligned}
&\left|\frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \sum_{j=1}^{\nu-1} \binom{\nu-1}{j} (-1)^j \left(\frac{\gamma e}{\lambda}\right)^{\lambda j} \right. \\
&\qquad \left. \times \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda j+\lambda/\gamma-1}[\gamma \lg x - \lambda]\, dx}{x^{\gamma j+1}} \right| \\
&\quad < \; C \int_{e^{\lambda/\gamma}}^{c_n} \frac{(\lg x)^{\lambda(\nu-1)+\lambda/\gamma}\, dx}{x^{\gamma+1}} = O(1)
\end{aligned}
$$

since $\gamma > 0$. Therefore

$$
\begin{aligned}
& \frac{\sum_{n=1}^{N} a_n E Y_n I(Y_n \le c_n)}{b_N} \\
\sim\quad & \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \frac{\sum_{n=1}^{N} \left[\frac{(\lg n)^{\beta-\lambda/\gamma-2}}{n}\right] [\lg n]^{\lambda/\gamma+1}}{[\lg N]^\beta} \\
=\quad & \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \frac{\sum_{n=1}^{N} \frac{[\lg n]^{\beta-1}}{n}}{[\lg N]^\beta} \\
\to\quad & \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)\beta(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}
\end{aligned}
$$

which concludes the proof.

The constants a_n and b_n in our Strong Law must be chosen in this fashion. We do have a little bit of freedom, but a_n must be a slowly varying function divided by n, while b_n needs to be slowly varying. If we want more conventional weight, a_n, then we either need to weaken our mode of convergence (see Section 3) or give up on trying to obtain an Exact Law. In Section 4, Generalized Laws of the Iterated Logarithm are obtained. That result show that the weights we used in Theorem 1 are necessary in order to establish an Exact Strong Law.

3. Exact weak law of large numbers. We are going to examine when our normalized partial sums converge in probability to a finite nonzero constant. We have a lot more freedom in our choice of weights and norms in this setting, since we are no longer working with almost sure convergence. Note that we are allowed in this situation to select $a_n = 1$, i.e., the unweighted case. That wasn't possible in the last section, as we will show in Section 4.

Theorem 2. *If $\gamma(m-\nu+1) = 1$ and $\alpha > -1$, then for any slowly varying function $L(n)$*

$$\frac{\sum_{n=1}^{N} n^{\alpha} L(n) Y_n}{N^{\alpha+1} L(N) [\lg N]^{\lambda/\gamma+1}} \xrightarrow{P} \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!} \left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}$$

as N goes to infinity.

Proof. Let $a_n = n^{\alpha} L(n)$ and $b_n = n^{\alpha+1} L(n) [\lg n]^{\lambda/\gamma+1}$. It is important to note that $\max_{1 \le n \le N} a_n = o(b_N)$. We will use the Degenerate Convergence Theorem from Chow and Teicher (1997). We first need to show that the tail probabilities are negligible

$$\begin{aligned}
\sum_{n=1}^{N} P\{Y_n > b_N/a_n\} &= \sum_{k=1}^{\nu} p_k \sum_{n=1}^{N} P\{X_{n(k)} > b_N/a_n\} \\
&< C \sum_{k=1}^{\nu} \sum_{n=1}^{N} \int_{b_N/a_n}^{\infty} \frac{(\lg x)^{\lambda(m-k+1)}\, dx}{x^{\gamma(m-k+1)+1}} \\
&= C \sum_{k=1}^{\nu} \sum_{n=1}^{N} \int_{b_N/a_n}^{\infty} \frac{(\lg x)^{\lambda(m-\nu+1)+\lambda(\nu-k)}\, dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)+1}}
\end{aligned}$$

$$\begin{aligned}
&= C\sum_{k=1}^{\nu}\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)+2}} \\
&= C\sum_{k=1}^{\nu-1}\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)+2}} \\
&\quad +C\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma}\ dx}{x^{2}}.
\end{aligned}$$

The last term is

$$\begin{aligned}
&C\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma}dx}{x^{2}} < C\sum_{n=1}^{N}\frac{[\lg(b_N/a_n)]^{\lambda/\gamma}}{b_N/a_n} = C\sum_{n=1}^{N}\frac{a_n}{b_N}[\lg(b_N/a_n)]^{\lambda/\gamma} \\
&= \frac{C\sum_{n=1}^{N}n^{\alpha}L(n)[(\alpha+1)\lg N+\lg L(N)+(\lambda/\gamma+1)\lg_2 N-\alpha\lg n-\lg L(n)]^{\lambda/\gamma}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \\
&< \frac{C\sum_{n=1}^{N}n^{\alpha}L(n)[\lg N]^{\lambda/\gamma}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} = \frac{C\sum_{n=1}^{N}n^{\alpha}L(n)}{N^{\alpha+1}L(N)\lg N} < \frac{C}{\lg N} \to 0
\end{aligned}$$

while the rest of the sum is

$$\begin{aligned}
&C\sum_{k=1}^{\nu-1}\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)+2}} < C\sum_{n=1}^{N}\int_{b_N/a_n}^{\infty}\frac{(\lg x)^{\lambda/\gamma+\lambda\nu}\ dx}{x^{\gamma+2}} \\
&< C\sum_{n=1}^{N}\frac{a_n^{\gamma+1}}{b_N^{\gamma+1}}[\lg(b_N/a_n)]^{\lambda/\gamma+\lambda\nu} \\
&= \Big[C\sum_{n=1}^{N}[n^{\alpha}L(n)]^{\gamma+1}[(\alpha+1)\lg N+\lg L(N)+(\lambda/\gamma+1)\lg_2 N \\
&\quad -\alpha\lg n-\lg L(n)]^{\lambda/\gamma+\lambda\nu}\Big]\Big/[N^{\alpha+1}L(N)(\lg N)^{\lambda/\gamma+1}]^{\gamma+1} \\
&< \frac{C\sum_{n=1}^{N}[n^{\alpha}L(n)]^{\gamma+1}[\lg N]^{\lambda/\gamma+\lambda\nu}}{[N^{\alpha+1}L(N)(\lg N)^{\lambda/\gamma+1}]^{\gamma+1}} \\
&= \frac{C\sum_{n=1}^{N}n^{\alpha(\gamma+1)}[L(n)]^{\gamma+1}[\lg N]^{\lambda\nu-\lambda-\gamma-1}}{N^{(\alpha+1)(\gamma+1)}[L(N)]^{\gamma+1}} \\
&< \frac{C[\lg N]^{\lambda\nu-\lambda-\gamma-1}}{N^{\gamma}} \to 0.
\end{aligned}$$

Next, we attack the variance term in the Degenerate Convergence Theorem

$$\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}V(Y_nI(Y_n\le b_N/a_n))<C\sum_{k=1}^{\nu}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda(m-k+1)}\,dx}{x^{\gamma(m-k+1)-1}}.$$

The last term in our sum, $k=\nu$, is

$$\begin{aligned}C\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}(\lg x)^{\lambda/\gamma}\,dx &< C\sum_{n=1}^{N}\left(\frac{a_n^2}{b_N^2}\right)\left(\frac{b_N}{a_n}\right)[\lg(b_N/a_n)]^{\lambda/\gamma}\\ &= \frac{C\sum_{n=1}^{N}a_n[\lg(b_N/a_n)]^{\lambda/\gamma}}{b_N}\to 0\end{aligned}$$

as before. As for the rest of the series we have

$$\begin{aligned}&\sum_{k=1}^{\nu-1}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda(m-k+1)}\,dx}{x^{\gamma(m-k+1)-1}}\\ &\quad= \sum_{k=1}^{\nu-1}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda(m-\nu+1)+\lambda(\nu-k)}\,dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)-1}}\\ &\quad= \sum_{k=1}^{\nu-1}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\,dx}{x^{\gamma(\nu-k)}}\\ &\quad< C\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma+\lambda\nu}\,dx}{x^{\gamma}}\\ &\quad< C\sum_{n=1}^{N}\left(\frac{a_n^2}{b_N^2}\right)\left(\frac{b_N}{a_n}\right)^{-\gamma+1}[\lg(b_N/a_n)]^{\lambda/\gamma+\lambda\nu}\\ &\quad= \frac{C\sum_{n=1}^{N}a_n^{\gamma+1}[\lg(b_N/a_n)]^{\lambda/\gamma+\lambda\nu}}{b_N^{\gamma+1}}\to 0\end{aligned}$$

as in an earlier calculation.

Our truncated first moment for the Weak Law is

$$\begin{aligned}&EY_nI(Y_n\le b_N/a_n)=\sum_{k=1}^{\nu}\frac{p_k m!}{(m-k)!(k-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)}\\ &\times\int_{e^{\lambda/\gamma}}^{b_N/a_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{k-1}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-k+1)}}.\end{aligned}$$

The last term, $k=\nu$, is

$$\begin{aligned}
&\frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-\nu+1)} \\
&\quad\times\int_{e^{\lambda/\gamma}}^{b_N/a_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^\lambda}{x^\gamma}\right]^{\nu-1}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-\nu+1)}} \\
&= \frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \\
&\quad\times\int_{e^{\lambda/\gamma}}^{b_N/a_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^\lambda}{x^\gamma}\right]^{\nu-1}\frac{(\lg x)^{\lambda/\gamma-1}[\gamma\lg x-\lambda]\,dx}{x} \\
&= \frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}(-1)^j\left(\frac{\gamma e}{\lambda}\right)^{\lambda j} \\
&\quad\times\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda j+\lambda/\gamma-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma j+1}} \\
&\sim \frac{p_\nu m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma-1}[\gamma\lg x-\lambda]\,dx}{x} \\
&\sim \frac{p_\nu \gamma m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma}\,dx}{x} \\
&\sim \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}[\lg(b_N/a_n)]^{\lambda/\gamma+1}
\end{aligned}$$

since

$$\begin{aligned}
&\left|\sum_{j=1}^{\nu-1}\binom{\nu-1}{j}(-1)^j\left(\frac{\gamma e}{\lambda}\right)^{\lambda j}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda j+\lambda/\gamma-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma j+1}}\right| \\
&\quad< C\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda\nu+\lambda/\gamma}\,dx}{x^{\gamma+1}}=O(1).
\end{aligned}$$

As for the rest of the sum, we have

$$\begin{aligned}
&\sum_{k=1}^{\nu-1}\frac{p_k m!}{(m-k)!(k-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)} \\
&\quad\times\int_{e^{\lambda/\gamma}}^{b_N/a_n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^\lambda}{x^\gamma}\right]^{k-1}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\,dx}{x^{\gamma(m-k+1)}}
\end{aligned}$$

$$\begin{aligned}
&< C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda(m-k+1)}dx}{x^{\gamma(m-k+1)}} = C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda(m-\nu+1)+\lambda(\nu-k)}dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)}}\\
&= C\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\,dx}{x^{\gamma(\nu-k)+1}} < C\int_{e^{\lambda/\gamma}}^{b_N/a_n}\frac{(\lg x)^{\lambda/\gamma+\lambda\nu}\,dx}{x^{\gamma+1}} = O(1).
\end{aligned}$$

All that remains to do is to find the limit of

$$\frac{\sum_{n=1}^N a_n[\lg(b_N/a_n)]^{\lambda/\gamma+1}}{b_N}. \tag{1}$$

We will apply the Mean Value Theorem twice. In both cases $f(x) = x^{\lambda/\gamma+1}$. In the first case

$x = (\alpha+1)\lg N - \alpha\lg n$ and $h = \lg L(N) + (\lambda/\gamma+1)\lg_2 N - \lg L(n)$. Hence (1) is equal to

$$\begin{aligned}
&\frac{\sum_{n=1}^N n^\alpha L(n)[(\alpha+1)\lg N + \lg L(N) + (\lambda/\gamma+1)\lg_2 N - \alpha\lg n - \lg L(n)]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}\\
&= \frac{\sum_{n=1}^N n^\alpha L(n)[(\alpha+1)\lg N - \alpha\lg n]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}\\
&\quad + \frac{(\lambda/\gamma+1)\sum_{n=1}^N n^\alpha L(n) z^{\lambda/\gamma}[\lg L(N) + (\lambda/\gamma+1)\lg_2 N - \lg L(n)]}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}
\end{aligned}$$

where $z \approx \lg N$. Thus the second term is bounded above by

$$\begin{aligned}
&\frac{C\sum_{n=1}^N n^\alpha L(n)[\lg N]^{\lambda/\gamma}[\lg L(N) + (\lambda/\gamma+1)\lg_2 N - \lg L(n)]}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}\\
&< C\left[\frac{\sum_{n=1}^N n^\alpha L(n)\lg L(N)}{N^{\alpha+1}L(N)\lg N} + \frac{\sum_{n=1}^N n^\alpha L(n)\lg_2 N}{N^{\alpha+1}L(N)\lg N} + \frac{\sum_{n=1}^N n^\alpha L(n)\lg L(n)}{N^{\alpha+1}L(N)\lg N}\right]\\
&< C\left[\frac{\lg L(N)}{\lg N} + \frac{\lg_2 N}{\lg N} + \frac{\lg L(N)}{\lg N}\right] \to 0.
\end{aligned}$$

Note that $\lg L(N) = o(\lg N)$ follows directly from Karamata's Representation Theorem, see Feller (1971). Applying the Mean Value Theorem once again, now with $x = \lg N$ and $h = \alpha(\lg N - \lg n)$ our first term from (1) becomes

$$\frac{\sum_{n=1}^N n^\alpha L(n)[\lg N]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} + \frac{(\lambda/\gamma+1)\alpha\sum_{n=1}^N n^\alpha L(n) z^{\lambda/\gamma}[\lg N - \lg n]}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}.$$

The first term leads us to our limit

$$\frac{\sum_{n=1}^N n^\alpha L(n)[\lg N]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} = \frac{\sum_{n=1}^N n^\alpha L(n)}{N^{\alpha+1}L(N)} \to \frac{1}{\alpha+1}$$

while

$$\begin{aligned} &\left|\frac{(\lambda/\gamma+1)\alpha\sum_{n=1}^N n^\alpha L(n) z^{\lambda/\gamma}[\lg N - \lg n]}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}\right| \\ < &\quad \frac{C\sum_{n=1}^N n^\alpha L(n)[\lg N]^{\lambda/\gamma}[\lg N - \lg n]}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \\ = &\quad C\left[\frac{\sum_{n=1}^N n^\alpha L(n)}{N^{\alpha+1}L(N)} - \frac{\sum_{n=1}^N n^\alpha L(n)\lg n}{N^{\alpha+1}L(N)\lg N}\right] \to C\left[\frac{1}{\alpha+1} - \frac{1}{\alpha+1}\right] = 0 \end{aligned}$$

since, once again $z \approx \lg N$.

Therefore

$$\begin{aligned} &\frac{\sum_{n=1}^N n^\alpha L(n) EY_n I(Y_n \le b_N/a_n)}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \\ \sim &\quad \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \frac{\sum_{n=1}^N n^\alpha L(n)[\lg(b_N/a_n)]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \\ \to &\quad \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \end{aligned}$$

which concludes this proof.

It is important to note that if the power in (1) was only λ/γ instead of $\lambda/\gamma+1$ the limit would be zero. This shows how delicate this theorem is. Also, it needs to be mentioned that these Weak Laws exist on their own. By that, we mean that we do not have almost sure convergence in Theorem 2. In the next section we will obtain the almost sure behavior of these normalized partial sums. It should be pointed out that in Theorem 1 we have $b_n/a_n = n[\lg n]^{\lambda/\gamma+2}$, while in Theorem 2 we have $b_n/a_n = n[\lg n]^{\lambda/\gamma+1}$. This will be explored further in Section 5.

4. Generalized laws of the iterated logarithm. In this section we use Theorem 2 to help us establish the almost sure lower limit of the normalized partial sums in our Weak Law. Since we use Kronecker's lemma in our proof we need to say that $N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}$ is eventually increasing. This is a mild condition given that $\alpha > -1$.

Theorem 3. *If $\gamma(m-\nu+1) = 1$ and $\alpha > -1$, then for any slowly varying function $L(n)$*

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N n^\alpha L(n) Y_n}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} = \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \quad \textit{almost surely}$$

and

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N n^\alpha L(n) Y_n}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} = \infty \qquad \textit{almost surely.}$$

Proof. From Theorem 2 we know that with probability one there is a subsequence of our normalized partial sums approaching the very same limit. Thus

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N n^\alpha L(n) Y_n}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \le \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \quad \text{almost surely.}$$

So our first goal is to show that our liminf is greater than or equal to this constant. As in Theorem 2, we set $a_n = n^\alpha L(n)$, $b_n = n^{\alpha+1}L(n)[\lg n]^{\lambda/\gamma+1}$, which now implies that $c_n = b_n/a_n = n[\lg n]^{\lambda/\gamma+1}$. The partition in this case is rather straighforward. We use

$$\frac{1}{b_N}\sum_{n=1}^N a_n Y_n \ge \frac{1}{b_N}\sum_{n=1}^N a_n Y_n I(Y_n \le n)$$

$$= \frac{1}{b_N}\sum_{n=1}^{N} a_n[Y_nI(Y_n \le n) - EY_nI(Y_n \le n)] + \frac{1}{b_N}\sum_{n=1}^{N} a_nEY_nI(Y_n \le n).$$

The first term will vanish with probability one, via the Khintchine-Kolmogorov theorem and Kronecker's lemma and the fact that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{c_n^2}EY_n^2I(Y_n \le n) &= \sum_{k=1}^{\nu}p_k\sum_{n=1}^{\infty}\frac{1}{c_n^2}EX_{n(k)}^2I(X_{n(k)} \le n)\\
&< C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda(m-k+1)}x^2\ dx}{x^{\gamma(m-k+1)+1}}\\
&= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda(m-v+1)+\lambda(\nu-k)}\ dx}{x^{\gamma(m-\nu+1)+\gamma(\nu-k)-1}}\\
&= C\sum_{k=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)}}\\
&= C\sum_{k=1}^{\nu-1}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)}} + C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}(\lg x)^{\lambda/\gamma}\ dx.
\end{aligned}$$

The last term is bounded above by

$$C\sum_{n=1}^{\infty}\frac{n[\lg n]^{\lambda/\gamma}}{c_n^2} = C\sum_{n=1}^{\infty}\frac{n[\lg n]^{\lambda/\gamma}}{n^2[\lg n]^{2\lambda/\gamma+2}} < C\sum_{n=1}^{\infty}\frac{1}{n[\lg n]^2} < \infty$$

while the other term is

$$\begin{aligned}
C\sum_{k=1}^{\nu-1}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma+\lambda(\nu-k)}\ dx}{x^{\gamma(\nu-k)}} &< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma+\lambda\nu}\ dx}{x^{\gamma}}\\
< C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda\nu}}{c_n^2n^{\gamma-1}} = C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda/\gamma+\lambda\nu}}{n^2[\lg n]^{2\lambda/\gamma+2}n^{\gamma-1}} &< C\sum_{n=1}^{\infty}\frac{[\lg n]^{\lambda\nu}}{n^{\gamma+1}} < \infty
\end{aligned}$$

since $\gamma > 0$, thus eliminating the first term. Likewise

$$\begin{aligned}
EY_nI(Y_n \le n) &= \sum_{k=1}^{\nu}\frac{p_km!}{(m-k)!(k-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda(m-k+1)}\\
&\quad\times\int_{e^{\lambda/\gamma}}^{n}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^{\lambda}}{x^{\gamma}}\right]^{k-1}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma\lg x-\lambda]\ dx}{x^{\gamma(m-k+1)}}
\end{aligned}$$

$$\sim \frac{p_\nu \gamma m!}{(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma}\,dx}{x}$$
$$\sim \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}[\lg n]^{\lambda/\gamma+1}$$

since

$$\sum_{k=1}^{\nu-1}\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda(m-k+1)-1}[\gamma \lg x-\lambda]\,dx}{x^{\gamma(m-k+1)}} < C\int_{e^{\lambda/\gamma}}^{n}\frac{(\lg x)^{\lambda/\gamma+\lambda\nu}\,dx}{x^{\gamma+1}} = O(1).$$

Therefore

$$\frac{\sum_{n=1}^{N} a_n E Y_n I(Y_n \le n)}{b_N}$$
$$\sim \frac{\frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(m-\nu)!(\nu-1)!}(\frac{\gamma e}{\lambda})^{\lambda/\gamma}\sum_{n=1}^{N} n^\alpha L(n)[\lg n]^{\lambda/\gamma+1}}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}}$$
$$\to \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma}.$$

This in turn implies that

$$\liminf_{N\to\infty}\frac{\sum_{n=1}^{N} n^\alpha L(n) Y_n}{N^{\alpha+1}L(N)[\lg N]^{\lambda/\gamma+1}} \ge \frac{p_\nu \gamma^2 m!}{(\lambda+\gamma)(\alpha+1)(m-\nu)!(\nu-1)!}\left(\frac{\gamma e}{\lambda}\right)^{\lambda/\gamma} \quad \text{almost surely}$$

which establishes the almost sure lower limit of our normalized partial sums.

All that remains to show is that the almost sure limit supremum is infinity. Let $M > 0$

$$\sum_{n=1}^{\infty} P\{Y_n > Mc_n\} = \sum_{n=1}^{\infty}\sum_{k=1}^{\nu} p_k P\{X_{n(k)} > Mc_n\} > C\sum_{n=1}^{\infty} P\{X_{n(\nu)} > Mc_n\}$$
$$> C\sum_{n=1}^{\infty}\int_{Mc_n}^{\infty}\left[1-\left(\frac{\gamma e}{\lambda}\right)^{\lambda}\frac{(\lg x)^\lambda}{x^\gamma}\right]^{\nu-1}\frac{(\lg x)^{\lambda/\gamma-1}[\gamma \lg x-\lambda]\,dx}{x^2}$$
$$= C\sum_{n=1}^{\infty}\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}(-1)^j\left(\frac{\gamma e}{\lambda}\right)^{\lambda j}\int_{Mc_n}^{\infty}\frac{(\lg x)^{\lambda j+\lambda/\gamma-1}[\gamma \lg x-\lambda]dx}{x^{\gamma j+2}} = \infty$$

since, when $j = 0$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{Mc_n}^{\infty} \frac{(\lg x)^{\lambda/\gamma}\ dx}{x^2} &> C \sum_{n=1}^{\infty} \frac{[\lg Mc_n]^{\lambda/\gamma}}{Mc_n} \\
&= C \sum_{n=1}^{\infty} \frac{[\lg M + \lg n + (\lambda/\gamma + 1)\lg_2 n]^{\lambda/\gamma}}{Mn[\lg n]^{\lambda/\gamma+1}} > C \sum_{n=1}^{\infty} \frac{[\lg n]^{\lambda/\gamma}}{n[\lg n]^{\lambda/\gamma+1}} \\
&= C \sum_{n=1}^{\infty} \frac{1}{n \lg n} = \infty
\end{aligned}$$

while

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{Mc_n}^{\infty} \frac{[\lg x]^{\lambda/\gamma-1}\ dx}{x^2} &< C \sum_{n=1}^{\infty} \frac{[\lg c_n]^{\lambda/\gamma-1}}{c_n} \\
&< C \sum_{n=1}^{\infty} \frac{[\lg n]^{\lambda/\gamma-1}}{n[\lg n]^{\lambda/\gamma+1}} = C \sum_{n=1}^{\infty} \frac{1}{n[\lg n]^2} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \sum_{j=1}^{\nu-1} \binom{\nu-1}{j} (-1)^j \left(\frac{\gamma e}{\lambda}\right)^{\lambda j} \int_{Mc_n}^{\infty} \frac{(\lg x)^{\lambda j + \lambda/\gamma - 1}[\gamma \lg x - \lambda]\ dx}{x^{\gamma j + 2}} \right| \\
&< C \sum_{n=1}^{\infty} \int_{Mc_n}^{\infty} \frac{(\lg x)^{\lambda\nu + \lambda/\gamma}\ dx}{x^{\gamma+2}} < C \sum_{n=1}^{\infty} \frac{[\lg Mc_n]^{\lambda\nu+\lambda/\gamma}}{[Mc_n]^{\gamma+1}} \\
&< C \sum_{n=1}^{\infty} \frac{[\lg n]^{\lambda\nu+\lambda/\gamma}}{n^{\gamma+1}[\lg n]^{(\lambda/\gamma+1)(\gamma+1)}} < C \sum_{n=1}^{\infty} \frac{[\lg n]^{\lambda\nu+\lambda/\gamma}}{n^{\gamma+1}[\lg n]^{\lambda+\gamma+\lambda/\gamma+1}} \\
&< C \sum_{n=1}^{\infty} \frac{[\lg n]^{\lambda\nu}}{n^{\gamma+1}} < \infty
\end{aligned}$$

since $\gamma > 0$. Now, from the inequality

$$\frac{\sum_{n=1}^{N} a_n Y_n}{b_N} > \frac{a_N Y_N}{b_N} = \frac{Y_N}{c_N}$$

we can claim that

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} n^{\alpha} L(n) Y_n}{N^{\alpha+1} L(N) [\lg N]^{\lambda/\gamma+1}} = \infty \qquad \text{almost surely}$$

which completes this proof.

5. Discussion. We conclude with a couple of examples that should shed some light into what has been accomplished in this paper.

Example 1. Let $\gamma = 1$. In this case we need $\nu = m$. So we can take a sample of any size, but we need to keep the largest order statistic as a viable candidate. In other words $p_m > 0$. Hence for all $\beta > 0$ and $\lambda \geq 0$ we have

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{[\lg n]^{\beta-\lambda-2}}{n} Y_n}{[\lg N]^{\beta}} = \frac{p_m m}{(\lambda+1)\beta} \left(\frac{e}{\lambda}\right)^{\lambda} \qquad \text{almost surely.}$$

or we can also say that for all $\alpha > -1$, $\lambda \geq 0$ and all slowly varying function $L(n)$

$$\frac{\sum_{n=1}^{N} n^{\alpha} L(n) Y_n}{N^{\alpha+1} L(N) [\lg N]^{\lambda+1}} \xrightarrow{P} \frac{p_m m}{(\lambda+1)(\alpha+1)} \left(\frac{e}{\lambda}\right)^{\lambda} \qquad \text{as } N \text{ goes to infinity.}$$

We can also investigate random variables that are far from being integrable.

Example 2. Let $\gamma = .2$. Here we will need $m - \nu = 4$. That gives us a lot of freedom in selecting our sample size and which order statistics we wish to observe. Our Strong Law is

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{[\lg n]^{\beta-5\lambda-2}}{n} Y_n}{[\lg N]^{\beta}} = \frac{p_\nu m!}{600(\lambda+.2)\beta(\nu-1)!} \left(\frac{e}{5\lambda}\right)^{5\lambda} \qquad \text{almost surely}$$

for all $\beta > 0$ and $\lambda \geq 0$, while our Weak Law is

$$\frac{\sum_{n=1}^{N} n^{\alpha} L(n) Y_n}{N^{\alpha+1} L(N) [\lg N]^{5\lambda+1}} \xrightarrow{P} \frac{p_\nu m!}{600(\lambda+.2)(\alpha+1)(\nu-1)!} \left(\frac{e}{5\lambda}\right)^{5\lambda} \qquad \text{as } N \text{ goes to infinity}$$

for all $\alpha > -1$, $\lambda \geq 0$ and all slowly varying functions $L(n)$.

It is important to note that these Weak and Strong Laws are slightly different. If one tries to set $\alpha = -1$ in Example 2 and thinks that the Weak Law follows directly from the Strong Law they are incorrect. For, if we let $L(n) = 1$ and set $\beta = 5\lambda + 1$ then the coefficient in the Strong Law would be $1/(n \lg n)$ while our Weak Law has the weight of just $1/n$. There is and always will be a difference of a factor of $\lg n$ in these two types of Exact Laws of Large Numbers. Naturally, there is an Exact Weak Law with a weight of $1/n$. It follows from Theorem 1, since almost sure convergence implies convergence in probability. But, there are no Exact Strong Laws for a more general class of weights than our type. What is meant by that is that we can change our a_n and b_n in Theorem 1, but only slightly. It has been shown that a_n needs to be a slowly varying function divided by n. While b_n must be slowly varying in order to obtain an Exact Strong Law of Large Numbers.

We conclude with a final comment. Observe that $\sum_{n=1}^{N} a_n = o(b_N)$ in both our Weak and Strong Laws. Hence, if we shift our original random variables, X_{nj}, by a constant all our theorems still hold.

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