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TWO-STOCK UNIVERSAL PORTFOLIOS WITH SIDE INFORMATION

BY

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Abstract. Cover and Ordentlich (1996) introduced the class of Dirichlet-weighted universal portfolios and studied in particular the beta-weighted (1,1) and (1/2,1/2) universal portfolios with side information by deriving bounds for the ratio of the optimal wealth to the wealth of the universal portfolio in the form of polynomials in the number of trading days. We shall extend the results of Cover and Ordentlich to the general parametric class of beta-weighted two-stock universal portfolios with side information by exploiting bounds on the ratio of gamma functions. An alternative way of deriving such bounds for the ratio of the optimal wealth to the universal wealth by using appropriate integrals is shown. The asymptotic behaviour of such bounds will be discussed.

1. Introduction and preliminaries. Universal portfolios with side information are studied in Cover and Ordentlich (1996). For Dirichlet-weighted $(1, 1, \dots, 1)$ and $(1/2, 1/2, \dots, 1/2)$ universal portfolios, polynomial bounds are derived for the ratio of the optimal wealth to the universal wealth. These bounds are extended to the general class of Dirichlet-weighted $(\alpha_1, \alpha_2, \dots, \alpha_m)$ universal portfolios by Tan (2002). It is well-known that with the aid of side information an investor can increase his wealth by a

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larger amount. The target of the optimal wealth is the maximum wealth achievable. For two-stock Dirichlet-weighted (α_1, α_2) universal portfolios (also known as beta-weighted (α_1, α_2) universal portfolios), we shall show that with a proper choice of α_1 and α_2 less than 1, there are smaller polynomial bounds for the ratio of the optimal wealth to the universal wealth.

Consider a two-stock market described by the *price-relative vector* $\mathbf{x} = (x_1, x_2)$ where the *price-relative* x_i is the ratio of the closing price of the ith stock to the opening price on a particular trading day, for $i = 1, 2$. For n trading days, $\mathbf{x}^n = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ describes the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of price-relative vectors, where $\mathbf{x}_j$ is the price-relative vector on the jth day, for $j = 1, 2, \ldots, n$. A *portfolio vector* or *strategy* $\mathbf{b} = (b_1, b_2)$ consists of fractions b_1, b_2 of the current wealth invested in the first and second stocks respectively, where $b_1 + b_2 = 1$. The *portfolio simplex* B is defined by:

$$B = \{(b_1, b_2) : b_i \geq 0 \text{ for } i = 1, 2,\ b_1 + b_2 = 1\}. \tag{1}$$

The *beta-weighted* or *Dirichlet-weighted* (α_1, α_2) *universal portfolio* is the sequence of portfolios $\{\hat{\mathbf{b}}_n\}$ on the nth trading day given by:

$$\hat{\mathbf{b}}_n(\mathbf{x}^{n-1}) = \frac{\displaystyle\int_B \mathbf{b} \prod_{i=1}^{n-1} \mathbf{b}^t \mathbf{x}_i \, d\mu(\mathbf{b})}{\displaystyle\int_B \prod_{i=1}^{n-1} \mathbf{b}^t \mathbf{x}_i \, d\mu(\mathbf{b})} \quad \text{for } n = 2, 3, \ldots, \tag{2}$$

where $\mathbf{x}^{n-1} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1})$ is the sequence of price-relative vectors before the nth trading day and $\mu(\mathbf{b})$ is the beta (α_1, α_2) distribution function defined over $0 \leq b_1 \leq 1$, namely

$$d\mu(\mathbf{b}) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} b_1^{\alpha_1 - 1}(1 - b_1)^{\alpha_2 - 1} db_1, \tag{3}$$

and $\alpha_1 > 0$, $\alpha_2 > 0$. For $n = 1$, $\hat{\mathbf{b}}_1 = \int_B \mathbf{b} \, d\mu(\mathbf{b})$. The *universal wealth* $\hat{S}_n(\mathbf{x}^n)$ achieved by the portfolio $\{\hat{\mathbf{b}}_n\}$ at the end of the nth trading day is given by:

$$\hat{S}_n(\mathbf{x}^n) = \prod_{i=1}^{n} \hat{\mathbf{b}}_i^t(\mathbf{x}^{i-1})\mathbf{x}_i, \tag{4}$$

assuming an initial wealth of 1 unit. The wealth $S_n(\mathbf{x}^n)$ achieved by a *constant rebalanced portfolio* $\mathbf{b}$ is given by

$$S_n(\mathbf{x}^n) = \prod_{i=1}^{n} \mathbf{b}^t\mathbf{x}_i, \tag{5}$$

assuming an initial wealth of 1 unit. The *best constant rebalanced portfolio* $\mathbf{b}^*$ is the portfolio that maximizes $S_n(\mathbf{x}^n)$ over all $\mathbf{b}$ in B.

Suppose the market is described by a set of M price-relative vectors $\mathbf{a}_i = (a_{i1}, a_{i2})$ for $i = 1, 2, \ldots, M$. If $\max(a_{j1}, a_{j2}) > 1$ for the price-relative vector (a_{j1}, a_{j2}), then at least one of the stock prices has risen in a one-day trading period. On the other hand if $\max(a_{j1},\ a_{j2}) < 1$, then both stocks have fallen in price in a one-day trading period. A price relative of 1 indicates that there is no change in price. Assume that there are two states of side information associated with the market. We represent these states as 1, 2 where 1 indicates a strong belief of the appearance of the current price-relative vector $\mathbf{a}_j$ where $a_{j1} \geq a_{j2}$ and 2 indicates a strong belief of the appearance of the current price-relative vector $\mathbf{a}_k$ where $a_{k1} < a_{k2}$. The portfolio $\mathbf{b} = (1, 0)$ is used if the current state is 1, otherwise $\mathbf{b} = (0, 1)$ is used if the current state is 2. Having known the $(n-1)$ price-relative vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$ appearing in the past $(n-1)$ trading days, the current state y_n on the nth trading day is a function of $\mathbf{x}^{n-1} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1})$, where $y_n \in \{1, 2\}$, $n = 2, 3, \ldots$. If there is some mathematical relationship among $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$, then this relationship is represented by the current state y_n. An example of such a relationship is given in Cover and Ordentlich (1996). The side-information vector $y^n = (y_1, y_2, \ldots, y_n)$ can predict with accuracy the appearance of $\mathbf{x}_n$ if the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$ appears in some predictable manner. In this case, by using the portfolios (1,0) and (0,1) according to the current states 1 and 2 respectively, the maximum wealth

$S_n^*(\mathbf{x}^n \mid y^n) = \prod_{j=1}^{n} \max(x_{j1}, x_{j2})$ is achievable, where each $\mathbf{x}_j = (x_{j1}, x_{j2}) = \mathbf{a}_k$ for some $k = 1, 2, \ldots, M$. More precisely,

$$(6) \qquad \begin{aligned} S_n^*(\mathbf{x}^n \mid y^n) &= \prod_{j=1}^{n} \mathbf{b}_j^t(y_j)\mathbf{x}_j \\ &= \left(\prod_{j\le n: y_j=1} (1,0)\mathbf{x}_j\right)\left(\prod_{j\le n: y_j=2} (0,1)\mathbf{x}_j\right) \\ &= \prod_{j=1}^{n} \max(x_{j1}, x_{j2}). \end{aligned}$$

We shall investigate the performance of the beta-weighted (α_1, α_2) universal portfolio (2) given the side-information y^n. In particular, we shall derive upper bounds for the ratio of the optimal wealth to the universal wealth, $S_n^*(\mathbf{x}^n \mid y^n)/\hat{S}_n(\mathbf{x}^n \mid y^n)$.

We proceed with some definitions. The *beta-weighted* (α_1, α_2) *universal portfolio with side-information* is the sequence of portfolios $\{\hat{\mathbf{b}}_n\}$ on the nth trading day given by:

$$(7) \qquad \hat{\mathbf{b}}_n(\mathbf{x}^{n-1} \mid y) = \frac{\int_B \mathbf{b}\left(\prod_{j\le (n-1): y_j=y} \mathbf{b}^t\mathbf{x}_j\right) d\mu(\mathbf{b})}{\int_B \left(\prod_{j\le (n-1): y_j=y} \mathbf{b}^t\mathbf{x}_j\right) d\mu(\mathbf{b})}$$

for $n = 2, 3, \ldots$, $y \in \{1, 2\}$ is the current state where $d\mu(\mathbf{b})$ is defined by (3), $\alpha_1 > 0$, $\alpha_2 > 0$ and $\hat{\mathbf{b}}_1 = \int_B \mathbf{b}\, d\mu(\mathbf{b})$. The *universal wealth with side information* $\hat{S}_n(\mathbf{x}^n \mid y^n)$ achieved by the portfolio $\{\hat{\mathbf{b}}_n\}$ is given by:

$$(8) \qquad \hat{S}_n(\mathbf{x}^n \mid y^n) = \prod_{i=1}^{n} \hat{\mathbf{b}}_i^t(\mathbf{x}^{i-1} \mid y_i)\mathbf{x}_i,$$

where $y^n = (y_1, y_2, \ldots, y_n)$ is the current side-information vector on the nth trading day. We can express (8) as:

$$(9)\quad \hat{S}_n(\mathbf{x}^n \mid y^n) = \left(\prod_{i\le n: y_i=1} \hat{\mathbf{b}}_i^t(\mathbf{x}^{i-1} \mid 1)\mathbf{x}_i\right)\left(\prod_{i\le n: y_i=2} \hat{\mathbf{b}}_i^t(\mathbf{x}^{i-1} \mid 2)\mathbf{x}_i\right).$$

In Cover and Ordentlich (1996), a simplified formula for calculating $\hat{S}_n(\mathbf{x}^n \mid y^n)$ is given as follows:

$$(10)\qquad \hat{S}_n(\mathbf{x}^n \mid y^n) = \prod_{y=1}^{2} \int_B \left(\prod_{i\le n: y_i=y} \mathbf{b}^t\mathbf{x}_i\right) d\mu(\mathbf{b}),$$

where as before, all initial wealths are assumed to be one unit.

2. Bounds for ratio of wealths. First, we have the following lemma from Tan (2002)

Lemma 1. *Let r be a nonnegative integer, $p > 0$ and $q > 0$. Then*

$$\frac{\Gamma(r+p)}{\Gamma(r+q)} \le (r+q)^{p-q} \text{ if } q \le p \le 1+q.$$

The proof of the above lemma follows from the fact that $\log\Gamma(x)$ is convex.

Lemma 2. *Let r be a nonnegative integer, $p > 0$, $q > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$. Then*

$$\text{(i)}\quad \frac{\Gamma(r+p)}{\Gamma(r+q)} \le (r+p-1)(r+p-2)\cdots(r+p-k)(r+q)^{p-k-q}$$

$$\textit{if}\ \ k+q \le p \le k+1+q\ \ \textit{for some integer}\ \ k \ge 1,$$

$$\text{(ii)}\quad \frac{\Gamma(r+\alpha_1+\alpha_2)}{\Gamma(r+\alpha_i)} = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_i)} \prod_{N=0}^{r-1} \left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right) \quad \textit{for } i = 1, 2,$$

$$\text{(iii)}\quad \frac{\Gamma(r+\alpha_1+\alpha_2)}{\Gamma(r+\alpha_i)} \le \begin{cases} (r+\alpha_i)^{\alpha_1+\alpha_2-\alpha_i} & \textit{if } \alpha_i < \alpha_1+\alpha_2 \le 1+\alpha_i \\ (r+\alpha_1+\alpha_2-1)(r+\alpha_1+\alpha_2-2)\cdots & \\ \cdots(r+\alpha_1+\alpha_2-k)(r+\alpha_i)^{\alpha_1+\alpha_2-k-\alpha_i} & \\ \qquad \textit{if } k+\alpha_i \le \alpha_1+\alpha_2 \le k+1+\alpha_i \textit{ for some} & \\ \qquad \textit{integer } k \ge 1, \textit{ for } i=1,2. & \end{cases}$$

Proof. (i) If $k+q \le p \le k+1+q$ for some integer $k \ge 1$, then

$$\begin{aligned} \Gamma(r+p) &= (r+p-1)\Gamma(r+p-1) \\ &= (r+p-1)(r+p-2)\cdots(r+p-k)\Gamma(r+p-k). \end{aligned}$$

Since $q \le p-k \le 1+q$, we can apply Lemma 1 to obtain

$$\frac{\Gamma(r+p-k)}{\Gamma(r+q)} \le (r+q)^{p-k-q} \qquad \text{and}$$

$$\frac{\Gamma(r+p)}{\Gamma(r+q)} \le (r+p-1)(r+p-2)\cdots(r+p-k)(r+q)^{p-k-q}.$$

(ii) For $i=1,2$,

$$\begin{aligned} \frac{\Gamma(r+\alpha_1+\alpha_2)}{\Gamma(r+\alpha_i)} &= \frac{(r-1+\alpha_1+\alpha_2)(r-2+\alpha_1+\alpha_2)\cdots(\alpha_1+\alpha_2)\Gamma(\alpha_1+\alpha_2)}{(r-1+\alpha_i)(r-2+\alpha_i)\cdots\alpha_i\Gamma(\alpha_i)} \\ &= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_i)}\prod_{N=0}^{r-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right). \end{aligned}$$

(iii) This result follows by applying Lemma 1 and Lemma 2(i).

Theorem 1. *Consider a two-stock market described by the M price-relative vectors $\mathbf{a}_i = (a_{i1}, a_{i2})$ for $i = 1,2,\ldots,M$, where $a_{i1} \ge a_{i2}$ for $i = 1,2,\ldots,M_1$ and $a_{i1} < a_{i2}$ for $i = M_1+1, M_1+2,\ldots,M_1+M_2$; $M_1+M_2 = M$ and $\max(a_{j1}, a_{j2}) > 1$ for some $j = 1,2,\ldots,M$ (see note at end of proof). For a sequence of price-relative vectors $\mathbf{x}^n$, suppose $\{\mathbf{a}_i\}_{i=1}^{M_1}$ appear a total of ℓ_1 times in $\mathbf{x}^n$ and $\{\mathbf{a}_i\}_{i=M_1+1}^{M_1+M_2}$ appear a total of ℓ_2 times in $\mathbf{x}^n$, where $\ell_1+\ell_2 = n$. Then for the beta-weighted (α_1,α_2) universal portfolio with side* *information,*

(i)

$$\frac{S_n^*(\mathbf{x}^n \mid y^n)}{\hat{S}_n(\mathbf{x}^n \mid y^n)} \le \prod_{N=0}^{\ell_1-1}\left(\frac{\alpha_1+\alpha_2+N)}{\alpha_1+N}\right)\prod_{K=0}^{\ell_2-1}\left(\frac{\alpha_1+\alpha_2+K)}{\alpha_2+K}\right), \tag{11}$$

(ii)

$$\frac{S_n^*(\mathbf{x}^n \mid y^n)}{\hat{S}_n(\mathbf{x}^n \mid y^n)} \le \begin{cases} \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha)^{\alpha_1+\alpha_2} & \text{if } 0<\alpha_1\le 1 \text{ or } 0<\alpha_2\le 1 \\ \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha_1+\alpha_2-1)^{\alpha_1+\alpha_2} & \text{if } 1<\alpha_1 \text{ and } 1<\alpha_2, \end{cases}$$

for all price-relative sequences $\mathbf{x}^n$ *achieving* $S_n^*(\mathbf{x}^n \mid y^n) = \prod_{i=1}^n \max(x_{i1}, x_{i2})$, $n = 1, 2, \ldots,$ $y^n \in \{1,2\}^n$ *where* $\alpha = \max(\alpha_1, \alpha_2)$.

Proof. (i) We assume that the price-relative vector $\mathbf{a}_i$ appears a total of n_i times in $\mathbf{x}^n$ for $i = 1, 2, \ldots, M$, where $n_1 + n_2 + \cdots + n_M = n$. From (10),

$$\begin{aligned}
&\hat{S}_n(\mathbf{x}^n \mid y^n) \\
&= \left\{\int_B \left(\prod_{i\le n: y_i=1} \mathbf{b}^t\mathbf{x}_i\right) d\mu(\mathbf{b})\right\}\left\{\int_B \left(\prod_{i\le n: y_i=2} \mathbf{b}^t\mathbf{x}_i\right) d\mu(\mathbf{b})\right\} \\
&= \left\{\int_B \prod_{i=1}^{M_1}[b_1a_{i1}+(1-b_1)a_{i2}]^{n_i} d\mu(\mathbf{b})\right\}\left\{\int_B \prod_{i=M_1+1}^{M_1+M_2}[b_1a_{i1}+(1-b_1)a_{i2}]^{n_i} d\mu(\mathbf{b})\right\} \\
&\ge \left\{\left(\prod_{i=1}^{M_1} a_{i1}^{n_i}\right)\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 b_1^{n_1+\cdots+n_{M_1}+\alpha_1-1}(1-b_1)^{\alpha_2-1}db_1\right\} \\
&\quad \times\left\{\left(\prod_{i=M_1+1}^{M_1+M_2} a_{i2}^{n_i}\right)\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 b_1^{\alpha_1-1}(1-b_1)^{n_{M_1+1}+\cdots+n_{M_1+M_2}+\alpha_2-1}db_1\right\} \\
&\ge S_n^*(\mathbf{x}^n \mid y^n)\left\{\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(n_1+\cdots+n_{M_1}+\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(n_1+\cdots+n_{M_1}+\alpha_1+\alpha_2)}\right\} \\
&\quad \times\left\{\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_1)\Gamma(n_{M_1+1}+\cdots+n_{M_1+M_2}+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(n_{M_1+1}+\cdots+n_{M_1+M_2}+\alpha_1+\alpha_2)}\right\}.
\end{aligned}$$

Therefore,

$$\frac{S_n^*(\mathbf{x}^n \mid y^n)}{\hat{S}_n(\mathbf{x}^n \mid y^n)} \leq \left\{\frac{\Gamma(\alpha_1)\Gamma(n_1+\cdots+n_{M_1}+\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)\Gamma(n_1+\cdots+n_{M_1}+\alpha_1)}\right\} \times \left\{\frac{\Gamma(\alpha_2)\Gamma(n_{M_1+1}+\cdots+n_{M_1+M_2}+\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)\Gamma(n_{M_1+1}+\cdots+n_{M_1+M_2}+\alpha_2)}\right\}. \tag{12}$$

Applying Lemma 2(ii) to (12), we obtain the desired result (11), where $n_1+\cdots+n_{M_1}=\ell_1$ and $n_{M_1+1}+\cdots+n_{M_1+M_2}=\ell_2$.

(ii) Applying Lemma 2(iii) to (12) and noting that $n_1+\cdots+n_{M_1}\leq n$ and $n_{M_1+1}+\cdots+n_{M_1+M_2}\leq n$, we have

$$\frac{S_n^*(\mathbf{x}^n \mid y^n)}{\hat{S}_n(\mathbf{x}^n \mid y^n)} \leq \begin{cases} \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha_1)^{\alpha_2}(n+\alpha_2)^{\alpha_1} & \text{if } 0<\alpha_1\leq 1,\ 0<\alpha_2\leq 1 \\ \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha_1+\alpha_2-1)^{\alpha_2}(n+\alpha_2)^{\alpha_1} & \text{if } 0<\alpha_1\leq 1,\ 1<\alpha_2 \\ \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha_1)^{\alpha_2}(n+\alpha_1+\alpha_2-1)^{\alpha_1} & \text{if } 1<\alpha_1,\ 0<\alpha_2\leq 1 \\ \dfrac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)^2}(n+\alpha_1+\alpha_2-1)^{\alpha_1+\alpha_2} & \text{if } 1<\alpha_1,\ 1<\alpha_2. \end{cases} \tag{13}$$

Since $(n+\alpha_1+\alpha_2-1)\leq(n+\alpha_2)$ if $\alpha_1\leq 1$ and $(n+\alpha_1+\alpha_2-1)\leq(n+\alpha_1)$ if $\alpha_2\leq 1$, we have the desired result from (13).

Note. The condition $\max(a_{j1},a_{j2})>1$ for some $j=1,2,\ldots,M$ is not required in the proof. The situation $\max(a_{i1},a_{i2})<1$ for all $i=1,2,\ldots,M$ indicates that both stock prices are falling on all trading days. In a real investment situation, an investor will not invest if there is no possibility of profit. This investment situation where profit is possible is represented by

$\max(a_{j1}, a_{j2}) > 1$ for some $j = 1, 2, \dots, M$.

We have the following remarks regarding the upper bound in (11).

Remark 1. Consider a two-stock market of M price-relative vectors $\{\mathbf{a}_i\}_{i=1}^{M}$, where $a_{i1} \geq a_{i2}$ for $i = 1, 2, \dots, M_1$ and $a_{i1} < a_{i2}$ for $i = M_1 + 1, M_1 + 2, \dots, M_1 + M_2$; $M_1 + M_2 = M$, $\max(a_{j1}, a_{j2}) > 1$ for some $j = 1, 2, \dots, M$. For a sequence of price-relative vectors $\mathbf{x}^n$, suppose $\{\mathbf{a}_i\}_{i=1}^{M_1}$ appear a total of ℓ_1 times in $\mathbf{x}^n$ and $\{\mathbf{a}_i\}_{i=M_1+1}^{M_1+M_2}$ appear a total of ℓ_2 times in $\mathbf{x}^n$, where $\ell_1 + \ell_2 = n$. Consider the upper bound (11), namely $\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$ for the ratio of wealths $S_n^*(\mathbf{x}^n \mid y^n)/\hat{S}_n(\mathbf{x}^n \mid y^n)$, where $S_n^*(\mathbf{x}^n \mid y^n) = \prod_{i=1}^{n}\max(x_{i1}, x_{i2})$ is the maximum achievable wealth and $\hat{S}_n(\mathbf{x}^n \mid y^n)$ is the wealth achieved by the beta-weighted (α_1, α_2) universal portfolio with side information $y^n \in \{1, 2\}^n$. Then

$$\begin{aligned}
&\prod_{i=1}^{2}\left\{\frac{(\alpha_1+\alpha_2)(\alpha_i+1)^{\alpha_i+1}}{\alpha_i(\alpha_1+\alpha_2+1)^{\alpha_1+\alpha_2+1}}\right\}\left\{(\alpha_1+\alpha_2+\ell_i)^{\alpha_{i^*}}\left[1+\frac{\alpha_{i^*}}{\alpha_i+\ell_i}\right]^{\alpha_i+\ell_i}\right\}\\
(14)\ \leq\ &\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)\\
\leq\ &\prod_{i=1}^{2}\left\{\frac{\alpha_i^{\alpha_i-1}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2-1}}\right\}\left\{(\alpha_1+\alpha_2+\ell_i-1)^{\alpha_{i^*}}\left[1+\frac{\alpha_{i^*}}{\alpha_i+\ell_i-1}\right]^{\alpha_i+\ell_i-1}\right\},
\end{aligned}$$

where we define i^*, the complement of i, by $i^* = 1$ if $i = 2$ and $i^* = 2$ if $i = 1$.

Remark 2. If the market is stationary and ergodic where $\{\mathbf{a}_i\}_{i=1}^{M_1}$ and $\{\mathbf{a}_i\}_{i=M_1+1}^{M_1+M_2}$ occur with probabilities p and q respectively with $p+q = 1$, then defining

$$(15)\qquad g(\alpha_1, \alpha_2, \ell_1, \ell_2, n) = n^{-(\alpha_1+\alpha_2)}\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$$

where $\ell_1+\ell_2=n$, we have

$$
\begin{aligned}
& (e^{\alpha_1+\alpha_2}p^{\alpha_2}q^{\alpha_1})\left[\frac{(\alpha_1+\alpha_2)^2(\alpha_1+1)^{\alpha_1+1}(\alpha_2+1)^{\alpha_2+1}}{(\alpha_1+\alpha_2+1)^{2(\alpha_1+\alpha_2+1)}\alpha_1\alpha_2}\right] \\
\le\ & \liminf_{n\to\infty} g(\alpha_1,\alpha_2,\ell_1,\ell_2,n) \\
\le\ & \limsup_{n\to\infty} g(\alpha_1,\alpha_2,\ell_1,\ell_2,n) \\
\le\ & (e^{\alpha_1+\alpha_2}p^{\alpha_2}q^{\alpha_1})\left[\frac{\alpha_1^{\alpha_1-1}\alpha_2^{\alpha_2-1}}{(\alpha_1+\alpha_2)^{2(\alpha_1+\alpha_2-1)}}\right] \quad \text{almost surely.}
\end{aligned} \tag{16}
$$

Proof. (i) First, we consider bounding the function $\prod_{N=1}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$ by integrating the function $h(t)=\ln\left(\frac{\alpha_1+\alpha_2+t}{\alpha_i+t}\right)$ for $i=1,2$. Since the derivative of $h(t)$ is negative for $t\ge 0$, $\alpha_1>0$, $\alpha_2>0$, the function $h(t)$ is a decreasing function of t for $t\ge 0$. We have

$$
\int_1^{\ell_i}\ln\left(\frac{\alpha_1+\alpha_2+t}{\alpha_i+t}\right)dt \le \sum_{N=1}^{\ell_i-1}\ln\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right) \le \int_0^{\ell_i-1}\ln\left(\frac{\alpha_1+\alpha_2+t}{\alpha_i+t}\right)dt \tag{17}
$$

for $i=1,2$. From the fact that

$$
\int \ln(\alpha+t)dt = (\alpha+t)\ln(\alpha+t)-t+c, \tag{18}
$$

we obtain

$$
\begin{aligned}
& \int_0^{\ell_i-1}\ln\left(\frac{\alpha_1+\alpha_2+t}{\alpha_i+t}\right)dt \\
=\ & \left[\ln\frac{(\alpha_1+\alpha_2+t)^{\alpha_1+\alpha_2+t}}{(\alpha_i+t)^{\alpha_i+t}}\right]_0^{\ell_i-1} \\
=\ & \ln\left\{\left[\frac{\alpha_i^{\alpha_i}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}\right]\left[\frac{(\alpha_1+\alpha_2+\ell_i-1)^{\alpha_1+\alpha_2+\ell_i-1}}{(\alpha_i+\ell_i-1)^{\alpha_i+\ell_i-1}}\right]\right\} \\
=\ & \ln\left\{\left[\frac{\alpha_i^{\alpha_i}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}\right](\alpha_1+\alpha_2+\ell_i-1)^{\alpha_{i^*}}\left[1+\frac{\alpha_{i^*}}{\alpha_i+\ell_i-1}\right]^{\alpha_i+\ell_i-1}\right\}.
\end{aligned} \tag{19}
$$

Similarly,

$$
\begin{aligned}
(20)\qquad & \int_1^{\ell_i} \ln\left(\frac{\alpha_1+\alpha_2+t}{\alpha_i+t}\right) dt \\
= & \left[\ln \frac{(\alpha_1+\alpha_2+t)^{\alpha_1+\alpha_2+t}}{(\alpha_i+t)^{\alpha_i+t}}\right]_1^{\ell_i} \\
= & \ln\left\{\frac{(\alpha_i+1)^{\alpha_i+1}}{(\alpha_1+\alpha_2+1)^{\alpha_1+\alpha_2+1}} \frac{(\alpha_1+\alpha_2+\ell_i)^{\alpha_1+\alpha_2+\ell_i}}{(\alpha_i+\ell_i)^{\alpha_i+\ell_i}}\right\} \\
= & \ln\left\{\left[\frac{(\alpha_i+1)^{\alpha_i+1}}{(\alpha_1+\alpha_2+1)^{\alpha_1+\alpha_2+1}}\right](\alpha_1+\alpha_2+\ell_i)^{\alpha_{i^*}}\left[1+\frac{\alpha_{i^*}}{\alpha_i+\ell_i}\right]^{\alpha_i+\ell_i}\right\}.
\end{aligned}
$$

Noting that

$$
(21)\qquad \prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right) = \prod_{i=1}^{2}\left(\frac{\alpha_1+\alpha_2}{\alpha_i}\right) e^{\sum_{N=1}^{\ell_i-1} \ln\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)},
$$

we obtain the desired result (14) from (17), (19) and (20).

(ii) Now, we assume that the market is stationary and ergodic. For large n, $\ell_1 \approx np$ and $\ell_2 \approx nq$. Noting that $\lim_{n\to\infty}\left(1+\frac{\lambda}{n}\right)^n = e^{\lambda}$, the asymptotic behaviour of the lower bound of $\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$ in (14) is

$$
\begin{aligned}
(22)\qquad & \left[\prod_{i=1}^{2}\frac{(\alpha_1+\alpha_2)(\alpha_i+1)^{\alpha_i+1}}{\alpha_i(\alpha_1+\alpha_2+1)^{\alpha_1+\alpha_2+1}}\right](np)^{\alpha_2}(nq)^{\alpha_1}e^{\alpha_1+\alpha_2} \\
= & (e^{\alpha_1+\alpha_2}p^{\alpha_2}q^{\alpha_1})\left[\frac{(\alpha_1+\alpha_2)^2(\alpha_1+1)^{\alpha_1+1}(\alpha_2+1)^{\alpha_2+1}}{(\alpha_1+\alpha_2+1)^{2(\alpha_1+\alpha_2+1)}\alpha_1\alpha_2}\right]n^{\alpha_1+\alpha_2}.
\end{aligned}
$$

Similarly, the asymptotic behaviour of the upper bound in (14) is

$$
\begin{aligned}
(23)\qquad & \left[\prod_{i=1}^{2}\frac{\alpha_i^{\alpha_i-1}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2-1}}\right](np)^{\alpha_2}(nq)^{\alpha_1}e^{\alpha_1+\alpha_2} \\
= & (e^{\alpha_1+\alpha_2}p^{\alpha_2}q^{\alpha_1})\left[\frac{\alpha_1^{\alpha_1-1}\alpha_2^{\alpha_2-1}}{(\alpha_1+\alpha_2)^{2(\alpha_1+\alpha_2-1)}}\right]n^{\alpha_1+\alpha_2}.
\end{aligned}
$$

Hence, the desired result (16) follows from (22) and (23).

Further Comments. (i) The bounds for the ratio of wealths given in Theorem 1(ii) are derived as upper bounds of $\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$ using Lemma 2 (iii). These are good bounds since they have the same kind of asymptotic behaviour as $\prod_{i=1}^{2}\prod_{N=0}^{\ell_i-1}\left(\frac{\alpha_1+\alpha_2+N}{\alpha_i+N}\right)$.
(ii) The generalizations of the results in this paper to the m-stock market with side information are straightforward for $m>2$. We shall not state these results here.

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