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ONE RELATOR QUOTIENTS OF THE MODULAR GROUP

BY

YÜCEL TÜRKER ULUTAŞ AND İSMAİL NACİ CANGÜL

Abstract. Modular group Γ is a discrete subgroup of PSL$(2, R)$ generated by two rotations of orders 2 and 3. Adding an extra relation to the existing two relations, one relator quotients of Γ are obtained.

1. Introduction. Hecke groups are the discrete subgroups of PSL$(2, R)$ generated by $a(z) = -1/z$ and $b(z) = -1/(z+\lambda_q)$ where $\lambda_q = 2\cos\frac{\pi}{q}$, $q \in N$, $q \geq 3$. Modular group Γ is one of the Hecke groups obtained for $q = 3$. It is therefore generated by

$$a(z) = -1/z \quad \text{and} \quad b(z) = -1/(z+1)$$

which are rotations of orders 2 and 3, respectively. It is well-known that Γ is isomorphic to the free product of two finite cyclic groups of orders 2 and 3, [1]. As a Fuchsian group, it has a signature $(0; 2, 3; 1)$, see [2].

Instead of $a(z)$ and $b(z)$, we shall briefly use a and b.

Γ can also be thought as a triangle group with an infinity. Recall that a triangle group $T(l, m, n)$ is a two generator group with representation

$$< a, b : a^l = b^m = (ab)^n = 1 > .$$

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If therefore has the signature $(0;1,m,n)$. It is known that $T(1,m,n)$ is finite precisely when $\frac{1}{1}+\frac{1}{m}+\frac{1}{n}>1$. The triangle groups satisfying this condition are as follows.

$T(1,n,n)\cong C_n$	the cyclic group of order n
$T(2,2,n)\cong D_n$	the dihedral group of order $2n$
$T(2,3,3)\cong A_4$	the tetrahedral group of order 12
$T(2,3,4)\cong S_4$	the octahedral group of order 24
$T(2,3,5)\cong A_5$	the icosahedral group of order 60

2. One relator quotients of Γ. Γ has a presentation $< a,b : a^2 = b^3 = 1 >$. Let $w = R(a,b) = 1$ be another relation in terms of a and b, for a cyclically reduced word $w = ab^{\varepsilon_1}ab^{\varepsilon_2}\cdots ab^{\varepsilon_n}$, where each ε_i is either 1 or 2. If we add such a relation to $a^2 = b^3 = 1$, we obtain a so-called one-relator quotient of Γ.

For a word w, we define two numbers $k = e_a(w)$ and $1 = e_b(w)$ as the sum of exponents of a's and b's in w, respectively. Of course $e_b(w) = \sum_{i=1}^{n}\varepsilon_i$. It is easy to see the following.

Theorem 2.1. *If* $e_a(w) = 0$ *then* $1 \le e_b(w) \le 2$ *and if* $e_a(w) = k$ *then* $k \le e_b(w) \le 2k$.

Two words w and w' are called equivalent if one of them is obtained from the other by cutting some part at the beginning and pasting it to the end at the same order. e.g. ab^2abab^2 is equivalent to $abab^2ab^2$.

We define $N_{k,l}$ as the total number of cyclically reduced non-equivalent words w with $e_a(w) = k$ and $e_b(w) = 1$. Then,

Theorem 2.2. $N_{n,n} = N_{n,2n} = N_{n,n+1} = N_{n,2n-1} = 1$.

3. Tables. In [4], the number of all cyclically reduced words and the number of non-equivalent cyclically reduced words in Γ with $e_a(w) = k$ and $e_b(w) = 1$ are given by

$$\alpha_{k,l} = \binom{k}{1-k} \tag{1}$$

and

$$N_{k,l} = \frac{1}{k} \sum_{d|(k,1-k)} \left[\varphi(d) \binom{k/d}{(1-k)/d} \right] \tag{2}$$

respectively; e.g. $N_{5,8} = 2$, $N_{6,9} = 4$ and $N_{8,13} = 7$.

Once having this formula, we can check that whether we have all words of the requried property or not.

We begin by giving an example of how we eliminate the words to obtain cyclically reduced ones for a given pair of integers $e_a(w)$ and $e_b(w)$.

Example 3.1. Let $e_a(w) = 7$ and $e_b(w) = 10$. Then by the formula (1), the number of all possible cyclically reduced words is 35. By formula (2) there are only 5 non-equivalent cyclically reduced ones. These are

$$abababababab^2ab^2ab^2$$
$$ababababab^2abab^2ab^2$$
$$ababab^2ababab^2ab^2$$
$$abab^2abababab^2ab^2$$

and

$$ababab^2abab^2abab^2.$$

We do not consider the words such as $ab^2ab^2ab^2abababab$ or $abab^2ab^2ab^2ababab$ as both are equivalent to the first word.

Let us add the first relation

$$abababababab^2ab^2ab^2 = 1,$$

to $a^2 = b^3 = 1$. Then as

$$aba = bababab^2ab^2ab^2,$$

we have

$$(aba)^2 = ab^2a = 1$$

and equivalently $b^2 = 1$. Then $a^2 = b^3 = b^2 = 1$ gives $a^2 = b = 1$. This means that the quotient group is $(2, 1, 2) \cong C_2$, the cyclic group of order two.

We similarly obtain C_2 as the quotient group for the other four words.

k	1	Possible Cyclically Reduced Words	Representation of Quotient Group	Abstract Structure of Quotient Group
0	1	b	$< a, b : a^2 = b^3 = b = 1 >$	C_2
0	2	b^2	$< a, b : a^2 = b^3 = b^2 = 1 >$	C_2
1	0	a	$< a, b : a^2 = b^3 = a = 1 >$	C_3
1	1	ab	$< a, b : a^2 = b^3 = ab = 1 >$	C_1
1	2	ab^2	$< a, b : a^2 = b^3 = ab^2 = 1 >$	C_1
2	2	$abab$	$< a, b : a^2 = b^3 = abab = 1 >$	D_3
2	3	$abab^2$	$< a, b : a^2 = b^3 = abab^2 = 1 >$	C_6
2	4	ab^2ab^2	$< a, b : a^2 = b^3 = ab^2ab^2 = 1 >$	D_3
3	3	$ababab$	$< a, b : a^2 = b^3 = ababab = 1 >$	D_3
3	4	$ababab^2$	$< a, b : a^2 = b^3 = ababab^2 = 1 >$	C_2
3	5	$abab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2 = 1 >$	C_2
3	6	$ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ab^2ab^2ab^2 = 1 >$	D_3
4	4	$abababab$	$< a, b : a^2 = b^3 = abababab = 1 >$	S_4
4	5	$abababab^2$	$< a, b : a^2 = b^3 = abababab^2 = 1 >$	D_3
4	6	$ababab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2 = 1 >$	C_6
4	6	$abab^2abab^2$	$< a, b : a^2 = b^3 = abab^2abab^2 = 1 >$	C_6
4	7	$abab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2ab^2 = 1 >$	D_3
4	8	$ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ab^2ab^2ab^2ab^2 = 1 >$	D_3
5	5	$ababababab$	$< a, b : a^2 = b^3 = ababababab = 1 >$	A_5
5	6	$ababababab^2$	$< a, b : a^2 = b^3 = ababababab^2 = 1 >$	A_4

5	7	$abababab^2ab^2$	$< a, b : a^2 = b^3 = abababab^2ab^2 = 1 >$	C_2
5	7	$abab^2abab^2ab$	$< a, b : a^2 = b^3 = abab^2abab^2ab = 1 >$	C_2
5	8	$ababab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2ab^2 = 1 >$	C_2
5	8	$abab^2ab^2abab^2$	$< a, b : a^2 = b^3 = abab^2ab^2abab^2 = 1 >$	C_2
5	9	$abab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2ab^2ab^2 = 1 >$	A_4
5	10	$ab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ab^2ab^2ab^2ab^2ab^2 = 1 >$	A_5
6	6	$abababababab$	$< a, b : a^2 = b^3 = abababababab = 1 >$	C_6
6	7	$abababababab^2$	$< a, b : a^2 = b^3 = abababababab^2 = 1 >$	C_2
6	8	$ababababab^2ab^2$	$< a, b : a^2 = b^3 = ababababab^2ab^2 = 1 >$	D_3
6	8	$abababab^2abab^2$	$< a, b : a^2 = b^3 = abababab^2abab^2 = 1 >$	D_3
6	8	$ababab^2ababab^2$	$< a, b : a^2 = b^3 = ababab^2ababab^2 = 1 >$	D_3
6	9	$abababab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abababab^2ab^2ab^2 = 1 >$	C_6
6	9	$ababab^2abab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2abab^2ab^2 = 1 >$	C_6
6	9	$ababab^2ab^2abab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2abab^2 = 1 >$	C_6
6	9	$abab^2abab^2abab^2$	$< a, b : a^2 = b^3 = abab^2abab^2abab^2 = 1 >$	C_6
6	10	$ababab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2ab^2ab^2 = 1 >$	D_3
6	10	$abab^2abab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2abab^2ab^2ab^2 = 1 >$	D_3
6	10	$abab^2ab^2abab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2abab^2ab^2 = 1 >$	D_3
6	11	$abab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2ab^2ab^2ab^2 = 1 >$	C_2
6	12	$ab^2ab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ab^2ab^2ab^2ab^2ab^2ab^2 = 1 >$	C_6
7	7	$ababababababab$	$< a, b : a^2 = b^3 = ababababababab = 1 >$	PSL(2, 7)
7	8	$ababababababab^2$	$< a, b : a^2 = b^3 = ababababababab^2 = 1 >$	C_1
7	9	$abababababab^2ab^2$	$< a, b : a^2 = b^3 = abababababab^2ab^2 = 1 >$	A_4
7	9	$ababababab^2abab^2$	$< a, b : a^2 = b^3 = ababababab^2abab^2 = 1 >$	A_4
7	9	$abababab^2ababab^2$	$< a, b : a^2 = b^3 = abababab^2ababab^2 = 1 >$	A_4
7	10	$ababababab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ababababab^2ab^2ab^2 = 1 >$	C_2
7	10	$abababab^2abab^2ab^2$	$< a, b : a^2 = b^3 = abababab^2abab^2ab^2 = 1 >$	C_2
7	10	$ababab^2ababab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ababab^2ab^2 = 1 >$	C_2
7	10	$abab^2abababab^2ab^2$	$< a, b : a^2 = b^3 = abab^2abababab^2ab^2 = 1 >$	C_2
7	10	$ababab^2abab^2abab^2$	$< a, b : a^2 = b^3 = ababab^2abab^2abab^2 = 1 >$	C_2
7	11	$abababab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abababab^2ab^2ab^2ab^2 = 1 >$	C_2
7	11	$ababab^2abab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2abab^2ab^2ab^2 = 1 >$	C_2
7	11	$abab^2ababab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ababab^2ab^2ab^2 = 1 >$	C_2
7	11	$ababab^2ab^2abab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2abab^2ab^2 = 1 >$	C_2
7	11	$abab^2ab^2abab^2abab^2$	$< a, b : a^2 = b^3 = abab^2ab^2abab^2abab^2 = 1 >$	C_2
7	12	$abab^2ab^2abab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2abab^2ab^2ab^2 = 1 >$	A_4
7	12	$abab^2abab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2abab^2ab^2ab^2ab^2 = 1 >$	A_4
7	12	$ababab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ababab^2ab^2ab^2ab^2ab^2 = 1 >$	A_4
7	13	$abab^2ab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = abab^2ab^2ab^2ab^2ab^2ab^2 = 1 >$	C_1
7	14	$ab^2ab^2ab^2ab^2ab^2ab^2ab^2$	$< a, b : a^2 = b^3 = ab^2ab^2ab^2ab^2ab^2ab^2ab^2 = 1 >$	PSL(2, 7)

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Department of Mathematics, Faculty of Science, University of Kocaeli, 41100 Kocaeli, Turkey.

E-mail: yturkerulutas@yahoo.com

Department of Pure Mathematics, Faculty of Science, University of Uludağ, Görükle 16059 Bursa, Turkey.

E-mail: cangul@uludag.edu.tr