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HARTOGS THEOREM FOR CR FUNCTIONS

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Abstract. The classical Hartogs theorem for holomorphic functions is shown to hold for CR functions on C^1 CR manifold without any convexity assumption.

I. Introduction. We recall the classical Hartogs theorem that we are going to discuss in this paper. Let Ω be a connected domain in $\mathbf{C}^m$, $m \geq 1$, $\triangle(a,b) = \{z \in \mathbf{C} \mid a < |z| < b\}$ for $0 \leq a < b$, and $\triangle(b) = \{z \in \mathbf{C} \mid |z| < b\}$. For $n \geq 1$, the variable of $\mathbf{C}^m \times \mathbf{C}^n$ is denoted by (z', z'') with $z' \in \mathbf{C}^m$, $z'' \in \mathbf{C}^n$. Let $D_1 = \Omega \times \triangle(a,b)$, $\tilde{D}_1 = \Omega \times \triangle(b)$.

Theorem 1.(Hartogs) *With Ω, D_1, $\tilde{D}_1$ as above, let $A \subset \Omega$ be a compact set not contained in any subvariety of codimension bigger than or equal to 1. Let f be a function holomorphic on D_1. If, for every $z' \in A$, f has a holomorphic extension to $\{z'\} \times \triangle(b)$, then f has a holomorphic extension to $\tilde{D}_1$.*

Now let ρ be a real C^1 function on $\mathbf{C}^{m+1}$ such that $\{\rho = 0\}$ defines a compact CR manifold. Let $M = \{\rho = 0\} \cap (\Omega \times \mathbf{C})$ be connected. Theorem 1 can be generalized as follows:

Theorem 2. *With Ω and A the same as in Theorem 1, let f be a C^1*

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CR function on M. If, for every $z' \in A$, f extends to a holomorphic function on $\{\rho \leq 0\} \cap \{z'\} \times \mathbf{C}$, then f extends holomorphically to $\{\rho \leq 0\} \cap (\Omega \times \mathbf{C})$.

In Theorem 1 one can also consider f meromorphic and its meromorphic extension, this is Levi's Theorem. Here A has to be replaced by a **thick set**, that is a set not contained in a countable union of proper subvarieties. The Levi's Theorem states as follows:

Theorem 1$'$(Levi) *With Ω, D_1, $\tilde{D}_1$ as in Theorem 1, let $A \subset \Omega$ be a thick set. Let f be a function meromorphic on D_1. If, for every $z' \in A$, f has a meromorphic extension to $\{z'\} \times \triangle(b)$, then f has a meromorphic extension to $\tilde{D}_1$.*

It has the following generalization for CR functions:

Theorem 2$'$ *Let M, Ω be as in Theorem 1. Let A be a thick subset of Ω and f be a C^1 CR function on M. If, for every $z' \in A$, f extends to a meromorphic function in $\{\rho \leq 0\} \cap (\{z'\} \times \mathbf{C})$, then f extends to a function meromorphic in $\{\rho \leq 0\} \cap (\Omega \times \mathbf{C})$.*

The key ingredient for the proofs of Theorems 2 and 2' is a jump formula (Proposition 3) which holds for quite general domains. We also remark that in Theorems 2 and 2' we do not require any convexity condition on $\{\rho \leq 0\}$. It is interesting to ask if Theorem 2' holds for meromorphic CR functions.

II. Preliminaries. We write $\zeta \in \mathbf{C}^{m+n}$ as (ζ', ζ'') where $\zeta' \in \mathbf{C}^m$ and $\zeta'' \in \mathbf{C}^n$. Similarly, for differential forms we write $\partial f = (\partial' f, \partial'' f)$, and $\bar{\partial} f = (\bar{\partial}' f, \bar{\partial}'' f)$, where $\partial' \cdot$, $\partial'' \cdot$ denote respectively the differentials with repect to the first m complex variables and those to the last n complex variables, likewise for $\bar{\partial}' \cdot$ and $\bar{\partial}'' \cdot$. Also we have $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_{m+n}$, $d\zeta' = d\zeta_1 \wedge \ldots \wedge d\zeta_m$ and $d\zeta'' = d\zeta_{m+1} \wedge \ldots \wedge d\zeta_{m+n}$, similarly for $d\bar{\zeta}$, $d\bar{\zeta}'$,

$d\bar{\zeta}''$, etc.. And $C^k_{(0,q)}(G)$ denotes the set of $(0,q)$-forms on a domain G with coefficients in $C^k(G)$.

Following the exterior calculus developed by Harvey-Polking [H-P], we let $E^1, \ldots, E^\alpha$ be a collection of $(m+n)$-tuples of C^2 functions in $(\zeta, z) \in \mathbf{C}^{m+n} \times \mathbf{C}^{m+n}$ i.e. $E^j(\zeta,z) = (E^j_1(\zeta,z), \cdots, E^j_{m+n}(\zeta,z))$, we define

$$\begin{aligned}\Omega(E^1,\ldots,E^\alpha) = {} & \frac{< E^1, d\zeta >}{< E^1, \zeta - z >} \wedge \cdots \wedge \frac{< E^\alpha, d\zeta >}{< E^\alpha, \zeta - z >} \wedge \\ & \sum_{\lambda_1+\ldots+\lambda_\alpha = m+n-\alpha} \left(\frac{< \bar\partial_{\zeta,z} E^1, d\zeta >}{< E^1, \zeta - z >}\right)^{\lambda_1} \wedge \cdots \wedge \left(\frac{< \bar\partial_{\zeta,z} E^\alpha, d\zeta >}{< E^\alpha, \zeta - z >}\right)^{\lambda_\alpha}\end{aligned} \tag{1}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = \sum \mathbf{x_i} \mathbf{y_i}$ for vectors x, y in $\mathbf{C}^l$, and $\mathbf{d\zeta}$ here is understood to be the $(m+n)$-vector $(\mathbf{d\zeta}_1, \ldots, \mathbf{d\zeta}_{m+n})$. Then $\Omega(E^1,\ldots,E^\alpha)$ is C^1 outside its singular set. We remark that if $E(\zeta,z) = b(\zeta,z) = (\bar\zeta_1 - \bar z_1, \cdots, \bar\zeta_{m+n} - \bar z_{m+n})$ the Bochner-Martinelli section then $\Omega(b)$ is the Bochner-Martinelli kernel in $\mathbf{C}^{m+n}$.

We can rewrite $\Omega(E^1,\ldots,E^\alpha)$ as

$$\Omega(E^1,\ldots,E^\alpha) = \sum_0^{m+n-1} \Omega_q(E^1,\ldots,E^\alpha) \tag{2}$$

where $\Omega_q(E^1,\ldots,E^\alpha)$ is the sum of components of $\Omega(E^1,\ldots,E^\alpha)$ which are of degree q in $d\bar z$. Outside the singular set we have the following identity

$$\bar\partial_{\zeta,z}\Omega(E^1,\ldots,E^\alpha) = \sum_{j=1}^{\alpha} (-1)^j \Omega(E^1,\ldots,\widehat{E^j},\ldots,E^\alpha). \tag{3}$$

III. The jump formula. The proofs of Theorems 2 and 2' depend on a jump formula for CR functions on a hypersurface M which is not necessarily pseudoconvex. Let ρ be a C^1 function in $\mathbf{C}^{m+n}$ such that $\{\rho < 0\}$ defines a bounded domain with C^1 boundary. Let Ω be a connected domain in $\mathbf{C}^m$.

Let $M = \{\rho = 0\} \cap (\Omega \times \mathbf{C}^n)$ which is assumed to be connected for simplicity. For a relatively compact connected open subset D of Ω we set

$$\begin{aligned} D^+ &= \{\rho < 0\} \cap (D \times \mathbf{C}^n) \\ D^- &= \{\rho > 0\} \cap (D \times \mathbf{C}^n) \\ M_D &= M \cap (D \times \mathbf{C}^n). \end{aligned}$$

Proposition 3. *With ρ, Ω, D, M, M_D, $D^\pm$ as above. For any $f \in C^1_{(0,q)}(M)$, $0 \le q \le n+m-1$, $\bar\partial_b f = 0$, there exist $\bar\partial$-closed forms $f^\pm \in C^1_{(0,q)}(D^\pm)$ which are continuous up to M_D and the jump formula $f = f^+ - f^-$ holds on M_D. If, in addition, f and ρ are C^∞ then $f^\pm$ are C^∞ up to M_D.*

Proof. We denote by $b(\zeta, z)$ the Bochner-Martinelli section in $\mathbf{C}^{m+n}$. We define the section $b'(\zeta, z) = (b_m(\zeta', z'), \underbrace{0, \dots, 0}_{n \text{ tuple}})$ where $b_m(\zeta', z')$ is the Bochner-Martinelli section in $\mathbf{C}^m$. Let $D' \Supset D$ be a subset of Ω with C^1 boundary and denote by $M' = \{\rho = 0\} \cap (D' \times \mathbf{C}^n)$. We choose D' such that M' is connected.

For any $f \in C^1_{(0,q)}(M)$ and $z \in D^+$, define

$$F^+(z) = \int_{M'} f \wedge \Omega(b)(\zeta, z), \quad \text{and} \quad f^+(z) = F^+(z) - \int_{\partial M'} f \wedge \Omega(b', b);$$

similarly, for $z \in D^-$ we define

$$F^-(z) = \int_{M'} f \wedge \Omega(b)(\zeta, z), \quad \text{and} \quad f^-(z) = F^-(z) - \int_{\partial M'} f \wedge \Omega(b', b).$$

Now well-known results for Bochner-Martinelli kernel give that $F^\pm(z)$ extends continuously up to $\overline{D^\pm}$ respectively, and the jump formula $F^+ - F^- = f$ holds on M_D.

The term $\int_{\partial M'} f \wedge \Omega(b', b)$ extends smoothly across M_D because for $z \in \overline{D^{\pm}}$ the integrand is regular. Hence $f^{\pm}$ extends continuously up to $\overline{D^{\pm}}$, respectively, and for $z \in M_D$ they have the following relation:

$$f^{+}(z) - f^{-}(z) = F^{+}(z) - F^{-}(z) = f.$$

by the jump formula for the Bochner-Martinelli kernel.

It remains to show $f^{\pm}$ is $\bar{\partial}$-closed. For $z \in D^{+}$ we calculate

$$\begin{aligned}
\bar{\partial} f^{+} &= \int_{M'} f \wedge \bar{\partial}_z \Omega_q(b) - \int_{\partial M'} f \wedge \bar{\partial}_z \Omega(b', b) \\
&= (-1)^{q+1} \int_{M'} f \wedge \bar{\partial}_\zeta \Omega_{q+1}(b) \\
&\quad - \int_{\partial M'} f \wedge \{-\bar{\partial}_\zeta \Omega(b', b) + \Omega_{q+1}(b') - \Omega_{q+1}(b)\} \\
&= -\int_{\partial M'} f \wedge \Omega_{q+1}(b) + \int_{\partial M'} f \wedge \{\bar{\partial}_\zeta \Omega(b', b) - \Omega_{q+1}(b') + \Omega_{q+1}(b)\} \\
&= \int_{\partial M'} f \wedge \Omega_{q+1}(b')
\end{aligned}$$

by using $\bar{\partial}_\zeta \Omega_{q+1}(b) = (-1)^{q+1} \bar{\partial}_z \Omega_q(b)$, (3) and the Stokes' theorem. But $\Omega(b') = 0$ by type considerations. When $z \in D^{-}$, $\bar{\partial} f^{-} = 0$ can be proved exactly the same way.

If f and ρ are C^{∞}, then from the fact that $F^{\pm}$ are C^{∞}, we conclude $f^{\pm}$ are C^{∞}.

IV. Proofs of Theorems 2 and 2′. The proofs of Theorems 2 and 2' are very similar. We first prove Theorem 2. We choose $D \Subset \Omega$ such that $A \subset D$ then by Proposition 3 there exist functions $f^{\pm}$ holomorphic in $D^{\pm}$ respectively and continuous up to M_D such that f is the jump of $f^{\pm}$ on M_D. We note the holomorphic extendability of f is equivalent to that of f^{-}. By the definition of f^{-} when $|z|$ gets large $|f^{-}|$ diminishes to zero. So for $z' \in A$, $f^-(z', \cdot)$ is an entire function on $\mathbf{C}$ which is zero at ∞, hence

$$f^-(z', \cdot) \equiv 0 \qquad \forall z' \in A. \tag{4}$$

On the other hand, as $\{\rho \leq 0\}$ is compact, there exists $N > 0$ such that $f^-(\cdot, z'')$ is a function holomorphic on $D \subset \mathbf{C}^m$ for each z'' with $|z''| > N$. By the hypothesis on A, (4) then implies that $f^-(\cdot, z'') \equiv 0$ for $|z''| > N$. This in turn implies $f^- \equiv 0$, i.e. f extends holomorphically to $\{\rho \leq 0\} \cap (D \times \mathbf{C})$ and the extension is given by f^+. Since D can be any relatively compact open connected subset of Ω containing A and the holomorphic extension of f is locally uniquely determined, the proof of Theorem 2 is complete.

As for Theorem 2', the meromorphic extension of f is equivalent to the meromorphic extension of f^-. By taking D such that $D \cap A$ is a thick set of D, we may apply the classical Levi's Theorem (Theorem 1') to f^- on $P = D \times \{z| \; z \in \mathbf{C}, R_1 < |z| < R_2\}$ where R_1 is large enough that $P \cap \{\rho \leq 0\} = \emptyset$. See [Siu] p.13-16. Hence f has a meromorphic extension to $(D \times \mathbf{C}) \cap \{\rho \leq 0\}$. It is easy to see that the subset Ω' where f extends meromorphically to a neighborhood of $(\{z'\} \times \mathbf{C}) \cap \{\rho \leq 0\}$, $z' \in \Omega'$, is both open and closed in Ω. Since $\Omega' \supset D$ and Ω is connected we conclude that $\Omega' = \Omega$. Theroem 2' is proved.

As an application, we have the following corollary of the proof of Theorem 2:

Corollary 4. *Let M be as in Theorem 2. Suppose in addition $\Omega \backslash \pi(\{\rho < 0\})$ contains an open subset of $\mathbf{C}^m$, where $\pi(\{\rho < 0\})$ is the projection of $\{\rho < 0\}$ to $\mathbf{C}^m$. Then every C^1 CR function on M extends holomophically to $\{\rho \leq 0\} \cap (\Omega \times \mathbf{C})$.*

Proof of Corollary 4. For any given CR function f on M. Let D be a relatively compact open connected subset of Ω, such that $D \setminus \pi(\{\rho < 0\})$ contains a non-empty open subset U of $\mathbf{C}^m$. Let $D^\pm$ and $f^\pm$ be as in

Proposition 3. Then as f^- is bounded and holomorphic on $\{z'\} \times \mathbf{C}$ for all $z' \in U$, by the same argument as in the proof for Theorem 2, we have $f^- \equiv 0$ on $U \times \mathbf{C}$, this in turn implies f^- is identically equal to zero. Thus f extends holomorphically by f^+ to the desired region.

Remark. We note that the conclusions for Theroems 2, 2' and Corollary 4 still hold if we only require $\{\rho = 0\}$ be a C^1 manifold such that for every $z_0' \in \Omega$ the set $\{z'' : \ \rho(z_0', z'') = 0\}$ is compact in $\mathbf{C}$.

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