BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 32, Number 4, December 2004

UNIONS OF LINES CONTAINED IN LOW DEGREE SUBVARIETIES AND WITH PRESCRIBED POSTULATION

BY

E. BALLICO

Abstract. Here we construct nodal and connected unions of lines $Y \subset \mathbf{P}^n$, $n \geq 2v+1 \geq 7$, with prescribed degree and arithmetic genus, Y contained in a union of s v-dimensional linear spaces and $h^1(\mathbf{P}^n, \mathcal{I}_Y(k)) = 0$

1. Introduction. An efficient way to produce curves equipped with a linear system (as needed for instance in coding theory) is to produce a finite union $Y \subset \mathbf{P}^n$ of lines. Indeed, to determine a line it is sufficient to give two points of it and if these points are defined over a field K, then the line is defined over K. Hence if K is a finite field, say $K = GF(q)$, then a line defined over $GF(q)$ has exactly $q^e + 1$ points defined over the extension $GF(q^e)$ of $GF(q)$ and these points are easily detected just knowing two of them. In our opinion this should be quite useful for a main problem in coding theory in which one evaluates a certain linear system induced by $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t))$, $t > 0$. To study the induced linear system one has to control the postulation of Y, i.e. its Hilbert function. If a curve $C \subset \mathbf{P}^n$ is contained in a closed subscheme V of $\mathbf{P}^n$, then the postulation of V imposes strong restrictions to the postulation of C because the restriction map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(C, \mathcal{O}_C(t))$ factors through the restriction map

Received by the editors April 23, 2003.
AMS 1991 Subject Classification: 14H99, 14N05.
Key words and phrases: Line, postulation, reducible curve.
The author was partially supported by MURST and GNSAGA of INdAM (Italy).

$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(V, \mathcal{O}_V(t))$. Here we consider union of lines, Y, contained in suitable unions of v-dimensional linear subspaces. In the literature suitable unions, V, of planes appear as degenerations of Veronese embeddings of $\mathbf{P}^2$ ([7], [8], [10], [4]), or ruled surfaces ([4]) or $K3$-surfaces ([5]). In [1] there is a degeneration of any Veronese embedding of $\mathbf{P}^v$, $v \geq 3$, to a union of v-dimensional linear spaces. In [4] unions of lines contained in V are used to study the postulation of V and the same may be done inductively for v-dimensional scrolls over a curve instead of two-dimensional scrolls over curves. The unions V of v-dimensional linear spaces we will consider here are degenerations of linear projections of minimal degree smooth rational scrolls (see Definition 1); the case $v = 2$ was considered in [4]. To state our first result we introduce the integers $r(n,k)$, $q(n,k)$, $n \geq 2$, $k \geq 1$, defined by the following relations:

$$(1) \qquad kr(n,k) + 1 + q(n,k) = \binom{n+k}{n}, 0 \leq q(n,k) < k$$

Theorem 1. *Fix integers n, k, s, v, d such that $n \geq 2v+1$, $k > s \geq 1$ and $1 \leq d \leq \sum_{i=1}^{s} r(v, k-i+1)$. Then there exist a union V of s v-dimensional linear subspaces of $\mathbf{P}^n$ and a nodal and connected union of d lines $Y \subset V$ such that $p_a(Y) = 0$ and $h^1(\mathbf{P}^n, \mathcal{I}_Y(k)) = 0$.*

A reduced curve $Y \subset \mathbf{P}^n$ is called a *tree* or a *stick figure* if it is connected, $p_a(Y) = 0$, all the irreducible components of Y are lines and it has only nodes as singularities. The integer $r(v,x)$ is the maximal degree of a tree $T \subset \mathbf{P}^v$ such that $h^0(\mathbf{P}^v, \mathcal{O}_{\mathbf{P}^v}(x)) \geq h^0(T, \mathcal{O}_T(x))$ and hence for which one can hope that $h^1(\mathbf{P}^v, \mathcal{I}_T(x)) = 0$ for some degree $r(v,x)$ tree $T \subset \mathbf{P}^v$. In the statement of Theorem 1 there is a telescoping sum: $\sum_{i=1}^{s} r(v, k-i+1)$. This telescoping sum makes not optimal the statement of Theorem 1 for general disjoint unions of v-dimensional linear spaces. We only claim that Theorem 1 is quite good for chains of v-planes in the sense of Definition 1 (and even optimal in the extremely particular case $k \le v$). To state our next result in full generality, we introduce the following notation. For all integers $n \ge 3$ and $t > 0$ set $A(n,t) := \{(d,g)$: there is a nodal and connected union of lines $Y \subset \mathbf{P}^n$ with $\deg(Y) = d$, $p_a(Y) = g$ and $h^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0\}$ and $B(n,t) := \{(d,g)$: there is a nodal and connected union of lines $Y \subset \mathbf{P}^n$ with $\deg(Y) = d$, $p_a(Y) = g$, $h^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0$ and $h^1(Y, \mathcal{O}_Y(t)) = 0\}$. For a discussion of the last condition in the definition of $B(n,t)$, see Remark 1.

Theorem 2. *Fix integers n, k, s and v such that $n \ge 2v + 1 \ge 7$ and $k > s \ge 1$. For every integer i with $1 \le i \le s$ fix a pair $(d_i, g_i) \in B(n, k-i+1)$ and set $d := d_1 + \cdots + d_s$, $g := g_1 + \cdots + g_s$. Then there exist a union V of s v-dimensional linear subspaces of $\mathbf{P}^n$ and a nodal and connected union of d lines $Y \subset V$ such that $p_a(Y) = g$ and $h^1(\mathbf{P}^n, \mathcal{I}_Y(k)) = 0$.*

Unions of lines with controlled postulation are also heavily used to construct curves with good cohomological properties (see [3], [6], [9] and refences therein).

In summary:

(a) certain unions of linear subspaces arise as degenerations of interesting varieties and we believe that they may be used to study geometric and cohomological properties of these interesting varieties;

(b) cohomological properties of unions of linear spaces quite often may be determined studying reducible curves (and very often union of lines) contained in them;

(c) the main drawback is that most properties of the union of linear spaces arising in (a) depend heavily from the varieties of which they are a limit (e.g. Veronese varieties and scrolls have different geometry) and hence it seems to us that the study in (b) must be carried independently for each interesting degeneration.

We work over an algebraically closed base field K. For the dependence of our results from the choice of K and for the case in which K is not algebraically closed, see Remark 2.

2. The Proofs. A closed subscheme $X \subset \mathbf{P}^n$ is said to have *maximal rank* if for every integer t the restriction map $\rho_{X,n,t} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(X, \mathcal{O}_X(t))$ has maximal rank, i.e. it is injective or surjective.

Definition 1. Fix integers $n \geq 2v+1 \geq 3$ and $s > 0$. A chain of s v-planes in $\mathbf{P}^n$ is a union $V \subset \mathbf{P}^n$ of s distinct v-dimensional linear subspaces of $\mathbf{P}^n$ such that there is an ordering $V_1, \dots, V_s$ of the irreducible components of V such that $\dim(V_i \cap V_j) = v - |i-j|$ for all i, j with $|i-j| \leq v$ and $V_i \cap V_j = \emptyset$ if $|i-j| > v$. Such an ordering of the irreducible components of V will be called a compatible ordering of the irreducible components or of the v-planes of V.

Obviously, for all integers $n \geq 2v+1 \geq 3$ and $s > 0$ there are several chains of s v-planes in $\mathbf{P}^n$.

Proof of Theorem 1. Fix a chain $V \subset \mathbf{P}^n$ of s v-planes and a compatible ordering $V_1, \dots, V_s$ of the irreducible components of V. By [2], case $g = 0$ of Theorem 1, for every integer $0 < d_i \leq r(v, k-i+1)$ there is a tree $Y_i \subset V_i$ with $\deg(Y_i) = d_i$ and Y_i with maximal rank as a subcurve of V_i. In particular by (1) the restriction map $\rho_{Y_i,v,k-i+1} : H^0(V_i, \mathcal{O}_{V_i}(k-i+1)) \to H^0(Y_i, \mathcal{O}_{Y_i}(k-i+1))$ is surjective. Moving each Y_i inside V_i with a suitable element of $\mathrm{Aut}(V_i)$ we may even assume $Y_i \cap Y_j \neq \emptyset$ if and only if $|i-j| \leq 1$, $\mathrm{card}(Y_i \cap Y_{i+1}) = 1$ for $i = 1, \dots, s-1$ and $Y := Y_1 \cup \cdots \cup Y_s$ nodal. Notice that Y is a tree. We may find positive integers $d_i \leq r(v, k-i+1)$, $1 \leq i \leq s$, such that $d_1 + \cdots + d_s = d$. For all integers i with $1 \leq i \leq s$ set $Y[i] := \cup_{i \leq j \leq s} Y_j$. Hence each curve $Y[j]$ is a tree. Since the restriction maps $\rho_{V_s,n,k-s+1} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k-s+1)) \to H^0(V_s, \mathcal{O}_{V_s}(k-s+1))$ and

$\rho_{Y_s,v,k-s+1} : H^0(V_s, \mathcal{O}_{V_s}(k-s+1)) \to H^0(Y_s, \mathcal{O}_{Y_s}(k-s+1))$ are surjective, the restriction map $\rho_{Y_s,n,k-s+1}$ is surjective. Fix an integer i with $1 \le i < s$ and take a general hyperplane H of $\mathbf{P}^n$ containing V_i. The residual scheme of $Y[i]$ with respect to H is the scheme $Y[i+1]$.

Claim. Fix a general $S_{i+1} \subset V_{i+1} \cap H$ with $\mathrm{card}(S_{i+1}) = d_{i+1} - 1$ and a general $S_j \subset V_j$, $i+2 \le j \le s$ with $\mathrm{card}(S_j) = d_j$. Set $S := \cup_{j \ge i+1} S_j$. We claim that $\rho_{Y_i \cup S,n,k-i+1}$ is surjective.

Proof of the Claim. For every $j \ge i+1$ set $S[j] := \cup_{a \ge j} S_a$. Fix an index $j > i+1$ and assume the surjectivity of $\rho_{S[j+1],n-1,k-j+1}$. By (1) we have $r(v,j) \le \binom{v+j-1}{v-1}$. Since S_j is general, we have $h^1(V_j, \mathcal{I}_{S_j}(k-s+j)) = 0$. Take a general hyperplane M of H containing $V_j \cap H$ and apply Horace Lemma with respect to the Cartier divisor M of H. We obtain the surjectivity of the map $\rho_{S[j],n-1,k-j+1}$. After $s-i-1$ steps we obtain the surjectivity of $\rho_{S,n-1,k-i}$. Then applying again Horace Lemma we get the surjectivity of $\rho_{Y_i \cup S,n,k-i+1}$. Using again Horace Lemma we obtain the surjectivity of $\rho_{Y_{i+1} \cup S[i+2],n,k-i+2}$. After finitely many steps we obtain the Claim.

The scheme $Y[i] \cap H$ is the union of Y_i and (if $i \le s-2$), the reduced set $Y[i+2] \cap H$ and the reduced set $Y_{i+1} \cap H \backslash Y_i \cap Y_{i+1}$ because each point of $Y_i \cap Y_{i+1}$ is a nodal point of $Y[i]$ and $V_{i+1} \cap H = V_{i+1} \cap V_i$. For each $j \ge i+2$ we may assume that $Y_j \cap H$ is formed by d_j general points of V_j. For general Y_{i+1} passing through the given point $Y_i \cap Y_{i+1}$ we may assume that $Y_{i+1} \cap H \backslash Y_i \cap Y_{i+1}$ is formed by d_{i+1} general points of V_{i+1}. Taking all V_j, $j > 1$, sufficiently general we may even assume that the restriction map $\rho_{Y_i \cup (H \cap Y[s+2],n,k-i+1}$ is surjective (use the Claim). Hence by Horace Lemma if $\rho_{Y[i+1],n,k-i}$ is surjective, then $\rho_{Y[i],k-i+1}$ is surjective. After $s-1$ steps we obtain the surjectivity of $\rho_{Y[1],n,k}$, proving the theorem.

Remark 1. Let $Y \subset \mathbf{P}^n$ be a reduced curve such that $h^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0$ and $H \subset \mathbf{P}^n$ a hyperplane containing no irreducible component of Y. We have $h^2(\mathbf{P}^n, \mathcal{I}_Y(t-1)) = h^1(Y, \mathcal{O}_Y(t-1))$. We have an exact sequence

$$0 \to \mathcal{I}_Y(t-1) \to \mathcal{I}_Y(t) \to \mathcal{I}_{Y\cap H,H}(t) \to 0 \tag{2}$$

Hence if $h^1(Y, \mathcal{O}_Y(t)) = 0$, then $h^1(H, \mathcal{I}_{H\cap Y,H}(t)) = 0$. By Castelnuovo-Mumford's lemma if $h^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = h^2(\mathbf{P}^n, \mathcal{I}_Y(t-1)) = 0$, then $h^1(\mathbf{P}^n, \mathcal{I}_Y(x)) = 0$ for every $x > t$ and the homogeneous ideal of Y is generate by forms of degree at most $t+1$. Now assume that Y is a union of lines. For every irreducible component T of Y let T' be the closure of $Y\backslash T$ in Y. Set $a(T) :=$ length$(T \cap T')$. If Y is nodal, then $a(T) =$ card$(T \cap T')$. Assume $a(T) \leq t$ for every line $T \subseteq Y$. The curve Y is obtained in $\deg(Y) - 1$ steps adding a line T to a finite union D of lines. We have a Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{D\cup T}(t-1) \to \mathcal{O}_D(t-1) \oplus \mathcal{O}_T(t-1) \to \mathcal{O}_{D\cap T}(t-1) \to 0 \tag{3}$$

Since length$(D \cap T) \leq a(T) \leq t$, the restriction map $H^0(T, \mathcal{O}_T(t-1)) \to H^0(D \cap T, \mathcal{O}_{D\cap T}(t-1))$ is surjective. Thus by (3) if $h^1(D, \mathcal{O}_D(t-1)) = 0$, then $h^1(D \cup T, \mathcal{O}_{D\cup T}(t-1)) = 0$. After finitely many steps we obtain $h^1(Y, \mathcal{O}_Y(t-1)) = 0$.

Proof of Theorem 2. We follow the proof of Theorem 1 just taking Y_i with $\deg(Y_i) = d_i$ and $p_a(Y_i) = g_i$. We only need to check that, setting $S_j := Y_j \cap H$, we have $h^1(V_j, \mathcal{I}_{S_j}(k-j-1)) = 0$. This follows from the last condition of the set of all pairs $B(v, s-k-1)$ by Remark 1.

Remark 2. Write $A(n,t)_K$ and $B(n,t)_K$ instead of $A(n,t)$ to stress that these sets a prori depend from the choice of the algebraically closed field K. By Lefschetz Principle these sets the same for all algebraically closed fields with the same charateristic. We do not know it these sets do not depend from the choice of K. When K is an infinite field Theorem 1 and Theorem 2 follow formally as in a standard proof of Lefschetz Principle from the corresponding result over the algebraic closure of K.

References

1. E. Ballico, *On projective degenerations of Veronese spaces*, in: Generalizations of Complex Analysis and their Applications in Physics, 45-51, Banach Center Publication Vol. 37, 1996.

2. E. Ballico, *Postulation of finite unions of lines in projective spaces*, Int. Journal of Pure and Applied Mathematics, **3**(4)(2002), 733-772.

3. E. Ballico, G. Bolondi, Ph. Ellia and R. M. Miro-Roig, *Curves of maximum genus in the range A and stick-figures*, Trans. Amer. Math. Soc., **349**(1996), 4589-4608.

4. E. Ballico and Ph. Ellia, *On projections of ruled and Veronese surfaces*, J. Algebra, **121**(2)(1989), 477-487.

5. C. Ciliberto, R. Miranda and M. Teicher, *Pillow degenerations of $K3$ surfaces*, in: Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 53-63, Nato Sci. Ser. II Math. Phys. Chem. 36, Kluwer Acad. Publ., Dordrecht, 2001.

6. G. Fløystad, *Construction of space curves with good properties*, Math. Ann., **289**(1991), 33-54.

7. B. Moishezon and M. Teicher, *Braid group techniques in complex geometry.* III. Projective degenerations of V_3, in: Classification of algebraic varieties (L'Aquila, 1992), 313-332, Contemp. Math. 162, Amer. Math. Soc., Providence, RI, 1994.

8. B. Moishezon and M. Teicher, *Braid group techniques in complex geometry.* IV. Braid monodromy of the branch curve S_3 of $V_3 \to \mathbf{CP}^2$ and application to $\pi_1(\mathbf{CP}^2, *)$, in: Classification of algebraic varieties (L'Aquila, 1992), 333-338, Contemp. Math. 162, Amer. Math. Soc., Providence, RI, 1994.

9. R. Hartshorne, *Families of curves in $\mathbf{P}^3$ and Zeuthen's problem*, Mem. Amer. Math. Soc., **130**(1997), no. 617, viii+96.

10. M. Teicher, *Hirzebruch surfaces: degenerations, related braid monodromy, Galois covers*, in: Algebraic Geometry. Hizebruch 70 (Warsaw, 1998), 305-325, Contemp. Math. 241, Amer. Math. Soc., Providence, RI, 1999.

Department of Mathematics, University of Trento, 38050 Povo (TN), Italy.

E-mail: ballico@science.unitn.it