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ON THE MODIFIED NAVIER-STOKES EQUATIONS IN n-DIMENSIONAL SPACES

BY

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1. Introduction. The $n\,(\geq\, 3)$-dimensional incompressible Navier-Stokes equations have been a challenging problem both in mathematics and in physics for a long time. There have been tremendous efforts in trying to solve the global strong solutions with arbitrarily large initial data in certain Sobolev spaces. Partial regularity for suitably weak solutions has been established very well by Caffarelli, Kohn and Nirenberg in [1] and by Fang-Hua Lin in [5]. See also Ladyzhenskaya [3], Serrin [11], Temam [13] and Zhang [19]. Necas, Ruzicka and Sverak proved the non-existence of a self-similar solution of the Navier-Stokes equations for 3-dimensional problem in [6]. Long time asymptotic behaviors of L^2-norm of the solutions has also been studied extensively by Schonbek in [7]-[10], by Wiegner in [14]-[16] and by the author in [17]-[20].

It is well known that the Cauchy problems for the 2-dimensional incompressible Navier-Stokes equations

$$u_t + u\cdot\nabla u - \triangle u + \nabla p = 0, \qquad \nabla\cdot u = 0, \tag{1}$$

with large initial data $u(x,0) = u_0(x) \in H^m(\mathbb{R}^2)$, have global smooth solutions in $L^\infty(\mathbb{R}^+; H^m(\mathbb{R}^2))$, where $m \geq 1$ is sufficiently large, see [4]. However, if we increase the spatial dimension by only one, namely, consider the

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3-dimensional Cauchy problems, then, for large initial data in $H^m(\mathbb{R}^3)$, no global strong result has been proved. It seems that the second order dissipation $-\triangle u$ is not "strong enough" to support the "nonlinear reaction terms $u \cdot \nabla u + \nabla p$". Even the presence of the "good term $-\frac{\partial}{\partial t}\triangle u$" cannot guarantee the global existence for n-dimensional problems, $n > 3$, although for $n = 3$ it is good enough. Actually, some people conjecture that $\|\nabla u(\cdot, t)\|^2$ of the solutions for $n = 3$ with large initial data blow up in finite time. It seems to us that the Cauchy problems with small initial data admits global strong solutions. We believe that when the dissipation and the nonlinearity attain some kind of balance, the strong solution will exist globally. To guarantee the global existence of the strong large solutions without breaking the physical meaning of the Navier-Stokes equations, i.e. keeping the nonlinearity $u \cdot \nabla u + \nabla p$ and the incompressible conditions $\nabla \cdot u = 0$, we can either increase the order σ of the dissipation $(-\triangle)^\sigma u$ or decrease the dimension n of the physical problems in $\mathbb{R}^n$, thus allowing n to be a positive real number. These suggestions make sense mathematically.

We are interested in the smallest power σ in $(-\triangle)^\sigma$ such that the Cauchy problems $u(x, 0) = u_0(x)$ for the modified Navier-Stokes equations

$$u_t + u \cdot \nabla u + \alpha(-\triangle)^\sigma u + \nabla p = 0, \qquad \nabla \cdot u = 0, \tag{2}$$

possesses a unique global strong solution $u \in L^\infty(\mathbb{R}^+; H^m(\mathbb{R}^n))$ for arbitrarily large initial data $u(x, 0) = u_0(x) \in H^m(\mathbb{R}^n)$, where $m \geq 1$ is sufficiently large.

Theorem 1. *Let $\sigma = \frac{n+2}{4}$ and $u_0 \in H^m(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Then the Cauchy problems $u(x, 0) = u_0(x)$ for the modified Navier-Stokes equations* (2) *have a unique smooth solution $u \in L^\infty(\mathbb{R}^+; H^m(\mathbb{R}^n)), \nabla u \in L^2(\mathbb{R}^+; H^m(\mathbb{R}^n))$.*

2. Proof of Theorem 1. Using the incompressible condition $\nabla \cdot u = 0$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, we see that for continuously differentiable functions u and $p \in C^1(\mathbb{R}^n; \mathbb{R}^+)$, there hold

$$\int_{\mathbb{R}^n} u(x,t) \cdot [u(x,t) \cdot \nabla u(x,t)]dx = \int_{\mathbb{R}^n} u(x,t) \cdot \nabla p(x,t)dx = 0.$$

The proof of Theorem 1 consists of four steps.

(A) First of all, if we make the scalar product of the vector $u(x,t)$ with the modified Navier-Stokes equations (2), we get

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^2 dx + 2\alpha \int_{\mathbb{R}^n} |(-\triangle)^{\sigma/2} u(x,t)|^2 dx = 0. \tag{3}$$

Hence we get the elementary estimates

$$\sup_{t\in\mathbb{R}^+} \|u(\cdot,t)\|^2 \le \|u_0\|^2, \qquad 2\alpha \int_0^\infty \|(-\triangle)^{\sigma/2} u(\cdot,t)\|^2 dt \le \|u_0\|^2. \tag{4}$$

(B) Next, we multiply the modified Navier-Stokes equations by $2(-\triangle)^\sigma u(x,t)$ and integrate with respect to $x \in \mathbb{R}^n$ to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} |(-\triangle)^{\sigma/2} u(x,t)|^2 dx + 2\alpha \int_{\mathbb{R}^n} |(-\triangle)^{\sigma} u(x,t)|^2 dx \\ = \; & -2\int_{\mathbb{R}^n} (-\triangle)^\sigma u(x,t) \cdot [u(x,t) \cdot \nabla u(x,t)]dx. \end{aligned} \tag{5}$$

The right hand side in the above equation is controlled by

$$\begin{aligned} & 2\left|\int_{\mathbb{R}^n} (-\triangle)^\sigma u(x,t) \cdot [u(x,t)\cdot\nabla u(x,t)]dx\right| \\ \le\; & \frac{\alpha}{2}\int_{\mathbb{R}^n} |(-\triangle)^\sigma u(x,t)|^2 dx + \frac{2}{\alpha}\int_{\mathbb{R}^n} |u(x,t)\cdot\nabla u(x,t)|^2 dx \\ \le\; & \frac{\alpha}{2}\int_{\mathbb{R}^n} |(-\triangle)^\sigma u(x,t)|^2 dx + \frac{2}{\alpha}\|u(\cdot,t)\|^2_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx \\ \le\; & \frac{\alpha}{2}\int_{\mathbb{R}^n} |(-\triangle)^\sigma u(x,t)|^2 dx \\ & +C\Big[\|u(\cdot,t)\|^{2-n/2\sigma}\|(-\triangle)^\sigma u(\cdot,t)\|^{n/2\sigma}\Big]\Big[\|u(\cdot,t)\|^{2-2/\sigma}\|(-\triangle)^{\sigma/2} u(\cdot,t)\|^{2/\sigma}\Big] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,t)|^2dx+\frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,t)|^2dx\\
&\quad +C\left[\int_{\mathbb{R}^n}|u(x,t)|^2dx\right]^{n/2}\left[\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx\right]^2\\
&\le \alpha\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,t)|^2dx+C(\|u_0\|^n)\left[\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx\right]^2.
\end{aligned}
$$

In the above estimates, we have implicitly used $4\sigma=n+2$. At last, we get the estimate

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx+\alpha\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,t)|^2dx\\
\le\ &C(\|u_0\|^n)\left[\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx\right]^2.
\end{aligned}
$$

Integrating in time to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx+\alpha\int_0^t\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,s)|^2dxds\\
\le\ &\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u_0(x)|^2dx+C(\|u_0\|^n)\int_0^t\left[\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,s)|^2dx\right]^2ds.
\end{aligned}
$$

Applying the generalized Gronwall's inequality, we have the global estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dx+\alpha\int_0^t\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,s)|^2dxds\\
\le\ &\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u_0(x)|^2dx\exp\left[C(\|u_0\|^n)\int_0^{\infty}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u(x,t)|^2dxdt\right]\\
\le\ &\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}u_0(x)|^2dx\exp\left[C(\|u_0\|^{n+2})\right]=C(\|u_0\|^{n+2},\|u_0\|_{\sigma}^2).
\end{aligned}
\tag{6}
$$

In the above discussions, notice that $\frac{n}{2\sigma}<2$ is always true if we choose $\sigma=\frac{n+2}{4}$. This is the key estimate for global strong solutions of (2) with large initial data.

(C) Finally, we multiply the modified Navier-Stokes equations by $2(-\triangle)^{2\sigma}u$ and integrate in x to obtain the following equation

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma}u(x,t)|^2dx+2\alpha\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx\\
= \quad & -2\int_{\mathbb{R}^n}(-\triangle)^{2\sigma}u(x,t)\cdot[u(x,t)\cdot\nabla u(x,t)]dx.
\end{aligned}
\tag{7}
$$

Applying the fractional interpolation inequality given in the Appendix, we see that the right hand side of the above energy equation is dominated by

$$
\begin{aligned}
& 2\left|\int_{\mathbb{R}^n}(-\triangle)^{2\sigma}u(x,t)\cdot[u(x,t)\cdot\nabla u(x,t)]dx\right|\\
= \; & 2\left|\int_{\mathbb{R}^n}(-\triangle)^{3\sigma/2}u(x,t)\cdot(-\triangle)^{\sigma/2}[u(x,t)\cdot\nabla u(x,t)]dx\right|\\
\leq \; & \frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx+\frac{2}{\alpha}\int_{\mathbb{R}^n}|(-\triangle)^{\sigma/2}[u(x,t)\cdot\nabla u(x,t)]|^2dx\\
\leq \; & \frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx+C\|u(\cdot,t)\|^2_{L^4(\mathbb{R}^n)}\|\nabla(-\triangle)^{\sigma/2}u(\cdot,t)\|^2_{L^4(\mathbb{R}^n)}\\
& +C\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^2_{L^4(\mathbb{R}^n)}\|\nabla u(\cdot,t)\|^2_{L^4(\mathbb{R}^n)}\\
\leq \; & \frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx\\
& +C\left[\|u(\cdot,t)\|^{2-n/2\sigma}\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{n/2\sigma}\right]\\
& \times\left[\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{2-(n+4)/(4\sigma)}\|(-\triangle)^{3\sigma/2}u(\cdot,t)\|^{(n+4)/(4\sigma)}\right]\\
& +C\left[\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{2-n/(4\sigma)}\|(-\triangle)^{3\sigma/2}u(\cdot,t)\|^{n/(4\sigma)}\right]\\
& \times\left[\|u(\cdot,t)\|^{2-(n+4)/(4\sigma)}\|(-\triangle)^{\sigma}u(\cdot,t)\|^{(n+4)/(4\sigma)}\right]\\
= \; & \frac{\alpha}{2}\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx\\
& +C\|u(\cdot,t)\|^{2-n/2\sigma}\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{2+(n-4)/(4\sigma)}\|(-\triangle)^{3\sigma/2}u(\cdot,t)\|^{(n+4)/(4\sigma)}\\
& +C\|u(\cdot,t)\|^{2-(n+4)/(4\sigma)}\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{2-n/(4\sigma)}\\
& \times\|(-\triangle)^{\sigma}u(\cdot,t)\|^{(n+4)/(4\sigma)}\|(-\triangle)^{3\sigma/2}u(\cdot,t)\|^{n/(4\sigma)}\\
\leq \; & \alpha\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx+C\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^6\\
& +C\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^{4-n/2\sigma}\|(-\triangle)^{\sigma}u(\cdot,t)\|^2\\
\leq \; & \alpha\int_{\mathbb{R}^n}|(-\triangle)^{3\sigma/2}u(x,t)|^2dx+C\|(-\triangle)^{\sigma/2}u(\cdot,t)\|^2+C\|(-\triangle)^{\sigma}u(\cdot,t)\|^2,
\end{aligned}
$$

where we have also applied the uniform estimates $\sup_{t\in\mathbb{R}^+}\|u(\cdot,t)\|\leq\|u_0\|$

and $\sup \|(-\triangle)^{\sigma/2}u(\cdot,t)\| \le C\|(-\triangle)^{\sigma/2}u_0\|$. Now equation (7) becomes

$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u(x,t)|^2dx + \alpha\int_{\mathbb{R}^n} |(-\triangle)^{3\sigma/2}u(x,t)|^2dx \\ \le\ & C\int_{\mathbb{R}^n} |(-\triangle)^{\sigma/2}u(x,t)|^2dx + C\int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u(x,t)|^2dx. \end{aligned}$$

Integrating in time gives

$$\begin{aligned} & \int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u(x,t)|^2dx + \alpha\int_0^t\int_{\mathbb{R}^n} |(-\triangle)^{3\sigma/2}u(x,s)|^2dxds \\ \le\ & \int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u_0(x)|^2dx + C\int_0^t\int_{\mathbb{R}^n} |(-\triangle)^{\sigma/2}u(x,s)|^2dxds \\ & +C\int_0^t\int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u(x,s)|^2dxds \\ \le\ & C\int_{\mathbb{R}^n} |(-\triangle)^{\sigma}u_0(x)|^2dx + C\int_{\mathbb{R}^n} |u_0(x)|^2dx. \end{aligned} \tag{8}$$

(D) The global L^2 estimates for higher order derivatives of the solutions and for the pressure p can be similarly obtained if $u_0 \in H^m(\mathbb{R}^n)$. The uniqueness follows from routine method. This completes the proof of Theorem 1.

Remark. Let $p > n \ge 3$ and $q > 2$ and $\frac{n}{p} + \frac{2}{q} = 1$, such that

$$\int_0^\infty \left[\int_{\mathbb{R}^n} |u|^p dx\right]^{q/p} dt < \infty.$$

Under this condition, Zhang [19] established that

$$u \in \left[\bigcap_{p\le r<\infty} L^{2r/(r-n)}(\mathbb{R}^+; L^r(\mathbb{R}^n))\right] \bigcap \left[\bigcap_{2\le s<\infty} L^\infty(\mathbb{R}^+; L^s(\mathbb{R}^n))\right].$$

3. Appendix. We list several well-known inequalities in dynamical systems.

1. (**Gronwall's inequality**) Let the positive continuous functions f, g and $h \in L^1[0,+\infty)$ satisfy the inequality

$$g(t) \le f(t) + \int_0^t g(s)h(s)ds,$$

for all $t > 0$. Then we have the estimate

$$g(t) \le f(0)\exp\left[\int_0^t h(s)ds\right] + \int_0^t f'(s)\exp\left[\int_s^t h(r)dr\right]ds.$$

Next, we present some classical interpolation inequalities in Sobolev spaces.

2. (**Gagliardo-Nirenberg's inequality**) For all $1 \le p,q,r \le +\infty$ and for all integers $n \ge 1$ and $m > k \ge 0$, there are positive constants $\alpha \in [k/m, 1]$ and C, such that for all $u \in C_0^\infty(\mathbb{R}^n)$, we have the estimate

$$\begin{aligned} \|D^k u\|_{L^p} &\le C\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \text{ where} \\ n/p - k &= \alpha(n/r - m) + (1-\alpha)n/q, \\ \left\|D^k u\right\|_{L^p}^p &= \sum_{\alpha_1+\cdots+\alpha_n=k}\left\|\frac{\partial^{\alpha_1+\cdots+\alpha_n}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right\|_{L^p}^p. \end{aligned}$$

The only exception is that $\alpha \ne 1$ if $m - n/r = k$ and $1 < p < \infty$.

The following interpolation inequality plays a role in the modified Navier-Stokes equations.

3. (**Fractional interpolation inequality**) Let $f, g \in W^{2m,p} \bigcap L^q(\mathbb{R}^n)$. Then for all p, q with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$

there holds the estimate

$$\begin{aligned} &\|(-\triangle)^m(fg)\|_{L^2(\mathbb{R}^n)} \\ \le\ & C\|(-\triangle)^m f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} + C\|f\|_{L^q(\mathbb{R}^n)}\|(-\triangle)^m g\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

4. (**Leray-Schauder fixed point principle**) Let X be a Banach space and let $A_\lambda : X \times [0,1] \to X$ be a well defined mapping. Suppose that
(1) For all fixed $\lambda : 0 \le \lambda \le 1$, A_λ is a completely continuous operator.
(2) For all bounded subset $E \subset X$, A_λ is uniformly continuous in λ.
(3) $A_0 X = \{a\}$, where $a \in X$ is a fixed point.
(4) There is a λ-independent constant $M > 0$, such that all possible solutions of the equation $A_\lambda x_\lambda = x_\lambda$ with $0 \le \lambda \le 1$ satisfy

$$\|x_\lambda\| \le M.$$

Then there exists a point $x_0 \in X$, such that

$$A_1 x_0 = x_0.$$

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