EMPIRICAL BAYES TEST FOR TRUNCATION PARAMETERS USING LINEX LOSS*

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Abstract. In this paper, an empirical Bayes test (EBT) for the truncation parameter is investigated under linex loss. The EBT is proposed, the asymptotic optimality and convergence rate are obtained. At last, an example satisfying conditions of theorems is given.

1. Introduction. As quadratic loss function or any other symmetric loss function may be inappropriate for some practical problem ([2], [8]). Varian introduced the asymmetric linex loss [9], which was employed by Zellner in the Bayes analysis of several statistical estimation and prediction problems [10]. Kuo and Dey considered the estimation of a Poisson mean under linex loss [6], Basu and Ebrahimi used the linex loss in lifetime testing and reliability estimation [1]. Also, see Huang for empirical Bayes testing procedures in a class of nonexponential families [3], Huang and Liang for the empirical Bayes estimation of the truncation parameter with linex loss [4]. However, the EBT problem for the truncation parameter of the following distribution family was not investigated.

Consider the truncation parameter with p.d.f. of the following form

(1) $$f(x|\theta) = u(x)A(\theta)I_{[\theta,m\theta]}(x)$$

where $m > 1$ is a constant, and $u(x)$ is positive, integrable and bounded with lower bound strictly greater than 0 on $[\theta, m\theta]$, $A(\theta) = [\int_{\theta}^{m\theta} u(x)dx]^{-1}$, $\theta \in (0, +\infty)$. The truncation parameter θ of our interest has a prior distribution $G(\theta)$ with p.d.f. $g(\theta)$ over $\theta \in (0, +\infty)$. Suppose that, given θ, the random variable X has the p.d.f. $f(x|\theta)$ of form (1). Then the marginal density of X is

$$f(x) = \int_0^\infty f(x|\theta)dG(\theta) = u(x)\int_{x/m}^x A(\theta)g(\theta)d\theta \triangleq u(x)v(x).$$

Let $s \geq 1$ be a given natural number and $f(x)$ be bounded and have sth bounded derivative. The hypothesis to be tested is

(2) $$H_0 : \theta \leq \theta_0 \leftrightarrow H_1 : \theta > \theta_0$$

where θ_0 is a known positive constant. Let the loss function be

(3) $$L(\theta, d_0) = l(\theta)I_{(\theta > \theta_0)}, \quad L(\theta, d_1) = l(\theta)I_{(\theta \leq \theta_0)}$$

where $l(\theta) = b[e^{c(\theta_0-\theta)} - c(\theta_0-\theta) - 1]$, d_i indicates accepting $H_i(i = 0, 1)$, $D = \{d_0, d_1\}$ is the decision space, $L(\theta, d_i)$ indicates the loss when the decision is in favor of H_i, while θ is the true parameter value. As b does not affect either Bayes rule or the empirical Bayes study and b is assumed to be 1. The constant c determines the shape of the loss function, when $c > 0$, the loss increases almost exponentially for wrongly decision of H_i as $|\theta_0 - \theta| \to \infty$. When $c < 0$, the linear-exponential increases are interchanged. That is, over-estimation is very different from under-estimation. As the value $|c|$ is small enough, the loss function is close to the squared error loss. Herewith we consider only $c > 0$ and the case of $c < 0$ is similar.

The paper is organized as follows. We construct an EBT for the truncation parameters using Linex loss and investigate the asymptotic optimality with convergence rate of the proposed EBT in Section 2. In Section 3 some lemmas are given to complete the proofs in Section 4. At last we give an example satisfying the conditions of the theorems.

2. Empirical Bayes test. Let $r(x) = P(\text{accept } H_0 \mid X = x)$ be a randomized decision rule. Then the Bayes risk of $r(x)$ can be written as,

$$
\begin{aligned}
R(r, G) &= \int_\Omega \int_\Delta \{L(\theta, d_0)r(x) + [1 - r(x)]L(\theta, d_1)\}f(x|\theta)dG(\theta)dx \\
&= \int_\Omega Q(x)r(x)dx + C_G
\end{aligned}
$$

where $\Omega = (0, +\infty)$, $\Delta = [x/m, x]$, $C_G = \int_\Omega \int_\Delta L(\theta, d_1)f(x|\theta)dG(\theta)dx$.

$$
\begin{aligned}
Q(x) &= \int_{\Delta \cap (\theta_0, \infty)} l(\theta)f(x|\theta)dG(\theta) - \int_{\Delta \cap (0, \theta_0)} l(\theta)f(x|\theta)dG(\theta) \\
&= \int_{\theta_0}^x l(\theta)f(x|\theta)dG(\theta) - \int_{x/m}^{\theta_0} l(\theta)f(x|\theta)dG(\theta) \\
&= 2\int_{\theta_0}^x l(\theta)f(x|\theta)dG(\theta) - \int_{x/m}^x l(\theta)f(x|\theta)dG(\theta) \\
&\triangleq e^{c\theta_0}[2S_1(x) - v_1(x)] + c[2S_2(x) - v_2(x)] - (c\theta_0 + 1)[2S_3(x) - f(x)],
\end{aligned}
$$

where

$$
\begin{aligned}
v_1(x) &= u(x)\int_\Delta e^{-c\theta}A(\theta)g(\theta)d\theta, \quad v_2(x) = u(x)\int_\Delta \theta A(\theta)g(\theta)d\theta, \\
S_1(x) &= u(x)\int_{\theta_0}^x e^{-c\theta}A(\theta)g(\theta)d\theta, \quad S_2(x) = u(x)\int_{\theta_0}^x \theta A(\theta)g(\theta)d\theta, \\
S_3(x) &= u(x)\int_{\theta_0}^x A(\theta)g(\theta)d\theta.
\end{aligned}
$$

Therefore the Bayes decision rule is

$$
r_G(x) = 1 \quad \text{when } Q(x) \le 0 \quad \text{and} \quad r_G(x) = 0 \quad \text{when } Q(x) > 0.
$$

It is easy to show that $r_G(x)$ is a Bayes test with respect to $G(\theta)$.

Since $G(\theta)$ is unknown and $r_G(x)$ cannot be applied, we introduce the EB approach. Suppose that $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ and (X, θ) are i.i.d. random samples. $X_1, \ldots, X_n$ (the past samples) and X (the present samples) are observable, have the same distribution density form (1), while θ_i and θ are not observable with the same prior distribution $G(\theta)(i = 1, 2, \ldots, n)$.

Let $Q_n(x) = Q_n(x_1, \ldots, x_n; x)$ be the estimator of $Q(x)$. We define the EBT of $r_G(x)$ by

$$
(4) \qquad r_n(x) = \begin{cases} 1, & \text{if } Q_n(x) \le 0 \\ 0, & \text{Otherwise} \end{cases}
$$

Then the over all Bayes risk of $r_n(x)$ is

$$
(5) \qquad R_n \triangleq R_n(r_n(x), G) = E_n \int Q(x) r_n(x) dx + C_G
$$

where E_n stands for the expectation with respect to the joint distribution of $(X_1, \ldots, X_n)$.

For any $G(\theta) \in F^1$, the EBT $r_n(x)$ is said to be asymptotically optimal (a.o) if $R_n \to R_G$ as $n \to \infty$. Moreover if for a $q > 0$, $R_n - R_G = O(n^{-q})$, then the EBT $r_n(x)$ is said to be asymptotically optimal with convergence rate of $O(n^{-q})$, where F^1 is the prior distribution family of θ, $R_G = R(r_G, G) = \inf_r R(r, G)$ is the Bayes risk of $r_G(x)$.

We can define the estimator of $f(x)$ by

$$
(6) \qquad f_n(x) = (nh)^{-1} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)
$$

where $h = h_n(> 0)$, and $\lim_{n \to \infty} h_n = 0$. $K(\cdot)$ is a kernel function, satisfying the following conditions:

(1) $K(y) = 0$, if $y \notin (0, 1)$.
(2) $|K(y)| \le M$ for all y, M is a positive number.
(3) $\int y^t K(y) dy = \begin{cases} 1, & \text{as } t = 0, \\ 0, & \text{as } t = 1, 2, \ldots, s - 1. \end{cases}$

Assume that the following conditions hold (for all $x > 0$),

(A1) $\sum\limits_{j=1}^{\infty} x^t v(m^j x) < \infty$, $t = 0, 1$

(A2) $m^j A(m^j x) g(m^j x) \to 0 (j \to \infty)$

(A3) $\sum\limits_{j=1}^{\infty} f^{(s)}(m^j x) < \infty$, $\sum\limits_{j=1}^{\infty} f(m^j x) < \infty$,

$\quad\quad x \sum\limits_{j=1}^{\infty} f^{(s)}(m^j x) < \infty$, $x^2 \sum\limits_{j=1}^{\infty} f(m^j x) < \infty$

(A4) $x \sum\limits_{j=1}^{\infty} \int_{x/m}^{x} f^{(s)}(m^j \theta) d\theta < \infty$

where $v(m^j x) = \int_{m^{j-1}x}^{m^j x} A(\theta) g(\theta) d\theta = f(m^j x)[u(m^j x)]^{-1}$.

Noting (A2) and $\frac{d}{dx} v(m^j x) = m^j A(m^j x) g(m^j x) - m^{j-1} A(m^{j-1}x) g(m^{j-1}x)$,

we get $A(x)g(x) = -\sum\limits_{j=1}^{\infty} \frac{dv(m^j x)}{dx}$. Therefore

$$v_2(x) = u(x) \int_{x/m}^{x} \theta A(\theta) g(\theta) d\theta = u(x) \int_{x/m}^{x} - \sum_{j=1}^{\infty} \theta \frac{dv(m^j \theta)}{d\theta} d\theta.$$

We need the following lemma to simplify $v_1(x)$, $v_2(x)$, $S_1(x)$, $S_2(x)$ and $S_3(x)$.

Lemma 2.1. (see [7]) *Let $\{f_n\}$ be a sequence of measurable functions on (Ω, F_1, μ), where (Ω, F_1, μ) is a measurable space, and μ is a σ-finite measure. If either $\sum_{n=1}^{\infty} \int_\Omega f_n^+ d\mu < \infty$ or $\sum_{n=1}^{\infty} \int_\Omega f_n^- d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ is integral for μ (i.e. $\int_\Omega \sum_{n=1}^{\infty} f_n d\mu < \infty$), and $\int_\Omega \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n d\mu$, where f_n^+ and f_n^- stand for the positive and negative parts of the f_n respectively. Since*

$$\sum_{j=1}^{\infty} \int_{x/m}^{x} \theta \left[\frac{dv(m^j \theta)}{d\theta} \right] d\theta = -\frac{xv(x)}{m} + \left(1 - \frac{1}{m}\right) x \sum_{j=1}^{\infty} v(m^j x) - \sum_{j=1}^{\infty} \int_{x/m}^{x} v(m^j \theta) d\theta.$$

From the condition (A1) $\sum\limits_{j=1}^{\infty} \int_{\frac{x}{m}}^{x} \left[\theta \frac{d}{d\theta} v(m^j \theta) \right]^+ d\theta = \left(1 - \frac{1}{m}\right) x \sum\limits_{j=1}^{\infty} v(m^j x) < \infty.$

By Lemma 2.1, we get

$$\int_{x/m}^{x} \sum_{j=1}^{\infty} \theta dv(m^j \theta) = \sum_{j=1}^{\infty} \int_{x/m}^{x} \theta \frac{d}{d\theta} v(m^j \theta) d\theta.$$

Hence

$$v_2(x) = -u(x) \sum_{j=1}^{\infty} \int_{x/m}^{x} \left[\theta \frac{dv(m^j \theta)}{d\theta} \right] d\theta$$

$$\underline{\underline{\Delta}} \, u(x)[xv(x)/m - (1 - 1/m)xT(x) + \psi_2(x)].$$

Similarly, one has,

$$v_1(x) \quad \underline{\underline{\Delta}} \quad u(x)[e^{-cx/m}v(x) - (e^{-cx} - e^{-cx/m})T(x) - c\psi_1(x)],$$

$$S_1(x) \quad \underline{\underline{\Delta}} \quad u(x)[-e^{-cx}T(x) + e^{-c\theta_0}T(\theta_0) - c\eta_1(x)],$$

$$S_2(x) \quad \underline{\underline{\Delta}} \quad u(x)[-xT(x) + \theta_0 T(\theta_0) + \eta_2(x)],$$

$$S_3(x) \quad \underline{\underline{\Delta}} \quad u(x)[-T(x) + T(\theta_0)]$$

where

$$T(x) = \sum_{j=1}^{\infty} f(m^j x)[u(m^j x)]^{-1},$$

$$\psi_1(x) = \sum_{j=1}^{\infty} \int_{x/m}^{x} e^{-c\theta} f(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\psi_2(x) = \sum_{j=1}^{\infty} \int_{x/m}^{x} f(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\eta_1(x) = \sum_{j=1}^{\infty} \int_{\theta_0}^{x} e^{-c\theta} f(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\eta_2(x) = \sum_{j=1}^{\infty} \int_{\theta_0}^{x} f(m^j \theta)[u(m^j \theta)]^{-1} d\theta.$$

We define the estimation of $v_i(x)$, $S_j(x)$ as $v_{in}(x)$, $S_{jn}(x)$ ($i = 1, 2$, $j = 1, 2, 3$), namely,

$$v_{1n}(x) = e^{-cx/m} f_n(x) - u(x)(e^{-cx} - e^{-cx/m})T_n(x) - cu(x)\psi_{1n}(x)$$

$$v_{2n}(x) = (x/m)f_n(x) - u(x)(1 - 1/m)xT_n(x) + u(x)\psi_{2n}(x)$$

$$S_{1n}(x) = u(x)[-e^{-cx}T_n(x) + e^{-c\theta_0}T_n(\theta_0) - c\eta_{1n}(x)]$$

$$S_{2n}(x) = u(x)[-xT_n(x) + \theta_0 T_n(\theta_0) + \eta_{2n}(x)]$$

$$S_{3n}(x) \ = \ u(x)[-T_n(x) + T_n(\theta_0)]$$

where $f_n(x)$ is defined by (6), and $T_n(x) = \sum_{j=1}^{\infty} f_n(m^j x)[u(m^j x)]^{-1}$,

$$\psi_{1n}(x) \ = \ \sum_{j=1}^{\infty} \int_{x/m}^{x} e^{-c\theta} f_n(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\psi_{2n}(x) \ = \ \sum_{j=1}^{\infty} \int_{x/m}^{x} f_n(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\eta_{1n}(x) \ = \ \sum_{j=1}^{\infty} \int_{\theta_0}^{x} e^{-c\theta} f_n(m^j \theta)[u(m^j \theta)]^{-1} d\theta,$$

$$\eta_{2n}(x) \ = \ \sum_{j=1}^{\infty} \int_{\theta_0}^{x} f_n(m^j \theta)[u(m^j \theta)]^{-1} d\theta.$$

Therefore, the estimator of $Q(x)$ can be constructed as follows,

$$Q_n(x) = e^{c\theta_0}[2S_{1n}(x) - v_{1n}(x)] + c[2S_{2n}(x) - v_{2n}(x)] - (c\theta_0 + 1)[2S_{3n}(x) - f_n(x)].$$

For the EBT $r_n(x)$, we claim the following Theorem 1 and Theorem 2.

Theorem 1. *Suppose the conditions* (A1)$\sim$(A5) *hold and* $f(x)$ *is* s *times differentiable,* $\sup_x f(x) < \infty$, $\sup_x |f^{(s)}(x)| < \infty$.

Then as $h_n = n^{-1/(2s+1)}$, *we have* $\lim_{n \to \infty} R_n = R_G$.

Theorem 2. *Suppose that the conditions of Theorem 1 hold, and there exists a constant* $\delta(0 < \delta < 1)$ *such that the following conditions are satisfied*

(B1) $\int_0^{\infty} x^{2\alpha s} f(x) dx < \infty, \quad \int_0^{\infty} [xf(x)]^{1-\delta} dx < \infty,$

for one such α, $1/2 < \alpha < 1 - 1/2s$;

(B2) $\int_0^{\infty} |x^k \sum_{j=1}^{\infty} f(m^j x)|^{1-\delta} dx < \infty, \quad k = 0, 1;$

(B3) $\int_0^{\infty} \left| \sum_{j=1}^{\infty} \int_{x/m}^{x} f(m^j \theta) d\theta \right|^{1-\delta} dx < \infty.$

Then as $h_n = n^{-1/(2s+1)}$, we obtain $R_n - R_G = O(n^{-q})$ with $q = \delta s/(2s+1)$.

3. Some Lemmas. In this paper, M denotes different positive constants in different cases, even in the same expression.

Lemma 3.1. *If R_G denotes the Bayes risk of r_G, and R_n is defined by (5). Then*

$$0 \leq R_n - R_G \leq \int |Q(x)| P\{|Q_n(x) - Q(x)| \geq |Q(x)|\} dx$$

Proof.

$$
\begin{aligned}
0 \leq R_n - R_G &= E_n \int Q(x) r_n(x) dx - \int Q(x) r_G(x) dx \\
&= \int Q(x) [E_n r_n(x) - r_G(x)] dx \\
&= \int Q(x) [P\{Q_n(x) \leq 0\} - r_G(x)] dx \triangleq \int Q(x) \Delta_n(x) dx.
\end{aligned}
$$

As $Q(x) \leq 0$, $\Delta_n(x) = P\{Q_n(x) \leq 0\} - 1 = -P\{Q_n(x) > 0\}$

$$(7) \qquad\qquad R_n - R_G = \int |Q(x)| P\{Q_n(x) > 0\} dx.$$

As $Q(x) > 0$, $\Delta_n(x) = P\{Q_n(x) \leq 0\} - 0 = P\{Q_n(x) \leq 0\}$

$$(8) \qquad\qquad R_n - R_G = \int |Q(x)| P\{Q_n(x) \leq 0\} dx.$$

From $Q(x) \leq 0$ and $Q_n(x) > 0$, we get $P\{Q_n(x) > 0\} \leq P\{|Q_n(x) - Q(x)| \geq |Q(x)|\}$,

From $Q(x) > 0$ and $Q_n(x) \leq 0$, we get $P\{Q_n(x) \leq 0\} \leq P\{|Q_n(x) - Q(x)| \geq |Q(x)|\}$.

Combining (7) and (8) we have $0 \leq R_n - R_G \leq \int |Q(x)| P\{|Q_n(x) - Q(x)| \geq |Q(x)|\} dx$.

Lemma 3.2. *Let $f_n(x)$ be defined by* (6). *If* $\sup_x f(x) < \infty$, $\sup_x |f^{(s)}(x)|$ $< \infty$, *then for* $0 < \lambda < 2$ *and* $h_n = n^{-1/(2s+1)}$, *we have* $|E_n f_n(x) - f(x)| \leq$ $Mn^{-s/(2s+1)} f^{(s)}(x)$,

$$\mathrm{Var}\,(f_n(x)) \quad \leq \quad Mn^{-2s/(2s+1)} f(x), \quad and$$

$$E_n |f_n(x) - f(x)|^\lambda \quad \leq \quad Mn^{-\lambda s/(2s+1)}[\{f(x)\}^{\lambda/2} + \{|f^{(s)}(x)|\}^\lambda].$$

Proof. From Lemma 3.1 in Karunamuni (1996) and C_r inequality, we can easily obtain the above inequalities.

Lemma 3.3. *Let* (A1)$\sim$(A5) *and the conditions of Lemma* 3.2 *hold and* $h_n = n^{-1/(2s+1)}$. *Then for any* $0 < \lambda \leq 2$,

$$|E_n v_{in}(x) - v_i(x)| \leq Mn^{-s/(2s+1)}, \quad \mathrm{Var}\,(v_{in}(x)) \leq Mn^{-2s/(2s+1)},$$

$$E_n |v_{in}(x) - v_i(x)|^\lambda \leq Mn^{-\lambda s/(2s+1)} \ (i = 1, 2),$$

$$|E_n S_{jn}(x) - S_j(x)| \leq Mn^{-s/(2s+1)}, \quad \mathrm{Var}\,(S_{jn}(x)) \leq Mn^{-2s/(2s+1)},$$

$$E_n |S_{jn}(x) - S_j(x)|^\lambda \leq Mn^{-\lambda s/(2s+1)} \ (j = 1, 2, 3).$$

Proof. We prove Lemma 3.3 only for $v_1(x)$, it is similar for others. Note that (A4), (A5) and Lemma 3.2, one has

$$
\begin{aligned}
&|E_n v_{1n}(x) - v_1(x)| \\
\leq\ & e^{-cx/m} |E_n f_n(x) - f(x)| + (e^{-cx} - e^{-cx/m})|E_n T_n(x) - T(x)| \\
& + c|E_n \psi_{1n}(x) - \psi_1(x)| \\
\leq\ & M\left\{ |E_n f_n(x) - f(x)| + \sum_{j=1}^{\infty} \frac{|E_n f_n(m^j x) - f(m^j x)|}{u(m^j x)} \right. \\
& \left. + \sum_{j=1}^{\infty} \int_{x/m}^{x} \frac{|E_n f_n(m^j \theta) - f(m^j \theta)|}{u(m^j \theta)} d\theta \right\} \\
\leq\ & M\left\{ |E_n f_n(x) - f(x)| + \sum_{j=1}^{\infty} f^{(s)}(m^j x) n^{-s/(2s+1)} \right.
\end{aligned}
$$

$$+n^{-s/(2s+1)}\sum_{j=1}^{\infty}\int_{x/m}^{x}f^{(s)}(m^j\theta)d\theta\Big\}$$

$$\leq\ Mn^{-s/(2s+1)}.$$

For Var $(v_{1n}(x))$, begin with

$$
\begin{aligned}
\operatorname{Var}[v_{1n}(x)] \ &=\ E_n(v_{1n}(x)-E_nv_{1n}(x))^2 \\
&\leq\ M\Bigg\{ e^{-2cx/m}[E_n(f_n(x)-E_nf_n(x))^2] \\
&\quad +[(e^{-cx}-e^{-cx/m})]^2 E_n\Bigg|\sum_{j=1}^{\infty}\frac{f_n(m^jx)-E_nf_n(m^jx)}{u(m^jx)}\Bigg|^2 \\
&\quad +c^2E_n\Bigg|\sum_{j=1}^{\infty}\int_{x/m}^{x}\frac{f_n(m^j\theta)-E_nf_n(m^j\theta)}{u(m^j\theta)}e^{-c\theta}d\theta\Bigg|^2\Bigg\}.
\end{aligned}
$$

By Fubini Theorem, (A4) and Lemma 3.2, we conclude

$$
\begin{aligned}
&E_n\Bigg|\sum_{j=1}^{\infty}\frac{f_n(m^jx)-E_nf_n(m^jx)}{u(m^jx)}\Bigg|^2 \\
(9)\quad &=\ \operatorname{Var}\sum_{j=1}^{\infty}\frac{f_n(m^jx)-E_nf_n(m^jx)}{u(m^jx)}+\Bigg[E_n\sum_{j=1}^{\infty}\frac{f_n(m^jx)-E_nf_n(m^jx)}{u(m^jx)}\Bigg]^2 \\
&\leq\ M\sum_{j=1}^{\infty}\frac{\operatorname{Var}f_n(m^jx)}{u^2(m^jx)}\leq\sum_{j=1}^{\infty}f(m^jx)n^{-2s/(2s+1)}\leq Mn^{-2s/(2s+1)}.
\end{aligned}
$$

From (A5), Lemma 3.2, Fubini Theorem and Cauchy inequality,

$$
\begin{aligned}
&E_n\Bigg|\sum_{j=1}^{\infty}\int_{x/m}^{x}\frac{f_n(m^j\theta)-E_nf_n(m^j\theta)}{u(m^j\theta)}e^{-c\theta}d\theta\Bigg|^2 \\
&\leq\ M\sum_{j=1}^{\infty}E_n\Bigg[\int_{x/m}^{x}\frac{f_n(m^j\theta)-E_nf_n(m^j\theta)}{u^{1/2}(m^j\theta)u^{1/2}(m^jx)}e^{-c\theta}d\theta\Bigg]^2 \\
(10)\quad &\leq\ M\sum_{j=1}^{\infty}E_n\Bigg[\int_{x/m}^{x}\frac{1}{u(m^j\theta)}d\theta\int_{x/m}^{x}\frac{[f_n(m^j\theta)-E_nf_n(m^j\theta)]^2}{u(m^j\theta)}e^{-2c\theta}d\theta\Bigg] \\
&\leq\ Mx\sum_{j=1}^{\infty}\int_{x/m}^{x}\operatorname{Var}f_n(m^j\theta)[u(m^j\theta)e^{2c\theta}]^{-1}d\theta
\end{aligned}
$$

$$\leq x \sum_{j=1}^{\infty} \int_{x/m}^{x} f(m^j \theta) d\theta n^{-2s/(2s+1)} \leq M n^{-2s/(2s+1)}.$$

By above two inequalities and Lemma 3.2, $\mathrm{Var}\,[v_{1n}(x)] \leq M n^{-2s/(2s+1)}$.

From C_r inequality and Jensen inequality, we get

$$
\begin{aligned}
E_n |v_{1n}(x) - v_1(x)|^\lambda &\leq M[E_n|E_n v_{1n}(x) - v_{1n}(x)|^\lambda + E_n|E_n v_{1n}(x) - v_1(x)|^\lambda] \\
&\leq M\{E_n|E_n v_{1n}(x) - v_{1n}(x)|^2\}^{\frac{\lambda}{2}} + M E_n|E_n v_{1n}(x) - v_1(x)|^\lambda \\
&\leq M\{\mathrm{Var}\,(v_{1n}(x))^{\lambda/2} + n^{-\lambda s/(2s+1)}\} \leq M n^{-\lambda s/(2s+1)}.
\end{aligned}
$$

Similarly we can get other expressions. End of proof.

Lemma 3.4. *If the conditions of Theorem* 1 *and the condition* (B1) *of Theorem* 2 *hold, then* $\int_0^\infty [f(x)]^{1-\alpha} dx < \infty$, $1/2 < \alpha < 1 - 1/2s$.

Proof. $\int_0^\infty [f(x)]^{1-\alpha} dx = \int_0^1 [f(x)]^{1-\alpha} dx + \int_1^\infty [f(x)]^{1-\alpha} dx \underline{\underline{\triangle}} I_1 + I_2.$
Obviously $I_1 = \int_0^1 [f(x)]^{1-\alpha} dx \leq M^{1-\alpha} < \infty.$

By Hölder inequality and conditions of Theorem 1, we have

$$
\begin{aligned}
I_2 &= \int_1^\infty [f(x)]^{1-\alpha} dx = \int_1^\infty |x|^{-(1+\xi)\alpha} [|x|^{(1+\xi)\alpha} [f(x)]^{1-\alpha}] dx \\
&\leq \Big[\int_1^\infty x^{-(1+\xi)} dx\Big]^\alpha \Big[\int_1^\infty x^{\frac{(1+\xi)\alpha}{1-\alpha}} f(x) dx\Big]^{1-\alpha} \leq M \Big[\int_1^\infty x^{\frac{(1+\xi)\alpha}{1-\alpha}} f(x) dx\Big]^{1-\alpha},
\end{aligned}
$$

where $\xi > 0$ is a constant. Since $2s(1-\alpha) > 1$, therefore there exists a $\xi > 0$, such that $1 + \xi \leq 2s(1-\alpha)$. Hence, $(1+\xi)\alpha/(1-\alpha) \leq 2s\alpha$.

From condition (B1) of Theorem 2, we get $\int_1^\infty x^{(1+\xi)\alpha/(1-\alpha)} f(x) dx \leq \int_1^\infty x^{2\alpha s} f(x) dx < \infty$. Thus $I_2 < \infty$ and $\int_0^\infty [f(x)]^{1-\alpha} dx = I_1 + I_2 < \infty$. This completes the proof.

4. Proof of Theorems.

Proof of Theorem 1. By Lemma 3.1 and Markov inequality, we get

$$0 \leq R_n - R_G \leq \int |Q(x)|P\{|Q_n(x) - Q(x)| \geq |Q(x)|\}dx \underline{\triangleq} \int B_n(x)dx$$

$$
\begin{aligned}
B_n(x) &= |Q(x)|P\{|Q_n(x) - Q(x)| \geq |Q(x)|\} \\
&\leq |Q(x)|E_n|Q_n(x) - Q(x)|/|Q(x)| = E_n|Q_n(x) - Q(x)|
\end{aligned}
$$

$$
\begin{aligned}
&E_n|Q_n(x) - Q(x)| \\
&\leq M\Big\{\sum_{j=1}^{3} E_n|S_{jn}(x) - S_j(x)| + \sum_{i=1}^{2} E_n|v_{1n}(x) - v_1(x)| + E_n|f_n(x) - f(x)|\Big\}.
\end{aligned}
$$

Take $\lambda = 1$ in Lemma 3.2, we get $E_n|Q_n(x) - Q(x)| \leq Mn^{-s/(2s+1)}$.
Therefore $0 \leq B_n(x) \leq Mn^{-s/(2s+1)}$. Using the dominated convergence
Theorem, we arrive at

$$\lim_{n\to\infty} \int B_n(x)dx = \int \lim_{n\to\infty} B_n(x)dx = 0.$$

So that $\lim_{n\to\infty} R_n - R_G = 0$. End of proof.

Proof of Theorem 2. By C_r inequality, Markov inequality, and Lemma
3.1 $\sim$ Lemma 3.4, we can obtain

$$
\begin{aligned}
0 \leq R_n - R_G &\leq \int |Q(x)|P\{|Q_n(x) - Q(x)| \geq |Q(x)|\}dx \\
&\leq \int |Q(x)|^{1-\delta} E_n|Q_n(x) - Q(x)|^{\delta} dx \\
&\leq M \int |Q(x)|^{1-\delta}\Big\{\sum_{i=1}^{2} E_n|v_{in}(x) - v_i(x)|^{\delta} + \sum_{j=1}^{3} E_n|S_{jn}(x) - S_j(x)|^{\delta} \\
&\qquad\qquad + |E_n f_n(x) - f(x)|^{\delta}\Big\}dx \\
&\leq Mn^{-\delta s/(2s+1)} \int |Q(x)|^{1-\delta} dx
\end{aligned}
$$

$$= Mn^{-\delta s/(2s+1)} \int \{|-e^{c\theta_0}v_1(x) - cv_2(x) + (c\theta_0 + 1)f(x) + 2e^{c\theta_0}S_1(x)$$

$$+ 2cS_2(x) - 2(c\theta_0 + 1)S_3(x)|\}^{1-\delta}dx$$

$$\le Mn^{-\delta s/(2s+1)}\left[\int \sum_{i=1}^{2}|v_i(x)|^{1-\delta}dx + \int \sum_{j=1}^{3}|S_j(x)|^{1-\delta}dx + \int [f(x)]^{1-\delta}dx\right]$$

Note that

$$\int |v_1(x)|^{1-\delta}dx$$

$$= \int \left\{\left|e^{-cx/m}f(x) - u(x)(e^{-cx} - e^{-cx/m})\sum_{j=1}^{\infty}\frac{f(m^j x)}{u(m^j x)}\right.\right.$$

$$\left.\left. - cu(x)\sum_{j=1}^{\infty}\int_{x/m}^{x}\frac{e^{-c\theta}f(m^j\theta)}{u(m^j\theta)}d\theta\right|\right\}^{1-\delta}dx$$

$$\le M\left\{\int |f(x)|^{1-\delta}dx + \int \left|\sum_{j=1}^{\infty}f(m^j x)\right|^{1-\delta}dx + \int \left|\sum_{j=1}^{\infty}\int_{x/m}^{x}f(m^j\theta)d\theta\right|^{1-\delta}dx\right\}$$

From Lemma 3.4 and conditions (B1) $\sim$ (B3) of Theorem 2, we get

$$\int |v_1(x)|^{1-\delta}d\delta < \infty \quad \text{and} \quad \int [f(x)]^{1-\delta}dx < \infty.$$

By the same computation we also obtain $\int |v_2(x)|^{1-\delta}dx < \infty$.

Similar computation yields $\int |S_j(x)|^{1-\delta}dx < \infty$, $j = 1, 2, 3$.

Combining above inequalities, we conclude $0 \le R_n - R_G \le Mn^{-\delta s/(2s+1)}$.

That is $R_n - R_G = O(n^{-\delta s/(2s+1)})$. End of proof.

5. An Example. Let $f(x|\theta) = \frac{1}{\theta}I_{[\theta,2\theta]}(x)$, $g(\theta) = \theta e^{-\theta}I_{(0,\infty)}(\theta)$. Then $u(x) = 1$, $A(\theta) = \frac{1}{\theta}$, $\theta \in (0, +\infty)$. Then $f(x) = v(x) = e^{-x/2} - e^{-x}$, straightforward calculating can yield

(A1) $\sum_{j=1}^{\infty} x^t[e^{-2^{j-1}x} - e^{-2^j x}] < \infty$, $t = 0, 1$

(A2) $m^j A(m^j x)g(m^j x) \to 0$ $(j \to \infty)$

(A3) $\sum_{j=1}^{\infty} f^{(s)}(m^j x) < \infty,\ \sum_{j=1}^{\infty} f(m^j x) < \infty,\ x\sum_{j=1}^{\infty} f^{(s)}(m^j x) < \infty,\ x^2\sum_{j=1}^{\infty} f(m^j x)$
$< \infty$

(A4) $x\sum_{j=1}^{\infty} \int_{x/m}^{x} f^{(s)}(m^j \theta)d\theta < \infty$

we can verify that (A1) $\sim$ (A5) and all conditions of Theorem 1, 2 hold.

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