BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 32, Number 4, December 2004

THE SECOND ORDER DIFFERENTIAL EQUATION WITH NONLINEAR DAMPING $u''(t) = u'(t)^q(c_1 + c_2 u(t)^p)$

BY

MENG RONG LI (李明融) AND X. Z. LIN (林脩竹)

Abstract. In this paper we study the following initial value problem for the nonlinear equation,

$$\begin{cases} u''(t) = u'(t)^q(c_1 + c_2 u(t)^p), \quad p,\ q \geq 1,\ c_1^2 + c_2^2 \neq 0, \\ u(0) = u_0,\ u'(0) = u_1. \end{cases}$$

We are interested in properties of positive solutions of the above problem. We have found blow-up phenomena and obtained some results on blow-up rates, blow-up constants and life-spans.

1. Introduction.

The Calligraphy Equation

Neglecting the friction force of the paper on which a calligrapher produces his work through a handwriting brush with mass $m(t)$ at time t, we suppose that $u(t)$ be the displacement of the motion of the brush at time t, then from the Newton's second law of motion we know that the force $F(t)$ at time t can be measured by $(m(t)u'(t))'$.

$$(m(t)u'(t))' = F(t). \tag{1.1}$$

Received by the editors September 15, 2003.

[1]There are more discussion which concern nonlinear differential equation in [3] and [4].

We assume that the force $F(t)$ depends on $u(t)$ and $u'(t)$, that is $F(t) = F(u(t), u'(t))$. From empirical studies, for some people, the change rate of the force is proportional to the change rate of velocity in a motion, that is, there is a positive real q so that

$$\frac{dF(u(t))}{dt} \diagup F(u(t)) = q\frac{du'(t)}{dt} \diagup u'(t). \tag{1.2}$$

This yields $F(u(t)) = cu'(t)^q$ for some constant c, and (1.1) becomes

$$(m(t)u'(t))' = cu'(t)^q. \tag{1.3}$$

Here q is called the temper-index of the equation (1.3). In this paper we consider fixed mass of the brush, $m(t) = m$ a positive constant. Consider a calligrapher with temper-index q creating his work, and being disturbed by another calligrapher with the same temper-index as well as characteristic p. Then we obtain equation (1.1) with external force $C_2(u')^q u^p$:

$$\begin{cases} u'' = (u')^q(c_1 + c_2 u^p), \\ u(0) = u_0,\ u'(0) = u_1. \end{cases} \tag{1.4}$$

We are interested in properties of solutions of the problem, particularly in phenomena on blow-up, blow-up rates, blow-up constants and life-spans. In next section, we separate q into three parts, $1 \le q < 2$, $q = 2$ and $q > 2$. And we find the blow-up time, blow-up rate and blow-up constant of u. For further informations on calligraphy equation we refer the reader to [1]

2. Existence and uniqueness of solution. To gain a rough estimate of the life-span of the solution for the initial value problem (2.1) below, in this subsection we reconsider the existence of the solutions of the following

[2]For results on the blow-up character of solution of the equation $(|\, u' \,|^{m-2}\, u')' = u^p$, see [2].

initial value problem for the nonlinear equation:

$$\begin{cases} u''(t) = u'(t)^q(c_1 + c_2u(t)^p), \quad p,\ q \geq 1,\ c_1^2 + c_2^2 \neq 0, \\ u(0) = u_0, u'(0) = u_1. \end{cases} \tag{2.1}$$

For $p \in \mathbb{Q}$, we say that p is odd (even, respectively) if $p = r/s$, $r \in \mathbb{N}$, $s \in 2\mathbb{N} + 1, (r, s) = 1$(common factor) and r is odd(even, respectively).

Define

$$T = \min \left\{ \begin{array}{l} \frac{1}{\mid u_1 \mid}, \ \frac{1}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}, \\ \frac{-\mid u_1 \mid + \sqrt{u_1^2 + 2(\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p)}}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}, \\ -1 + \sqrt{1 + \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}} \end{array} \right\},$$

where

$$N = \mid u_0 \mid + 1,\ M = \mid u_1 \mid + 1,$$
$$\alpha_1 = \mid c_1 \mid M^q p N^{p-1},\ \alpha_2 = \mid c_1 \mid q M^{q-1},\ \alpha_3 = \mid c_2 \mid q N^p M^{q-1},$$

and

$$\mathbb{X}_T = \{u \in C^2[0, T) : \parallel u \parallel_\infty \leq N \text{ and } \parallel u' \parallel_\infty \leq M\}.$$

By the standard arguments of existence of solutions to ordinary differential equations, one can easily prove the following result:

Theorem 2.1. *For any initial values u_0 and u_1, there exists a constant T given as above such that the problem* (2.1) *possesses exactly one solution u in $\mathbb{X}_T$.*

3. Blow-up phenomena. Consider the initial value problem (2.1).

Definition 3.1. A function $g:\mathbb{R}\to\mathbb{R}$ has blow-up time T^* and a blow-up rate q means that there is a finite number T^* such that $g(t)$ exists for $t<T^*$, and

$$\lim_{t\to T^*} g(t)^{-1}=0 \tag{3.1}$$

and there exists a nonzero $\beta\in\mathbb{R}$ with

$$\lim_{t\to T^*}(T^*-t)^q g(t)=\beta, \tag{3.2}$$

in this case β is called the blow-up constant of g.

For $u_1=0$, the solution u of problem (2.1) must be constant.

For $u_1\neq 0$ and $t\in[0,T^*)$, where $T^*=\inf\{t>0:u'(t)=0\}$, we have

$$\begin{cases} u'(t)^{2-q}=(2-q)(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E(0)) \quad \text{if } q\neq 2,\\[2ex] E(0)=\frac{u_1^{2-q}}{2-q}-(c_1u_0+\frac{c_2}{p+1}u_0^{p+1}) \end{cases} \tag{3.3}$$

and

$$\begin{cases} \ln|\,u'(t)\,|=(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E_1(0)) \quad \text{if } q=2,\\[2ex] E_1(0)=\ln|\,u_1\,|-(c_1u_0+\frac{c_2}{p+1}u_0^{p+1}). \end{cases} \tag{3.4}$$

Here yield the relations between $u(t)$ and $u'(t)$.

Lemma 3.2. *Suppose that $f\in C^1[t_0,\infty)\cap C^2(t_0,\infty)$, $f(t_0)>0$, $f'(t_0)<0$ and $f''(t)\le 0$ for $t>t_0$. Then there exists a finite positive number $T>t_0$ such that $f(T)=0$.*

Proof. Since $f\in C^1[t_0,\infty)$ and $f''(t)\le 0$ for $t>t_0$, we obtain that $f'(t)\le f'(t_0)<0$ and $f(t)\le f(t_0)+f'(t_0)(t-t_0)$. Hence there exists $t_1>t_0$ such that $f(t_1)<0$. By the continuity of f in $[t_0,\infty)$, there exists $T\in(t_0,t_1)$ such that $f(T)=0$.

Lemma 3.3. *Suppose that u is the classical solution of* (2.1). *If $u_0 \geq 0$, $c_2, u_1 > 0$, and $u_0^p \geq -\frac{c_1}{c_2}$, then $u(t), u'(t), u''(t) > 0$ for $t \in [0, T)$, where T is the life-span of u.*

Proof. Suppose that there exists a positive number t_0 such that $u'(t_0) \leq 0$. Since $u \in C^2$ and $u_1 > 0$, there exists a positive number t_1, defined by

$$t_1 = \inf\{t \in (0, t_0] : u'(t) = 0\},$$

such that $u'(t_1) = 0$ and $u'(t) \geq 0$ for $t \in [0, t_1]$. For $t \in [0, t_1]$, $u'(t) \geq 0$, we have

$$u(t)^p \geq -\frac{c_1}{c_2}, \ u''(t) \geq 0.$$

Therefore, $u'(t_1) \geq u_1 > 0$. This result contradicts with $u'(t_1) = 0$; thus we conclude that

$$u'(t) > 0 \quad \text{for} \quad t \in [0, T),$$

where T is the life-span of u. Together the equation (2.1) and the continuities of u, u' and u'', the lemma follows.

By Theorem 2.1, there exists the unique solution to the (2.1) on $[0, T)$, where T depends on the initial values as follows

$$T(u_0, u_1) = \min \left\{ \begin{array}{l} \frac{1}{|u_1|}, \ \frac{1}{|c_1|M^q + |c_2|M^q N^p}, \\ \\ \frac{-|u_1| + \sqrt{u_1^2 + 2(|c_1|M^q + |c_2|M^q N^p)}}{|c_1|M^q + |c_2|M^q N^p}, \\ \\ -1 + \sqrt{1 + \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}} \end{array} \right\}$$

and

$$N = | u_0 | + 1, \ M = | u_1 | + 1,$$

$$\alpha_1 = | c_1 | M^q p N^{p-1}, \ \alpha_2 = | c_1 | q M^{q-1}, \ \alpha_3 = | c_2 | q N^p M^{q-1}.$$

The function T has the following monotonicity property.

Lemma 3.4. *If* $u_0 \leq u_0^*$ *and* $u_1 \leq u_1^*$, *then* $T(u_0, u_1) \geq T(u_0^*, u_1^*)$.

Proof. Let

$$N^* =| u_0^* | +1,\ M^* =| u_1^* | +1,$$

$$\alpha_1^* =| c_1 | M^{*q} p N^{*p-1},\ \alpha_2^* =| c_1 | q M^{*q-1},\ \alpha_3^* =| c_2 | q N^{*p} M^{*q-1}.$$

(1) If $T(u_0, u_1) = \frac{1}{|u_1|}$, then by $u_1 \leq u_1^*$,

$$T(u_0, u_1) \geq \frac{1}{| u_1^* |} \geq T(u_0^*, u_1^*).$$

(2) If $T(u_0, u_1) = -1 + \sqrt{1 + \frac{1}{\alpha_1+\alpha_2+\alpha_3}}$, using the fact that $u_1 \leq u_1^*$, $p, q \geq 1$, we have $\alpha_1^* \geq \alpha_1 \geq 0$, $\alpha_2^* \geq \alpha_2 \geq 0$ and $\alpha_3^* \geq \alpha_3 \geq 0$. Therefore

$$T(u_0, u_1) \geq -1 + \sqrt{1 + \frac{1}{\alpha_1^* + \alpha_2^* + \alpha_3^*}} \geq T(u_0^*, u_1^*).$$

(3) If $T(u_0, u_1) = \frac{1}{|c_1|M^q + |c_2|M^q N^p}$, then by the conditions $u_0 \leq u_0^*$, $u_1 \leq u_1^*$ and $p, q \geq 1$, we obtain that $M^{*q} \geq M^q$ and $N^{*p} \geq N^p$. Thus

$$T(u_0, u_1) \geq \frac{1}{| c_1 | M^{*q} + | c_2 | M^{*q} N^{*p}} \geq T(u_0^*, u_1^*).$$

(4) If $T(u_0, u_1) = \frac{-|u_1| + \sqrt{u_1^2 + 2(|c_1|M^q + |c_2|M^q N^p)}}{|c_1|M^q + |c_2|M^q N^p}$, then from $u_0 \leq u_0^*$ and $u_1 \leq u_1^*$, it follows that $M^{*q} \geq M^q$, $N^{*p} \geq N^p$ and

$$\begin{aligned} T(u_0, u_1) &= \frac{2}{| u_1 | + \sqrt{{u_1}^2 + 2(| c_1 | M^q + | c_2 | M^q N^p)}} \\ &\geq \frac{2}{| u_1^* | + \sqrt{u_1^{*2} + 2(| c_1 | M^{*q} + | c_2 | M^{*q} N^{*p})}} \\ &\geq T(u_0^*, u_1^*). \end{aligned}$$

Lemma 3.5. *Suppose that* u *is the classical solution of* (2.1) *for* $q \in [1,2]$. *If* u *exists locally and* ${t_1}^*$ *is the life-span of* u, *then* u *blows up at* $t = {t_1}^*$.

Proof. Assume that $\lim_{t \to t_1^{*-}} u(t) = M < \infty$. By (3.3), (3.4) and $q \in [1,2]$, we have

$$\lim_{t \to t_1^{*-}} u'(t) = \begin{cases} [(2-q)(c_1 M + \frac{c_2}{p+1} M^{p+1} + E(0))]^{\frac{1}{2-q}} & \text{if } 1 \le q < 2, \\ \exp\{c_1 M + \frac{c_2}{p+1} M^{p+1} + E_1(0)\} & \text{if } q = 2. \end{cases}$$

Now we consider the following differential equation

$$\begin{cases} v''(t) = v'(t)^q (c_1 + c_2 v(t)^p), \\ v(0) = u(t_1^{*-}), v'(0) = u'(t_1^{*-}). \end{cases}$$

Let $v(t)$ be the existing unique solution to the above equation on $[0, T_v)$. Since $u(t_1^{*-})$ and $u'(t_1^{*-})$ are finite, so $T_v > 0$. Let

$$U(t) = \begin{cases} u(t) & \text{if } t \in [0, t_1^{*-}), \\ v(t - t_1^{*-}) & \text{if } t \in [t_1^{*-}, t_1^{*-} + T_v), \end{cases}$$

the problem(2.1) can be solved beyond the time ${t_1}^*$, this contradicts with the assumption of ${t_1}^*$. Therefore, u blows up at $t = {t_1}^*$.

3.1. Blow-up phenomena of u**.** To discuss blow-up phenomena of u with $u_1 \neq 0$, we separate this subsection into three parts $1 \le q < 2$, $q > 2$ and $q = 2$.

Case1. Blow-up phenomena for $1 \le q < 2$**.** In this situation, we have some blow-up results.

Theorem 3.6. *Suppose that* u *is the positive solution of* (2.1) *and*

$q \in [1,2)$, $c_2 > 0$, $u_0 \ge 0$, $u_1 > 0$, $u_0^p \ge -\frac{c_1}{c_2}$, then u blows up at time $t = T_{11}$ for some finite real number $T_{11} > 0$.

Remark 3.6. If we don't restrict ourself to the positiveness of the solution u to the equation (2.1), then we also have the following blow-up results:
If u is the solution of equation (2.1), $q \in [1,2]$ and one of the followings is valid:

(1) p is even, q is odd, $c_2 > 0$, $u_0 \le 0$, $u_1 < 0$, $u_0^p \ge -\frac{c_1}{c_2}$,
(2) p is odd, q is even, $c_2 > 0$, $u_0 \le 0$, $u_1 < 0$, $u_0^p \le -\frac{c_1}{c_2}$,
(3) p is even, q is even, $c_2 < 0$, $u_0 \le 0$, $u_1 < 0$, $u_0^p \ge -\frac{c_1}{c_2}$,
(4) p is odd, q is odd, $c_2 < 0$, $u_0 \le 0$, $u_1 < 0$, $u_0^p \le -\frac{c_1}{c_2}$,

then u blows up in finite time.

For a given function u in this work we use the following abbreviations

$$a(t) = u(t)^2,\ J(t) = a(t)^{-m},\ m = \frac{1}{2}(\frac{1}{2-q} - 1).$$

Proof of Theorem 3.6. Suppose that u is a global solution of equation (2.1).

(I) For $q = 1$, $u''(t) = u'(t)(c_1 + c_2 u(t)^p)$, by (3.5) and Lemma 3.3, we obtain that

$$\int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr = t \quad \text{for all } t > 0$$

and

$$u(t) > u_0 \text{ for } t > 0.$$

Using the fact that $c_1 + \frac{c_2}{p+1} r^{p+1} + E(0) > 0$ for $r \ge u_0$ (see the proof of Theorem 4.2), we get

$$\int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)} dr \le \int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)} dr \quad \text{for all } t > 0$$

and then

$$\int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)} dr \geq \lim_{t\to\infty} \int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\ dr = \lim_{t\to\infty} t.$$

Since the integral $\int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\ dr$ is finite (see the proof of Theorem 4.2), this leads to a contradictory conclusion with the above last estimate. Hence we can conclude that u only exists on $[0, T_{11})$, where T_{11} is the life-span of u. By Lemma 3.5, we obtain that u blows up at $t = T_{11}$.

(II) For $1 < q < 2$, $m = \frac{1}{2}(\frac{1}{2-q} - 1) > 0$, and we claim that there exists a finite time $T_{11} > 0$ such that

$$J(T_{11}) = 0.$$

According to Lemma 4.1, we find that u' and u blow up simultaneously. Thus $u \in C^2[0, T)$, where T is a blow-up time of u. By (3.3) and Lemma 3.3,

$$u'(t)^{2-q} = (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)) \quad \text{for all } t > 0.$$

By direct computation, we obtain that

$$J'(t) = -ma(t)^{-(m+1)} a'(t) = -ma(t)^{-(m+1)} 2u(t)u'(t),$$

$$\begin{aligned} a''(t) &= 2u'(t)^2 + 2u'(t)^q (c_1 u(t) + c_2 u(t)^{p+1}) \\ &= 2(1 + \frac{1}{2-q}) a'(t)^2 + 2u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0)) \end{aligned}$$

and

$$a(t)a''(t) = \frac{1}{2}(1 + \frac{1}{2-q}) a'(t)^2 + 2a(t)u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0)).$$

Hence we have

$$\begin{aligned} J''(t) &= -ma(t)^{-(m+2)}(a(t)a''(t)-(m+1)a'(t)^2) \\ &= -ma(t)^{-(m+2)}2a(t)u'(t)^q\Big(\frac{c_2p}{p+1}u(t)^{p+1}-E(0)\Big). \end{aligned}$$

With the help of Lemma 3.3, $u(t), u'(t), u''(t) > 0$ for all $t > 0$, and there exists a finite time $t_1 > 0$ such that

$$\frac{c_2p}{p+1}u(t_1)^{p+1}-E(0)\geq 0.$$

Herewith, $J(t_1) > 0$, $J'(t_1) < 0$ and $J''(t) \leq 0$ for $t \geq t_1$. These and Lemma 3.2 imply that there exists a finite positive number $T_{11} > t_1$ such that $J(T_{11}) = 0$. Thus u blows up in finite time. This leads to contradiction and we have shown that u exists locally and by Lemma 3.5, u blows up in finite time.

Proof of Remark 3.6. The arguments are similar to the proof of Theorem 3.6, we only mention the case (1).

Let $v(t) = -u(t)$. By the fact that p is even and q is odd, we have $v(t)^p = u(t)^p$ and $v'(t)^q = -u'(t)^q$. We get

$$\begin{cases} v''(t) = -u''(t) = -u'(t)^q(c_1+c_2u(t)^p) = v'(t)^q(c_1+c_2v(t)^p), \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

Since $u_0 \leq 0, u_0^p \geq -\frac{c_1}{c_2}, u_1 < 0$ and p is even, we have $v_0 \geq 0, v_1 > 0$ and $v_0^p = u_0^p \geq -\frac{c_1}{c_2}$. By Theorem 3.6 and Theorem 3.8 below, v blows up, so does u.

Next we estimate the blow-up rate and blow-up constant.

Theorem 3.7. *Suppose that u is a solution of* (2.1) *and* $1 \leq q < 2$. *If u blows up in finite time, then the blow-up rate of u is $\frac{2-q}{p+q-1}$ and the blow-up* *constant of* u *is* $(\frac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\frac{c_2}{p+1}]^{\frac{-1}{p+q-1}}$.

Proof. Set $i=\frac{p+q-1}{2-q}$. By some calculations on (2.1) using L. Hôpital's rule we obtain

$$\begin{aligned}\lim_{t\to T_{11}^-}\frac{u^{-i}}{T_{11}-t} &= \lim_{t\to T_{11}^-} iu(t)^{-(i+1)}u'(t)\\ &= \lim_{t\to T_{11}^-} i\frac{[(2-q)(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E(0))]^{\frac{1}{2-q}}}{u(t)^{i+1}}\\ &= \frac{p+q-1}{2-q}[(2-q)\frac{c_2}{p+1}]^{\frac{1}{2-q}}.\end{aligned}$$

Thus

$$\lim_{t\to T_{11}^-}(T-t)^{\frac{2-q}{p+q-1}}u(t)=(\frac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\frac{c_2}{p+1}]^{\frac{-1}{p+q-1}}.$$

Case2. Blow-up phenomena for $q=2$. In the particular case of $q=2$, we obtain an interesting blow-up result and special blow-up constant.

Theorem 3.8. *For $q=2$, if u is the positive solution of* (2.1) *and* $c_2>0,\ u_0\geq 0,\ u_1>0,\ u_0^p\geq -\frac{c_1}{c_2}$, *then u blows up logarithmically at finite time $t=T_{12}$ and*

$$\lim_{t\to T_{12}^-}[\frac{1}{-\ln(T_{12}-t)}]^{\frac{1}{p+1}}u(t)=[\frac{c_2}{p+1}]^{-\frac{1}{p+1}}.$$

Proof. Assume that u is a global solution of (2.1). By (3.4) and Lemma 3.3,

$$\ln|\ u'(t)\ |=(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E_1(0))\quad\text{for all } t>0.$$

Since $u(t)$, $u'(t)$ blow up simultaneosly (see Lemma 4.1), $u\in C^2[0,T_{12})$, where T_{12} is blow-up time of u.

Let $K(t) = a(t)^{-1}$, then

$$K'(t) = -a(t)^{-2}a'(t) = -2a(t)^{-2}u(t)u'(t)$$

and

$$\begin{aligned} K''(t) &= -a(t)^{-3}(a(t)a''(t) - 2a'(t)^2) \\ &= -a(t)^{-3}a'(t)^2\{\frac{1}{2}[1 + u(t)(c_1 + c_2u(t)^p)] - 2\}. \end{aligned}$$

By Lemma 3.3, $u(t), u'(t), u''(t) > 0$ for $t > 0$. Hence there exists $t_0 > 0$ such that

$$u(t) \geq (\frac{|\ c_1\ | + 3}{c_2})^{\frac{1}{p}} + 1 \quad \text{for } t \geq t_0$$

and

$$\frac{1}{2}[(1 + u(t)(c_1 + c_2u(t)^p)] - 2 \geq 0 \quad \text{for } t \geq t_0.$$

We conclude that

$$K(t_0) > 0, K'(t) < 0 \text{ and } \quad K''(t) < 0 \text{ for } t \geq t_0,$$

thus by Lemma 3.2 there exists positive number T_{12} such that $K(T_{12}) = 0$ and u blows up at time $t = T_{12}$. This result contradicts with our assumption that u is a global solution of problem (2.1). Therefore u can exist only locally. By Lemma 3.5, u blows up in finite time. After some computations we get

$$\begin{aligned} \lim_{t\to T_{12}^-} -\ln(T_{12} - t)u(t)^{-(p+1)} &= \lim_{t\to T_{12}^-} \frac{u(t)^{-p}u'(t)^{-1}}{(p+1)(T_{12} - t)} \\ &= \lim_{t\to T_{12}^-} \frac{pu(t)^{-(p+1)} + u(t)^{-p}u'(t)^{-2}u''(t)}{p+1}. \end{aligned}$$

Using (2.1), we obtain $u''(t)=u'(t)^2(c_1+c_2u(t)^p)$ and

$$\begin{aligned}\lim_{t\to T_{12}^-} -\ln(T_{12}-t)u(t)^{-(p+1)} &= \lim_{t\to T_{12}^-}\frac{pu(t)^{-(p+1)}+u(t)^{-p}(c_1+c_2u(t)^p)}{p+1}\\ &= \frac{c_2}{p+1}.\end{aligned}$$

Herefrom follows the conclusion of Theorem 3.8.

Case3. Blow-up phenomena for $q>2$. Under $q>2$ we have the boundedness for the solution.

Theorem 3.9. *For $q>2$, if u is the positive solution of* (2.1) *and $c_2>0$, $u_0\geq 0$, $u_1>0$, $u_0^p\geq -\frac{c_1}{c_2}$, then u is bounded in $[0,T)$, where T is the life span of u.*

Proof. Integrating the equation (2.1) from 0 to t to obtain

$$\frac{u'(t)^{2-q}}{2-q}-\frac{u_1^{2-q}}{2-q}=c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}-c_1u_0-\frac{c_2}{p+1}u_0^{p+1}.$$

For $t\in[0,T)$, by Lemma 3.3, $u(t),u'(t)>0$ and

$$\frac{u_1^{2-q}}{q-2}>c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}-c_1u_0-\frac{c_2}{p+1}u_0^{p+1}.$$

Since that $c_2>0$ and $u(t)>0$ for $t\in[0,T)$, u is bounded in $[0,T)$.

3.2. Blow-up phenomena of u'. In this subsection we come back to the consideration of blow-up phenomena of u'.

Theorem 3.10. *For $q\geq 1$, if u is a positive solution of* (2.1) *and $c_2>0$, $u_0\geq 0$, $u_1>0$, $u_0^p\geq -\frac{c_1}{c_2}$, then u' blows up at time $t=T_2$.*

Proof. We separate this proof into three parts: $1\leq q<2$, $q=2$ and $q>2$.

(I) For $1 \leq q < 2$, by Theorem 3.6 and Lemma 4.1 below, u and u' blow up in finite time.

(II) For $q = 2$, using Theorem 3.8 and Lemma 4.1 below, then u and u' blow up in finite time.

(III) In the case $q > 2$, let

$$b(t) = u'(t)^2, \;\; L(t) = b(t)^{-\alpha},$$

where $\alpha = \frac{1}{2}(q-1)$, we have

$$L'(t) = -\alpha b(t)^{-(\alpha+1)} b'(t) = -2\alpha b(t)^{-(\alpha+1)} u'(t)u''(t),$$

and

$$\begin{aligned} L''(t) &= -\alpha b(t)^{-(\alpha+2)} [(\frac{1}{2}(1+q)-(\alpha+1))b'(t)^2 + 2c_2 p b(t) u(t)^{p-1} u'(t)^{q+2}] \\ &= -2pc_2 \alpha b(t)^{-(\alpha+1)} u(t)^{p-1} u'(t)^{q+2}. \end{aligned}$$

From Lemma 3.3, $u(t) > 0$, $u'(t) > 0$ and $u''(t) > 0$ for $t > 0$, we obtain that $L'(t), L''(t) < 0$ for $t > 0$. Now we need to check that u doesn't blow up earlier than u'. By Theorem 3.9, u is bounded. Using Lemma 3.2, there exists a finite number T_2 such that $L(T_2) = 0$. Since $q > 2$, we $\alpha > 0$, we obtain that u' blows up at finite time $t = T_2$.

We want to calculate blow-up rate and blow-up constant of u', again under three cases: $1 \leq q \leq 2$, $q = 2$ and $q > 2$.

Theorem 3.11. *Under the conditions in Theorem* 3.10, *for* $1 \leq q < 2$, u' *blows up in finite time with blow-up rate* $\frac{p+1}{p+q-1}$ *and blow-up constant*

$$\Big[\frac{c_2(p+q-1)}{p+1}\Big(\frac{c_2(2-q)}{p+1}\Big)^{\frac{-p}{p+1}}\Big]^{\frac{-(p+1)}{p+q-1}}.$$

Proof. By Lemma 4.1 u and u' have the same blow-up time. According to (2.1), L. Hôpital's rule and Theorem 3.7 we have

$$\begin{aligned}
&\lim_{t\to T_2^-}\frac{u'(t)^{\frac{1-p-q}{p+1}}}{(T_2-t)}\\
=&\lim_{t\to T_2^-}\frac{p+q-1}{p+1}u'(t)^{\frac{-(2p+q)}{p+1}}u''(t)\\
=&\lim_{t\to T_2^-}\frac{c_2(p+q-1)}{p+1}[(2-q)(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E(0))]^{\frac{-p}{p+1}}u(t)^p\\
=&\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}.
\end{aligned}$$

Thus

$$\lim_{t\to T_2^-}(T_2-t)^{\frac{p+1}{p+q-1}}u'(t)=[\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}]^{\frac{-(p+1)}{p+q-1}}.$$

Theorem 3.12. *Under the conditions in Theorem* 3.10, *for* $q=2$, u' *blows up in finite time* T_2, *and*

$$\lim_{t\to T_2^-}[-\ln(T_2-t)]^{\frac{p}{p+1}}(T_2-t)u'(t)=c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}.$$

Proof. According to Lemma 4.1 u and u' have the same life-span. By (2.1), L. Hôpital's rule and Theorem 3.8 we have

$$\begin{aligned}
&\lim_{t\to T_2^-}[-\ln(T_2-t)]^{\frac{p}{p+1}}(T_2-t)u'(t)\\
=&\lim_{t\to T_2^-}\frac{\frac{p}{p+1}[-\ln(T_2-t)]^{\frac{-1}{p+1}}(T_2-t)-[-\ln(T_2-t)]^{\frac{p}{p+1}}}{-(c_1+c_2u(t)^p)}\\
=&c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}.
\end{aligned}$$

Theorem 3.13. *Under the conditions in Theorem* 3.10, *for* $q>2$, u' *blows up in finite time with blow-up rate* $\frac{1}{q-1}$ *and blow-up constant* $[(q-$

$1)(c_1+c_2u(T_2)^p)]^{\frac{1}{1-q}}$.

Proof. For $q>2$, by (2.1) and L. Hôpital's rule we have

$$\begin{aligned}\lim_{t\to T_2^-}\frac{u'(t)^{1-q}}{(T_2-t)} &= \lim_{t\to T_2^-}(1-q)u'(t)^{-q}u''(t)(-1)\\ &= (q-1)(c_1+c_2u(T_2)^p).\end{aligned}$$

Thus

$$\lim_{t\to T_2^-}(T_2-t)^{\frac{1}{q-1}}u'(t)=[(q-1)(c_1+c_2u(T_2)^p)]^{\frac{1}{1-q}}.$$

3.3. Blow-up phenomena of u''. We want to calculate blow-up rate and blow-up constant of u'' in the this subsection.

Theorem 3.14. *Under the conditions in Theorem* 3.10 *suppose that* u *is a positive solution of* (2.1). *For* $q\geq 1$, *then* u'' *blows up at time* $t=T_3$ *for some* $T_3>0$. *Furthermore, for*

(I) $q\in[1,2)$, *the blow-up rate of* u'' *is* $\frac{q(p+1)}{p+q-1}+\frac{p(2-q)}{p+q-1}$ *and the blow-up constant is*

$$c_2^{\frac{-1}{p+q-1}}(2-q)^{\frac{p}{p+q-1}}(p+1)^{\frac{p+q}{p+q-1}}(p+q-1)^{\frac{-(2p+q)}{p+q-1}}.$$

(II) $q=2$, *then* u'' *blows up logarithmically and*

$$\begin{aligned}&\lim_{t\to T_3^-}\{[-\ln(T_3-t)]^{\frac{p}{p+1}}(T_3-t)\}^q\{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p u''(t)\\ =\ & c_2^{\frac{1-q}{p+1}}(p+1)^{\frac{p(1-q)}{p+1}}.\end{aligned}$$

(III) $q>2$, *the blow-up rate of* u'' *is* $\frac{q}{q-1}$ *and the blow-up constant is*

$$(q-1)^{\frac{q}{1-q}}(c_1+c_2u(T_3)^p)^{\frac{1}{1-q}}.$$

Proof. According to Theorem 3.10 and Lemma 4.1 below, u' and u'' blow up at the same time $t = T_3$.

(I) For $1 \le q < 2$, by Lemma 4.1, u, u' and u'' possess the same blow-up time . Using (2.1), Theorem 3.7 and Theorem 3.11, we conclude that

$$\begin{aligned}
&\lim_{t\to T_3^-} (T_3-t)^{\frac{q(p+1)}{p+q-1}+\frac{p(2-q)}{p+q-1}} u''(t) \\
= &\lim_{t\to T_3^-} (T_3-t)^{\frac{q(p+1)}{p+q-1}} u'(t)^q (T_3-t)^{\frac{p(2-q)}{p+q-1}} (c_1 + c_2 u(t)^p) \\
= &\; c_2^{\frac{-1}{p+q-1}} (2-q)^{\frac{p}{p+q-1}} (p+1)^{\frac{p+q}{p+q-1}} (p+q-1)^{\frac{-(2p+q)}{p+q-1}}.
\end{aligned}$$

(II) For $q = 2$, using Lemma 4.1, u, u' and u'' have the same blow-up time. Thus T_3 is also blow-up time of u and u'. By (2.1), Theorem 3.8 and Theorem 3.12, we conclude that

$$\begin{aligned}
&\lim_{t\to T_3^-} \{[-\ln(T_3-t)]^{\frac{p}{p+1}} (T_3-t)\}^q \{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p u''(t) \\
= &\lim_{t\to T_3^-} \{[-\ln(T_3-t)]^{\frac{p}{p+1}} (T_3-t)\}^q u'(t)^q \{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p (c_1 + c_2 u(t)^p) \\
= &\; c_2^{\frac{1-q}{p+1}} (p+1)^{\frac{p(1-q)}{p+1}}.
\end{aligned}$$

(III) For $q > 2$, by Lemma 4.1, u'' and u' blow up contemporaneously in finite time. Thanks to Lemma 3.3 we have $u(t) > 0$ and $u(t)^p \ge -\frac{c_1}{c_2}$. Since $c_2 > 0$, $c_1 + c_2 u(t)^p > 0$. By (2.1) and Theorem 3.13, we conclude that

$$\begin{aligned}
&\lim_{t\to T_3^-} (T_3-t)^{\frac{q}{q-1}} u''(t) \\
= &\lim_{t\to T_3^-} (T_3-t)^{\frac{q}{q-1}} u'(t)^q (c_1 + c_2 u(t)^p) \\
= &\; (q-1)^{\frac{q}{1-q}} (c_1 + c_2 u(T_3)^p)^{\frac{1}{1-q}}.
\end{aligned}$$

4. Estimations for the life-spans. To estimate the life-span of the solution of the equation (2.1), we separate this section into two parts, $1 \leq q < 2$ and $q = 2$. Here the life-span T of u means that u is the solution of problem (2.1) and the existence interval of u is contained only in $[0, T)$ so that the problem (2.1) has the solution $u \in C^2[0, T)$. We have the following results.

Lemma 4.1. *Suppose that $u \in C^2[0, T)$ is a positive solution of problem (2.1) and that $c_2 > 0$, $u_0 \geq 0$, $u_1 > 0$, $u_0^p \geq \frac{-c_1}{c_2}$. For $1 \leq q \leq 2$, $u(t)$ and $u'(t)$ blow up simultaneously; and so does u''. For $q > 2$, $u'(t)$ and u'' blow up at the same time.*

Proof. (I) For $1 \leq q < 2$, by (3.3) we have

$$u'(t)^{2-q} = (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)).$$

(1) First, we claim that if u blows up in finite time, then so does u'. According to Theorem 3.6, u blows up at time $t = T_{11}$. Since $\lim_{t \to T_{11}^-} \frac{1}{u(t)} = 0$, we have

$$\begin{aligned}
\lim_{t \to T_{11}^-} \frac{1}{u'(t)^{2-q}} &= \lim_{t \to T_{11}^-} \frac{1}{(2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0))} \\
&= \lim_{t \to T_{11}^-} \frac{\frac{1}{u(t)^{p+1}}}{(2-q)(\frac{c_1}{u(t)^p} + \frac{c_2}{p+1} + \frac{E(0)}{u(t)^{p+1}})} \\
&= 0.
\end{aligned}$$

Therefore,

$$\lim_{t \to T_{11}^-} \frac{1}{u'(t)} = 0.$$

Thus, u and u' blows up at the same finite time.

(2) We claim that if u' blows up in finite time, then so does u. With the help of Theorem 3.10, u' blows up at time $t = T_2$. Assume that u doesn't

blow up at time $t = T_2$. Let

$$\lim_{t \to T_2^-} u(t) = M < \infty.$$

Then

$$\begin{aligned}
\lim_{t \to T_2^-} u'(t)^{2-q} &= \lim_{t \to T_2^-} (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)) \\
&= (2-q)(c_1 M + \frac{c_2}{p+1} M^{p+1} + E(0)) \\
&< \infty.
\end{aligned}$$

This result contradicts with the fact that $u'(t)$ blows up at time $t = T_2$. It deduces that u blows up at time $t = T_2$. Combining (1) with (2), we conclude that u and u' blow up simultaneously.

(II) For the case $q = 2$, by (3.4), we have

$$\ln | u'(t) |= c_1 u(t) + \frac{c_2}{p+1} u(t)^p + E_1(0).$$

(3) We claim that if u blows up in finite time, then so does u'.

By Theorem 3.8 and Lemma 3.3, u blows up at time $t = T_{12}$ and $u(t), u'(t) > 0$ for $0 \le t < T_{12}$. Since that $c_2 > 0$ and u blows up toward positive direction, $\ln | u' |$ also blows up toward positive direction. Thus u' blow up at time $t = T_{12}$.

(4) We now prove that u' blows up then so does u. Using Theorem 3.10 and Lemma 3.3, u' blows up at time $t = T_2$ and $u(t), u'(t) > 0$ for $0 \le t < T_{12}$. Assume that u doesn't blow up at time $t = T_2$. Set

$$\lim_{t \to T_2^-} u(t) = M < \infty.$$

Then

$$\begin{aligned}\lim_{t\to T_2^-}\ln|\,u'(t)\,| &= \lim_{t\to T_2^-}(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E_1(0))\\ &= (2-q)(c_1M+\frac{c_2}{p+1}M^{p+1}+E_1(0))\\ &< \infty.\end{aligned}$$

This result is contradictory to the fact that u' blows up in finite time. It deduces that u blows up at time $t=T_2$. Together (3) and (4), we conclude that u and u' blow up simultaneously. From (2.1), we have

$$u''(t)=u'(t)^q(c_1+c_2u(t)^P).$$

Since that u and u' blow up toward positive direction at the same time and $c_2>0$, u'' blows up toward positive direction.

(III) Under $q>2$, according to Theorem 3.10, u' blows up at time $t=T_2$. By Theorem 3.9, we obtain that u is bounded in $[0,T_2)$, and, by Lemma 3.3, we have $u'(t)>0$ for $t\in[0,T_2)$. Thus the following limit exists,

$$\lim_{t\to T_2^-} c_1+c_2u(t)^p.$$

Since $u_0\geq\frac{-c_1}{c_2}$ and $u'(t)>0$ for $t\in[0,T_2)$, we have

$$\lim_{t\to T_2^-} c_1+c_2u(t)^p>0.$$

From $u''(t)=u'(t)^q(c_1+c_2u(t)^p)$, it deduces that u' and u'' blow up simultaneously.

We have the following estimates for the life-span of solution to the equation (2.1).

Theorem 4.2. *Suppose that $u\in C^2[0,T)$ is the positive solution of (2.1) and T is life-span of u and that T_{11}^* is blow-up time of u. Under the* *same conditions as in Theorem* 3.6, T *is bounded. For* $1 \le q < 2$, *we have the estimation*

$$T \le T_{11}^* = (2-q)^{\frac{1}{q-2}} \int_{u_0}^{\infty} (c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}}\ dr.$$

For $q = 2$, *we have*

$$T \le T_{12}^* := \int_{u_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\ dr,$$

where $E_1(0) = \ln \mid u_1 \mid - (c_1 u_0 + \frac{c_2}{p+1} u_0^{p+1})$.

Proof. (I) For $1 \le q < 2$, by (3.3),

$$u'(t)^{2-q} = (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)).$$

Using the fact that $u'(t) > 0$ for $t \in [0, T_{11}^*)$ and $u'(t) = [(2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0))]^{\frac{1}{2-q}}$, we have

$$\frac{u'(t)}{(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0))^{\frac{1}{2-q}}} = (2-q)^{\frac{1}{2-q}},$$

$$\int_0^t \frac{u'(r)}{(c_1 u + \frac{c_2}{p+1} u^{p+1} + E(0))^{\frac{1}{2-q}}(r)}\ dr = (2-q)^{\frac{1}{2-q}} t$$

and

$$\int_{u_0}^{u(t)} \frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{2-q}}}\ dr = (2-q)^{\frac{1}{2-q}} t. \tag{3.5}$$

We claim that $T_{11}^* < \infty$. By $u_0 \ge (\frac{-c_1}{c_2})^{\frac{1}{p}}$ and

$$c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0) = \int_{u_0}^{r} (c_1 + c_2 s^p)\ ds + \frac{u_1^{2-q}}{2-q},$$

we obtain that $c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0) > 0$ for $r \ge u_0$. And it is continuous on

$[u_0, a]$ for $a \geq u_0$. Therefore the function $\frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{2-q}}}$ is integrable and positive on $[u_0, a]$ for $a \geq u_0$. Thus $T_{11}^* < \infty$. Since $u \in C^2[0, T)$, $T \leq T_{11}^*$.

(II) For $q = 2$, by (3.4),

$$\ln |\, u'(t)\, | = c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0).$$

Seeing that $u'(t) > 0$, we have

$$\frac{u'(t)}{\exp\left(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0)\right)} = 1,$$

$$\int_0^t \frac{u'(t)}{\exp\left(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0)\right)}\, dr = t$$

and

$$\int_{u_0}^{u(t)} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr = t.$$

We next claim that $T_{12}^* < \infty$. Set

$$f(r) = c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0).$$

Then $f'(r) \geq 0$ for $r^p \geq \frac{-c_1}{c_2}$ and $f''(r) \geq 0$ for $r \geq 0$. So there exists $r_0 > 0$, $r_0^p \geq \frac{-c_1}{c_2}$, such that $f(r) > 0$ for $r \geq r_0$.

We calculate

$$\begin{aligned} &\int_{u_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} dr \\ = \;& \int_{u_0}^{r_0} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} dr + \int_{r_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} dr \end{aligned}$$

Since $\frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}$ is a continuous function on $[u_0, r_0]$, the first integrand is bounded. From

$$\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right) > c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0) > 0 \text{ for } r \geq r_0,$$

we obtain

$$\frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} < \frac{1}{\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} \text{ for } r \geq r_0.$$

By $\int_{r_0}^{\infty} \frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0))}\, dr < \infty$, and the comparison test, the second integrand is bounded. Therefore, T_{12}^* is bounded. Since $u \in C^2(0,T)$, we conclude $T \leq T_{12}^*$.

5. Conclusion. Finally, we summarize the results from all preceding discussions.

5.1. Tables of results. Suppose that u is a classical positive solution of (2.1) and that $c_2 > 0$, $u_0 \geq 0$, $u_1 > 0$ and $u_0^p \geq -\frac{c_1}{c_2}$. We have the following tables:

Blow-up Phenomena for $1 \leq q < 2$

	Behavior	Blow-up time
u	Blow-up	$(2-q)^{\frac{1}{q-2}} \int_{u_0}^{\infty} (c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}}\, dr$
u'	Blow-up	The same as u
u''	Blow-up	The same as u

	Blow-up rate	Blow-up constant
u	$\frac{2-q}{p+q-1}$	$\left(\frac{p+q-1}{2-q}\right)^{-\frac{2-q}{p+q-1}} \left[(2-q)\frac{c_2}{p+1}\right]^{\frac{-1}{p+q-1}}$
u'	$\frac{p+1}{p+q-1}$	$\left[\frac{c_2(p+q-1)}{p+1}\left(\frac{c_2(2-q)}{p+1}\right)^{\frac{-p}{p+1}}\right]^{\frac{-(p+1)}{p+q-1}}$
u''	$\frac{q(p+1)}{p+q-1} + \frac{p(2-q)}{p+q-1}$	$c_2^{\frac{-1}{p+q-1}} (2-q)^{\frac{p}{p+q-1}} (p+1)^{\frac{p+q}{p+q-1}} (p+q-1)^{\frac{-(2p+q)}{p+q-1}}$

Blow-up Phenomena for $q = 2$

	Behavior	Blow-up time
u	Blow-up	$\int_{u_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr$
u'	Blow-up	The same as u
u''	Blow-up	The same as u

u	$\lim_{t\to T_{12}^-} [\frac{1}{-\ln(T_{12}-t)}]^{\frac{1}{p+1}} u(t) = [\frac{c_2}{p+1}]^{-\frac{1}{p+1}}$
u'	$\lim_{t\to T_2^-} [-\ln(T_2-t)]^{\frac{p}{p+1}} (T_2-t)u'(t) = c_2^{\frac{-1}{p+1}} (\frac{1}{p+1})^{\frac{p}{p+1}}$
u''	$\lim_{t\to T_3^-} \{[-\ln(T_3-t)]^{\frac{p}{p+1}}(T_3-t)\}^q \{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p u''(t) = c_2^{\frac{1-q}{p+1}} (p+1)^{\frac{p(1-q)}{p+1}}$

Blow-up Phenomena for $q > 2$

	Behavior	Blow-up rate	Blow-up constant
u	Bounded	None	None
u'	Blow-up	$\frac{1}{q-1}$	$[(q-1)(c_1 + c_2 u(T_2)^p)]^{\frac{1}{1-q}}$
u''	Blow-up	$\frac{q}{q-1}$	$(q-1)^{\frac{q}{1-q}} (c_1 + c_2 u(T_3)^p)^{\frac{1}{1-q}}$

5.2. Properties of blow-up rates and blow-up constants of u.

Property 5.1. *Suppose that $1 \le q < 2$, the blow-up rate α is strictly decreasing and convex in p(Figure 1); α is strictly decreasing and convex in q(Figure 1). If $[\frac{p+q-1}{2-q}]^{2-q} > \frac{e(p+1)}{c_2(2-q)}$, the blow-up constant β is strictly increasing in p(Figure 2); otherwise β is decreasing in p.(Figure 2).*

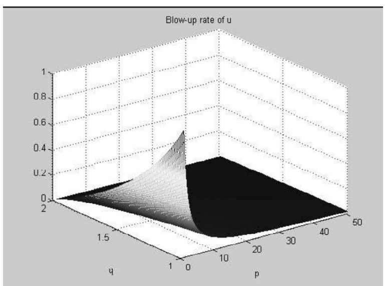

Figure 1. $1 \le q < 2$.

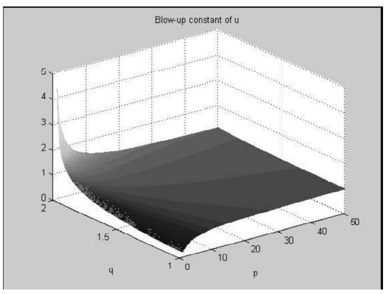

Figure 2. $1 \le q < 2,\ c_2 = 10$.

References

1. Meng-Rong Li, *On the Differential Equation* $u'' - u^p = 0$, Preprint, National Chengchi University, 1999.

2. I-Chen Chen, *Some Studies in Differential Equation*, Preprint, National Chengchi University, 1999.

3. C. Corduneanu, *Principle of Differential and Integral Equations*, Allyn and Bacon,Inc., Boston, 1971.

4. D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, Clarendon Press, Oxford, 1977.

Department of Mathematics Sciences, National Chengchi University, Taipei, Taiwan, Republic of China.