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OSCILLATION IN LINEAR NEUTRAL DIFFERENTIAL SYSTEMS WITH SEVERAL DELAYS

BY

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Abstract. Algebraic sufficient conditions are obtained for all nontrivial solutions of the linear system

$$\frac{d}{dt}\left[x_i(t)-\sum_{j=1}^{n}c_{ij}(t)x_j(t-\tau)\right]+\sum_{j=1}^{n}a_{ij}(t)x_j(t-\sigma_j)=0,\qquad i=1,2,\ldots,n$$

to be oscillatory.

1. Introduction. Oscillations of scalar neutral differential equations have been investigated by many authors (Gammatikopoulos et al [6-8], Kulenovic [10], Gopalsamy and Zhang [5, 14], Lu [11], Ruan [12]). The purpose of this article is to derive a set of sufficient conditions for all nontrivial solutions of the neutral system

$$\frac{d}{dt}\left[x_i(t)-\sum_{j=1}^{n}c_{ij}(t)x_j(t-\tau)\right]+\sum_{j=1}^{n}a_{ij}(t)x_j(t-\sigma_j)=0,\quad i=1,2,\ldots,n \tag{1}$$

to be oscillatory. It has been established by Arino and Gyon [1] that a

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necessary and sufficient condition for the linear system

$$\frac{d}{dt}\left[y(t)-\sum_{j=1}^{m}B_j y(t-\sigma_j)\right]=\sum_{i=1}^{m}A_i y(t-\tau_i) \tag{2}$$

(where A_i, B_j are $n\times n$ matrices, σ_j and τ_j are nonnegative constants and y is an n vector) to be oscillatory is that the associated characteristic equation

$$\det\left[\lambda\left(I-\sum_{j=1}^{m}B_j e^{-\lambda\sigma_j}\right)-\sum_{i=1}^{m}A_i e^{-\lambda\tau_i}\right]=0 \tag{3}$$

has no real roots. In applications it is often desirable to derive sufficient conditions expressed in terms of parameters (coefficients, delays etc.) of the equations themselves. This involves further analysis of the characteristic equations such as (3). In fact, it is a nontrivial task to obtain conditions for (3) to have or not to have real roots. A special case of (2) has been considered by Gyori and Ladas [9] in the form

$$\frac{d}{dt}[x_i(t)-p_i x_i(t-\tau)]+\sum_{k=1}^{m}\left[\sum_{j=1}^{n}q_{ij}^{(k)}x_j(t-\sigma_k)\right]=0,\quad i=1,2,\ldots,n, \tag{4}$$

and easily verifiable sufficient conditions for the oscillation of the system (4) have been obtained. Our result obtained below differs from the result of Gyori and Ladas [9] in two different respects; firstly we consider the coefficient of the neutral term to be a general matrix rather than a diagonal one; secondly we reduce our result to the oscillation of a scalar neutral differential equation whereas in [9] the result has been reduced to a nonneutral delay differential equation and thereby the effects of the coefficients p_i $(i=1,2,\ldots,n)$ are lost.

2. Oscillation criteria. A nontrivial solution of system (1) is said to be oscillatory if at least one component of solution is oscillatory in the sense of oscillation of scalar valued functions. A solution $x = (x_1, x_2, \ldots, x_n)$ of (1) is said to be nonoscillatory if all the compenents of the solution are nonoscillatory; that is there exists a $t_0 \in R$ such that

$$|x_i(t)| > 0 \quad \text{for} \quad t \geq t_0, \quad i = 1, 2, \ldots, n.$$

Theorem 1. *Suppose that the following conditions are satisfied:*

1. $\sigma_1, \sigma_2, \ldots, \sigma_n$ *are positive real numbers and* τ *is a nonnegative number;*
2. a_{ij} $(i, j = 1, 2, \ldots, n)$ *are bounded continuous functions defined for all* $t \geq 0$;
3. c_{ij} $(i, j = 1, 2, \ldots, n)$ *are bounded continuous functions with bounded derivatives such that*

$$\begin{aligned} c &= \max_{1\leq i\leq n} \sup_{t\geq 0} \sum_{j=1}^{n} |c_{ji}(t)| < 1, \\ C &= \min_{1\leq i\leq n} \inf_{t\geq 0} c_{ii}(t) - \max_{1\leq i\leq n} \sup_{t\geq 0} \sum_{\substack{j=1 \\ j\neq i}}^{n} |c_{ji}(t)| > 0; \end{aligned} \tag{5}$$

4. *all nontrivial solutions of the scalar neutral differential equation*

$$\frac{d}{dt}[u(t) - Cu(t-\tau)] + \mu u(t - \sigma_0) = 0 \tag{6}$$

are oscillatory where

$$\begin{aligned} \sigma_0 &= \min_{1\leq i\leq n} \{\sigma_i\}, \quad \mu = \min_{1\leq i\leq n} \left[\alpha_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ji}\right], \\ \alpha_{ii} &= \inf_{t\geq 0} a_{ii}(t), \quad \beta_{ji} = \sup_{t\geq 0} |a_{ji}(t)|, \quad j \neq i. \end{aligned}$$

Then all nontrivial solutions of the linear neutral system (1) *are oscillatory.*

Proof. Our strategy of proof is to assume the existence of a nonoscillatory solution of (1) and then derive a contradiction. Accordingly suppose that (1) has a solution such that

$$|x_i(t)| > 0 \quad \text{for} \quad t \geq T_0, \quad i = 1, 2, \ldots, n.$$

Then for $t \geq T_1 = T_0 + \sigma^* + \tau$ ($\sigma^* = \max\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$), we have

$$|x_i(t)| > 0; \quad |x_i(t-\tau)| > 0, \quad |x_i(t-\sigma_i)| > 0; \quad i = 1, 2, \ldots, n.$$

Let $\delta_i = \text{sign} x_i(t)$ for $t \geq T_1$, then for $t \geq T_1$, we have

$$\frac{d}{dt}\left[\delta_i x_i(t) - \sum_{j=1}^{n} c_{ij}(t)\delta_i x_j(t-\tau)\right] + \sum_{j=1}^{n} a_{ij}(t)\delta_i x_j(t-\sigma_j) = 0,$$

which leads to

$$\begin{aligned} &\frac{d}{dt}|x_i(t)| + \frac{d}{dt}\left[-\sum_{j=1}^{n} c_{ij}(t)\delta_i x_j(t-\tau)\right] + a_{ii}(t)|x_i(t-\sigma_i)| \\ \leq &\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}(t)|\,|x_j(t-\sigma_j)|, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{7}$$

One can simplify (7) to the form

$$\begin{aligned} \frac{d}{dt}\left[|x_i(t)| - \sum_{j=1}^{n} c_{ij}(t)\delta_i x_j(t-\tau)\right] + \alpha_{ii}|x_i(t-\sigma_i)| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ij}|x_j(t-\sigma_j)|, \\ i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Adding the respective sides of (8) from $i = 1$ to $i = n$ and then simplifying, we obtain

$$\frac{d}{dt}\left[\sum_{i=1}^{n}|x_i(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(t)\delta_i x_j(t-\tau)\right] + \sum_{i=1}^{n}\left[\alpha_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n}\beta_{ji}\right]|x_i(t-\sigma_i)| \leq 0. \tag{9}$$

We further simplify (9) to the form

$$\frac{d}{dt}\left[\sum_{i=1}^{n}|x_i(t)|-\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}(t)\delta_i x_j(t-\tau)\right]+\mu\sum_{i=1}^{n}|x_i(t-\sigma_i)|\leq 0. \tag{10}$$

Integrating (10) on $[T_1,t]$, we have

$$\begin{aligned}&\sum_{i=1}^{n}|x_i(t)|-\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}(t)\delta_i x_j(t-\tau)+\mu\sum_{i=1}^{n}\int_{T_1}^{t}|x_i(s-\sigma_i)|ds\\ \leq\ &\sum_{i=1}^{n}|x_i(T_1)|+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}(T_1)||x_j(T_1-\tau)|.\end{aligned} \tag{11}$$

Let

$$p=\sum_{i=1}^{n}|x_i(T_1)|+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}(T_1)||x_j(T_1-\tau)|,$$

then we can rewrite (11) as follows:

$$\begin{aligned}\sum_{i=1}^{n}|x_i(t)|+\sum_{i=1}^{n}\mu\int_{T_1}^{t}|x_i(s-\sigma_i)|ds &\leq p+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}(t)||x_j(t-\tau)|\\ &\leq p+c\sum_{i=1}^{n}|x_i(t-\tau)|,\end{aligned} \tag{12}$$

and therefore

$$\sum_{i=1}^{n}|x_i(t)|\leq p+c\sum_{i=1}^{n}|x_i(t-\tau)| \quad \text{for} \quad t\geq T_1. \tag{13}$$

Define $v(t)=\sum_{i=1}^{n}|x_i(t)|$ and note from (13) that

$$\begin{aligned}v(t)&\leq p+cv(t-\tau)\\ &\leq p+c[p+cv(t-2\tau)]\\ &\leq p+cp+c^2[p+cv(t-3\tau)]\\ &\ \ \vdots\end{aligned} \tag{14}$$

$$\begin{aligned} &\leq p[1+c+c^2+\cdots+c^{k-1}]+pc^k v(t-k\tau) \\ &< \frac{p}{1-c}+pc^k v(T_1+t_*), \end{aligned}$$

where $t_* = t-T_1-k\tau \in [0,\tau)$. We can conclude from (14) and the assumption $0<c<1$ that v is uniformly bounded for all $t \geq 0$. Since $v(t)=\sum_{i=1}^n |x_i(t)|$, it follows that any nonoscillatory solution $x=(x_1,x_2,\ldots,x_n)$ is uniformly bounded for $t \geq 0$. We shall now show that $\dot{x}_i$ $(i=1,2,\ldots,n)$ are also uniformly bounded for $t \geq 0$. We have directly from (1) that

$$\begin{aligned} |\dot{x}_i(t)| &\leq \sum_{j=1}^n |a_{ij}(t)||x_j(t-\sigma_j)| + \sum_{j=1}^n |\dot{c}_{ij}(t)||x_j(t-\tau)| \\ &\quad + \sum_{j=1}^n |c_{ij}(t)||\dot{x}_j(t-\tau)|, \quad i=1,2,\ldots,n, \end{aligned} \tag{15}$$

which by addition leads to

$$\begin{aligned} \sum_{i=1}^n |\dot{x}_i(t)| &\leq \sum_{i=1}^n \left[\sum_{j=1}^n |a_{ij}(t)||x_j(t-\sigma_j)| + \sum_{j=1}^n |\dot{c}_{ij}(t)||x_j(t-\tau)|\right] \\ &\quad + \sum_{i=1}^n\sum_{j=1}^n |c_{ij}(t)||\dot{x}_j(t-\tau)| \\ &\leq c\sum_{i=1}^n |\dot{x}_i(t-\tau)| + \sum_{i=1}^n\sum_{j=1}^n \left[|a_{ij}(t)||x_j(t-\sigma_j)| + |\dot{c}_{ij}(t)||x_j(t-\tau)|\right]. \end{aligned} \tag{16}$$

If we let

$$\rho(t) = \sup_{s\leq t}\sum_{i=1}^n |\dot{x}_i(s)|, \tag{17}$$

then we have

$$\rho(t) \leq \frac{1}{1-c}\sup_{s\leq t}\sum_{i=1}^n\sum_{j=1}^n \left[|a_{ij}(t)||x_j(s)| + |\dot{c}_{ij}(t)||x_j(s)|\right]. \tag{18}$$

Since x_i $(i=1,2,\ldots,n)$ are uniformly bounded and $\dot{c}_{ij}$ $(i,j=1,2,\ldots,n)$

are bounded by our assumptions, we conclude from (17)-(18) that $\dot{x}_i$ ($i = 1, 2, \ldots$) are uniformly bounded.

We shall now verify that

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, 2, \ldots, n. \tag{19}$$

Using the uniform boundedness of $\sum_{i=1}^{n} |x_i(t)|$, we can derive from (12) that

$$\mu \int_{T_1}^{t} |x_i(s-\sigma_i)| ds \le \beta_i < \infty, \quad i = 1, 2, \ldots, n, \tag{20}$$

showing that $x_i \in L_1(T_1, \infty)$, $i = 1, 2, \ldots, n$. The uniform boundedness of x_i and $\dot{x}_i$ together with Barbalat's lemma ([2,3]) implies (19).

Integrating both sides of (10) on $[t, \infty)$, $t \ge T_1$ and using (19), we have

$$-\left[\sum_{i=1}^{n} |x_i(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(t)\delta_i x_j(t-\tau)\right] + \mu \int_{t}^{\infty} \sum_{i=1}^{n} |x_i(s-\sigma_i)| ds \le 0. \tag{21}$$

We estimate the terms of (21) so that

$$\begin{aligned} &-\sum_{i=1}^{n} |x_i(t)| + \min_{1\le i\le n} \inf_{t\ge 0} c_{ii}(t) \sum_{i=1}^{n} |x_i(t-\tau)| + \mu \int_{t}^{\infty} \sum_{i=1}^{n} |x_i(s-\sigma_i)| ds \\ &\le \left(\max_{1\le i\le n} \sup_{t\ge 0} \sum_{\substack{j=1 \\ j\ne i}}^{n} |c_{ji}(t)|\right) \sum_{i=1}^{n} |x_i(t-\tau)| \end{aligned} \tag{22}$$

which implies

$$\sum_{i=1}^{n} |x_i(t)| - C\sum_{i=1}^{n} |x_i(t-\tau)| \ge \mu \int_{t}^{\infty} \sum_{i=1}^{n} |x_i(s-\sigma_i)| ds, \tag{23}$$

where

$$C = \min_{1\le i\le n} \inf_{t\ge 0} c_{ii}(t) - \max_{1\le i\le n} \sup_{t\ge 0} \sum_{\substack{j=1 \\ j\ne i}}^{n} |c_{ji}(t)|. \tag{24}$$

We can further simplify (23) so that

$$\sum_{i=1}^{n}|x_i(t)| \geq C\sum_{i=1}^{n}|x_i(t-\tau)| + \mu\int_{t-\sigma_0}^{\infty}\sum_{i=1}^{n}|x_i(s)|ds, \tag{25}$$

where $\sigma_0 = \min\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. We note $v(t) = \sum_{i=1}^{n}|x_i(t)|$ and derive from (25) that

$$v(t) \geq Cv(t-\tau) + \mu\int_{t-\sigma_0}^{\infty} v(s)ds, \quad t \geq T_1. \tag{26}$$

Let $\sigma = \max\{\sigma_0, \tau\}$. We define a sequence $\{\varphi_k\}$ as follows:

$$\begin{aligned}
\varphi_0(t) &= \begin{cases} v(T_1), & T_1 - \sigma \leq t < T_1, \\ v(t), & t \geq T_1, \end{cases} \\
\varphi_{k+1}(t) &= \begin{cases} \varphi_{k+1}(T_1), & T_1 - \sigma \leq t < T_1, \\ C\varphi_k(t-\tau) + \mu\int_{t-\sigma_0}^{\infty}\varphi_k(s)ds, & t \geq T_1. \end{cases}
\end{aligned} \tag{27}$$

It is found from (27) and (26) that

$$0 \leq \cdots \leq \varphi_{k+2} \leq \varphi_{k+1} \leq \varphi_k \leq \cdots \leq \varphi_2 \leq \varphi_1 \leq \varphi_0, \tag{28}$$

which shows that the limit $\lim_{k\to\infty}\varphi_k(t) = \varphi^*(t)$ exists in a pointwise sense. By Lebesgue's convergence theorem, φ^* satisfies

$$\varphi^*(t) = C\varphi^*(t-\tau) + \mu\int_{t-\sigma_0}^{\infty}\varphi^*(s)ds, \quad t \geq T_1$$

or equivalently

$$\frac{d}{dt}[\varphi^*(t) - C\varphi^*(t-\tau)] + \mu\varphi^*(t-\sigma_0) = 0, \quad t \geq T_1. \tag{29}$$

It is now easy to see from (27) that

$$\begin{aligned}
\varphi_{k+1}(t) &\geq C\varphi_k(t-\tau)\\
&\geq C^2\varphi_{k-1}(t-2\tau)\\
&\geq C^3\varphi_{k-2}(t-3\tau)\\
&\quad \vdots\\
&\geq C^{k+1}\varphi_0(t-(k+1)\tau)\\
&= \exp\{\frac{t-t^*}{\tau}\ln C\}\varphi_0(t^*),
\end{aligned}$$

where $t=(k+1)\tau+t^*$, $t^*\in[T_1-\sigma,T_1)$ from which we can conclude that φ^* is an eventually positive solution of (29); but this is a contradiction. The proof is complete.

3. Some Remarks. We have obtained sufficient conditions for the oscillation of a vector system of neutral differential equations to those of the oscillation of a scalar neutral equation; whereas Gyori and Ladas [9] have obtained a similar reduction to a certain nonneutral delay equation. We remark that there are numerous type of sufficient conditions for scalar neutral equations to be oscillatory (for instance see Ruan [12], Gopalsamy and Zhang [5], Stavroulakis [13]). For completeness, we derive briefly one such condition in the following.

Corollary 2. *Let μ, σ_0, C be as before. A sufficient condition for* (6) *to be oscillatory is that*

$$\mu\sigma_0>\frac{1-C}{e}. \tag{30}$$

Proof. Suppose that the assertion is not true. Then (6) has a nonoscillatory solution which can be shown to be bounded as in the proof of our theorem above. It is well known that the characteristic equation

$$\lambda(1-Ce^{-\lambda\tau})+\mu e^{-\lambda\sigma_0}=0 \tag{31}$$

associated with (6) has a real root. Since the nonoscillatory solution is bounded, the real characteristic root of (31) cannot be positive. But $\lambda = 0$ is not a root of (31). Thus $\lambda = -\eta$ ($\eta > 0$) is a root and therefore

$$\eta(1 - Ce^{\eta\tau}) = \mu e^{\eta\sigma_0}. \tag{32}$$

It is readily seen from (32) that

$$1 - Ce^{\eta\tau} = \frac{\mu e^{\eta\sigma_0}}{\eta}$$

and hence

$$1 - C > \mu\sigma_0 \frac{e^{\eta\sigma_0}}{\eta\sigma_0} \geq \mu e\sigma_0$$

implying

$$\mu\sigma_0 \leq \frac{1 - C}{e},$$

which contradicts (30). Thus we conclude that (6) is oscillatory when (30) holds.

It should be noted that (30) explicitly contains C, the coefficient of the neutral term of the equation (6). For other sufficient conditions which involve both C and τ, we refer to Gopalsamy and Zhang [5].

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References

1. O. Arino and I. Gyori., *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Diff. Equations, **81**(1989), 98-105.

2. I. Barbalat, *Systemes d'equations differentielle d'oscillations nonlinearires*, Rev. Roumaine Math. Pures Appl., **4**(1959), 267-270.

3. C. Corduneana, *Integral Equations and Stability of Systems*, Academic Press, New York, 1973.

4. Roger A. Horn and Charles R. Johnson, *Matrix Analysis.* Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985, 344.

5. K. Gopalsamy and B. G. Zhang, *Oscillation and nonoscillation in first order neutral differential equations*, J. Math. Anal. Appl., **150**(1)(1990), 42-57.

6. M. K. Grammatikopoulos, E. A. Grove and G. Ladas, *Oscillation and asymptotic behaviour of neutral differential equations with deviating arguments*, Appl. Anal., **22**(1986), 1-19.

7. M. K. Grammatikopoulos, E. A. Grove and G. Ladas, *Oscillation of first order neutral delay differential equations*, J. Math. Anal. Appl., **120**(1986), 510-520.

8. M. K. Grammatikopoulos, Y. G. Sficas and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillations of neutral equations with several coefficients*, J. Diff. Eqns., **76**(1988), 294-311.

9. I. Gyori and G. Ladas, *Oscillations of systems of neutral differential equations*, Diff. Int. Equations, **1**(1988), 281-287.

10. M. R. S. Kulenovic, G. Ladas and A. Meimaridou, *Necessary and sufficient conditions for the oscillation of neutral differential equations*, J. Austral. Math. Soc., Ser. B, **28**(1987), 362-375.

11. W. D. Lu, *Existence of nonoscillatory solutions of first order nonlinear neutral equations*, J. Austral. Math. Soc., Ser. B, **32**(1990), 180.

12. J. Ruan, *On the oscillation of neutral differential difference equations with several retarded arguments*, Scientia Sinica, Ser. A, **29**(5)(1986), 1132-1144.

13. I. P. Stavroulakis, *Oscillations of mixed neutral equations*, Hiroshima Math. J., **19**(3)(1989), 441-456.

14. B. G. Zhang and K. Gopalsamy, *Oscillation and nonoscillation of a class of neutral differential equations*, Proceedings of the First World Congress of Nonlinear Analysis (eds V. Lakshmikantham, Walter de Gruyter), Berlin, New York, 1996, 1515.

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