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ULM-KAPLANSKY INVARIANTS FOR $S(RG)/G_p$

BY

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Abstract. Suppose G is an arbitrary abelian group with p-component G_p and R is a perfect commutative ring with identity in prime characteristic p without nilpotent elements. Then the Ulm-Kaplansky invariants of the factor-group $S(RG)/G_p$ are computed. This gives an useful strategy for the full description of the structure of the group $S(RG)$, thus exhausting this major classical problem for the theory of commutative group algebras.

1. Notation and known results. Throughout the text, let RG be the group ring (usually regarded as an R-algebra) of an abelian group G over a commutative unitary (i.e. with unity) ring R of prime characteristic p. As usual, $V(RG)$ designates the group of all normed invertible elements in RG, and $S(RG)$ denotes its Sylow p-subgroup. For the most part, G_p will denote the p-torsion component of the maximal torsion subgroup G_t in G. All other notation and terminology to the abelian group theory and commutative group algebras theory are standard and will follow those from the excellent classical books of L. Fuchs [13] and G. Karpilovsky [16].

In 1967, S. D. Berman has first computed in [1] the Ulm-Kaplansky functions of $S(RG)$ when the group ring RG is finite plus the additional conditions that R is a field and G is a p-primary group. After this, N. A. Nachev and T.Zh.Mollov have calculated in [20] these functions when G is a p-group, and Mollov extends this result in [17] for an arbitrary group G but

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only when R is a field. More recently, Nachev in [19] completely computed the Ulm-Kaplansky invariants for $S(RG)$ in the case of absolute arbitrary objects R and G, thus he exhausted this problem for the modular case.

Nevertheless, these calculations have no global impotrance for the complete descriptions of the structure of the group $S(RG)$. This is so because of the following arguments: In the abstract abelian group theory it is a well-known and documented fact that the Ulm-Kaplansky(-Mackey) functions determine up to an isomorphism only the classes of torsion-complete abelian p-groups and totally projective (simply presented) abelian p-groups plus some variants and specifications of the last group class. On an other hand, it is also well-documented by W.L.May that $V(RG)$ is a totally projective p-group if and only if G is a totally projective p-group provided R is a perfect field (see for instance [16] or [5]). Moreover, invoking to ([18] and [3, 8, 9]), $S(RG)$ is torsion-complete if and only if G_p is bounded provided R is with no nilpotents; or $S(RG)$ is torsion-complete if and only if G is a bounded p-group provided R is perfect which possesses nilpotents. Thus we obviously observe that the torsion completeness is not an invariant property for $S(RG)$, i.e. in other words it is not preserved in the large group $S(RG)$ in one way, and it not guaranteed the existence of group elements in G with infinite orders in another way. Thus, in conjuction with our papers [2, 4, 6, 7, 10] where are studied group algebras of abelian groups which can not be characterized by their Ulm-Kaplansky invariants, we establish the above mentioned negative claim about the connection between the Ulm-Kaplansky functions and the structure of $S(RG)$. Well, the results due to Nachev-Mollov on the computation of the Ulm-Kaplansky invariants for $S(RG)$ determine the structure of $S(RG)$ almost only when G is totally projective p-primary and R is a perfect field, that is why they are not enough.

In the spirit of the long-standing and very difficult hypothesis going from May that $S(RG)/G_p$ is totally projective whenever R is perfect (often called the *Generalized Direct Factor Problem* – **GDFP**) whence $S(RG) \cong$

$G_p \times S(RG)/G_p$ (see for example [16]), it is natural to calculate the Ulm-Kaplansky invariants of $S(RG)/G_p$. If we make this calculation successfully, in view of the above conjecture, the isomorphic class of $S(RG)$ will be completely described, thus solving this old problem in general after all. Although our method (that is as all different to these in [17, 19, 20] and [16]) may be applied without serious problems and for the general situations, we restrict our attention on the case of perfect ring with trivial nilradical, because of the applications to the group structure and some technical difficulties.

2. Ulm-Kaplansky functions of $S(RG)/G_p$. Before stating and proving in all details the main result that motivates the present article, we need a series of some preliminary assertions. We start with key technical matters, namely

Lemma 0([2]). *For each ordinal* α

$$S^{p^\alpha}(RG) = S(R^{p^\alpha} G^{p^\alpha}).$$

Lemma 1. *The subgroup* G_p *is balanced in* $S(RG)$*, that is, for every ordinal* α

(1) $[S(RG)/G_p]^{p^\alpha} = S^{p^\alpha}(RG)G_p/G_p = S(R^{p^\alpha} G^{p^\alpha})G_p/G_p$;

(2) $G_p \cap S^{p^\alpha}(RG) = (G_p)^{p^\alpha}$.

Proof. Foremost, we shall prove the first equality. Owind to a lemma due to P. D. Hill (see for instance [13], p.91, Lemma 79.2), (1) is equivalent to $\bigcap_{\tau<\alpha}[S^{p^\tau}(RG)G_p] = S^{p^\alpha}(RG)G_p$ for every limit ordinal α. And so, given arbitrary element x that lies in the left hand-side. Hence from the first lemma $x = g_p \sum_i r_i g_i = h_p \sum_i \alpha_i h_i = \cdots$, where g_p, $h_p \in G_p$; $\sum_i r_i = 1$, $r_i \in R^{p^\tau}$, $g_i \in G^{p^\tau}$; $\sum_i \alpha_i = 1$, $\alpha_i \in R^{p^\delta}$, $h_i \in G^{p^\delta}$; $\tau < \delta \le \alpha$; $i \in \mathbb{N}$. The canonical forms ensure $r_i = \alpha_i$ and $g_p g_i = h_p h_i$ for all i whence $g_i g_\ell^{-1} = h_i h_\ell^{-1}$ for any $i \neq \ell$. Furthermore, since $\sum_i r_i g_i \in S(RG)$, there is $g_j \in G_p$ for some index j. Finally, write $x = g_p g_j \sum_i r_i g_i g_j^{-1} \in G_p S(R^{p^\delta} G^{p^\delta})$, and since δ is arbitrary such that we have a finite support whereas the equalities are infinite the last means that $x \in G_p S(R^{p^\alpha} G^{p^\alpha})$, as required. The proof of this point is completed.

(2) According to Lemma 0, $G_p \cap S^{p^\alpha}(RG) = G_p \cap S(R^{p^\alpha} G^{p^\alpha}) = (G^{p^\alpha})_p = (G_p)^{p^\alpha}$, as well. The proof is over.

We continue with

Lemma 2. *Let C be a p-pure subgroup of the abelian group A. Then*

$$(A/C)[p] = A[p]C/C.$$

Proof. It is evident that the left hand-side contains the right hand-side. For the converse, choose x to belongs to $(A/C)[p]$. Hence $x = aC$ for some $a \in A$ and $a^p \in C$. Consequently $a^p \in C \cap A^p = C^p$ whence $a^p = c^p$ for some $c \in C$. From this it follows that $(ac^{-1})^p = 1$, i.e. $a \in cA[p]$. Finally, $x = a_p C$ for some $a_p \in A[p]$ or equivalently $x \in CA[p]/C$. The proof is argued.

The next technical attainments are crucial for our further investigation.

Lemma 3. *Let $C \leq G$. Then $[S(RC)G_p][p] = S(RC)[p]G[p]$.*

Proof. Certainly, the left hand-side contains the right hand-side. For the reverse, take x to belongs to $S(RC)G_p$ such that $x^p = 1$. Write down $x = g_p \sum_i r_i c_i$, where $g_p \in G_p$, $0 \neq r_i \in R$, $c_i \in C$ and $i \in \mathbb{N}$. The relation $x^p = 1$ implies $\sum_i r_i^p g_p^p c_i^p = 1$. With no harm of generality, we can presume that $g_p^p c_1^p = 1$, i.e. $g_p c_1 \in G[p]$. Thus $x = g_p c_1 \sum_i r_i c_i c_1^{-1} \in G[p]S(RC)[p]$, because it is clear that $\sum_i r_i c_i c_1^{-1} \in S(RC)[p]$. This establishes the proof.

The following group isomorphisms are well-known, but for the sake of completeness and for the convenience of the reader, we will formulate their.

Lemma 4. *Assume $D \le B \le A$ and C are abelian groups. Then*

$$AC/BC \cong A/B(A \cap C) \quad \text{and} \quad A/D/B/D \cong A/B.$$

Proof. The reader can see it in [15].

We conclude the preliminaries with one valuable

Proposition 5. *For $1 \neq G$ an abelian group and R without nilradical is fulfilled*

$$S(RG)[p] = S(R^pG^p)[p]G[p] \iff \begin{array}{l} (1)\ G \neq G[p] \neq 1,\ G = G^p \text{ and } R = R^p; \\ (2)\ |G| = |G[p]| = 2,\ |R| = 2; \\ (3)\ G[p] = 1. \end{array}$$

Proof. We will attack the necessity. For this purpose, we foremost assume that $G \neq G[p] \neq 1$. Choose then arbitrary $0 \neq r \in R$, $1 \neq g \in G \setminus G[p]$ and $1 \neq g_p \in G[p]$. Thus, since $g_p \neq g^{-1}$, the element $1 + rg - rgg_p$ is in canonical form and lies in $S(RG)[p]$ hence in $S(R^pG^p)[p]G[p]$. Therefore $1 + rg - rgg_p = (r_1g_1 + \cdots + r_sg_s)a_p$ whence $r = r_j \in R^p$, $g = g_ja_p$ and $gg_p = g_ka_p$. That is why, $g_p = g_kg_j^{-1} \in G^p$. Because of these equalities, $R = R^p$ and $G[p] = G^p[p]$. Finally, $g = g_ja_p$ with $g_j \in G^p$ and $a_p \in G[p]$ do imply $g \in G^p$, i.e. $G = G^p$, as claimed. Let us now $G = G[p] \neq 1$. So, $G^p = 1$ and $S(RG)[p] = G[p]$. First consider the element $1 + g(1 - g_p)$ where $g \in G$ is arbitrary and $1 \neq g_p \in G[p]$. It is easy to observe that either $g = g_p^{-1} \in G[p]$ or in the remaining case the stated element in the group ring is in canonical form and so it again follows that $g \in G[p]$. As a final, $G = G[p]$. Further, select now the element $1 + rg - rh$, where $g \neq h$ lie in G, $g^p = h^p (\neq 1)$ and $r \in R$. Evidently the present group ring's element belongs to $S(RG)[p] \setminus G[p]$, which is false. Then $G = \langle 1, g\rangle$ and $p = 2$. By the same token, $1+r(1-g_p) \in S(RG)[p] \setminus G[p]$ whenever $r \neq 0, 1$. Finally $R = \{0, 1\}$, as well.

Next, we treat the sufficiency. In this direction, (1) or (3) elementarily imply the relation from the left hand-side. Moreover, the dependence (2) obviously implies $S(RG)[p] = G[p]$, i.e. the desired equality. The proof is verified.

Next, we come to the central statement in the present paper formulated below.

Theorem 6. *Suppose* $1 \neq G$ *is an abelian group and* R *is an unitary perfect commutative ring without nilpotent elements in prime characteristic* p. *Then*

(i) *if* $|R| < \aleph_0$ *and* $|G^{p^\sigma}| < \aleph_0$ *for any ordinal* σ,

$$f_\sigma(S(RG)/G_p) = \begin{cases} (|G^{p^\sigma}| - 2|G^{p^{\sigma+1}}| + |G^{p^{\sigma+2}}|)\log_p |R| - f_\sigma(G_p) \\ \quad \text{when } G_p^{p^\sigma} \neq 1 \text{ and } |G^{p^\sigma}| \neq |G^{p^\sigma}[p]| \neq 2 \text{ or } |R| \neq 2; \\ \\ 0 \text{ when } G_p^{p^\sigma} = 1 \text{ or } |G^{p^\sigma}| = |G^{p^\sigma}[p]| = 2 \text{ and } |R| = 2. \end{cases}$$

(ii) *if* $|R| \geq \aleph_0$ *or* $|G^{p^\sigma}| \geq \aleph_0$ *for some ordinal* σ,

$$f_\sigma(S(RG)/G_p) = \begin{cases} \max(|R|, |G^{p^\sigma}|) & \text{when } G_p^{p^\sigma} \neq 1 \text{ and } G^{p^\sigma} \neq G^{p^{\sigma+1}}; \\ 0 & \text{when } G_p^{p^\sigma} = 1 \text{ or } G^{p^\sigma} = G^{p^{\sigma+1}}. \end{cases}$$

Proof. By definition (see [13]),

$$f_\sigma(S(RG)/G_p) = \text{rank}\Big((S(RG)/G_p)^{p^\sigma}[p]/(S(RG)/G_p)^{p^{\sigma+1}}[p]\Big).$$

By making use of our preliminary assertions from Lemma 1 to Lemma 4, we detect step by step that

$$\begin{aligned}
& (S(RG)/G_p)^{p^\sigma}[p]/(S(RG)/G_p)^{p^\sigma}[p] \\
= \;& [S(RG^{p^\sigma})G_p/G_p][p]/[S(RG^{p^{\sigma+1}})G_p/G_p][p] \\
= \;& [S(RG^{p^\sigma})G_p][p]G_p/G_p/[S(RG^{p^{\sigma+1}})G_p][p]G_p/G_p \\
\cong \;& S(RG^{p^\sigma})[p]G_p/S(RG^{p^{\sigma+1}})[p]G_p \\
\cong \;& S(RG^{p^\sigma})[p]/S(RG^{p^{\sigma+1}})[p]G^{p^\sigma}[p].
\end{aligned}$$

Well,

$$\begin{aligned}
& f_\sigma(S(RG)/G_p) \\
= \;& \dim_{\mathbb{F}_p}\Big[S(RG^{p^\sigma})[p]/S(RG^{p^{\sigma+1}})[p]G^{p^\sigma}[p]\Big] \\
= \;& \begin{cases} |S(RG^{p^\sigma})[p]/S(RG^{p^{\sigma+1}})[p]G^{p^\sigma}[p]| & \text{if the dimension is infinite} \\ \log_p|S(RG^{p^\sigma})[p]/S(RG^{p^{\sigma+1}})[p]G^{p^\sigma}[p]| & \text{if the dimension is finite.} \end{cases}
\end{aligned}$$

Further, because of the substitution $G \to G^{p^\sigma}$ together with Lemma 0, we need only calculate $f_0(S(RG)/G_p)$. And so, we will differ in the sequel the two basic cases.

(i) Since G_p is finite, utilizing our result in [2], we exstract $S(RG)/G_p$ is a direct sum of cyclics and thus $S(RG) \cong G_p \times S(RG)/G_p$. Therefore by virtue of ([13], p.185, Exercise 8), we yield $f_0(S(RG)) = f_0(G_p) + f_0(S(RG)/G_p)$, hence $f_0(S(RG)/G_p) = f_0(S(RG)) - f_0(G_p)$. Consequently applying Proposition 5 together with [19], we argue the first case.

(ii) The application of [2] plus the last Proposition 5 ensure that the factor-group $S(RG)/G_p$ has a zero Ulm-Kaplansky functions in the cases when $G_p = 1$ or G is p-divisible, i.e. in other words $G = G^p$. Let us now we presume that $G_p \neq 1$ and $G \neq G^p$.

Further, we distinguish the next major cases:

Case 1. $G_p = G_p^p$, i.e. G_p is divisible.

Write down $G = G_d \times G_r$, the normal decomposition of G into a direct product of a divisible part and a reduced part. Therefore, $G_p = (G_d)_p \times (G_r)_p$ whence $(G_r)_p = 1$. Complying with our proposition argued in [5], $S(RG) = S(RG_d) \times [1 + I_p(RG; G_r)]$ hence $S(RG)/G_p \cong S(RG_d)/(G_d)_p \times [1 + I_p(RG; G_r)]$. Because it is a simple matter to establish that $S(RG_d)$ and $S(RG_d)/(G_d)_p$ are both divisible, $f_\sigma(S(RG)/G_p) = f_\sigma(1 + I_p(RG; G_r)) = f_\sigma(S(RG))$ and so [19] guarantees our assertion.

Case 2. $G_p \neq G_p^p$, i.e. G_p is not divisible.

Consequently, we may choose $g_p \in G_p \setminus G_p^p$ of order p^t for $t \geq 1$. After this, we consider the following elements of $S(RG)[p]$:

$$x_{rg} = 1 + rg(1 - g_p)^{p^t - 1}, \quad \text{where} \quad r \in R,\ g \in G$$

and such that they satisfies the conditions:

(a) if $|R| \geq |G|$, then $g = 1$ and r ranges over all elements of R;

(b) if $|R| < |G|$, then $r = 1$ and g ranges a transversal T for $\langle g_p \rangle$ in G with the elements of $\langle g_p \rangle$ deleted. Apparently $|T| = |G|$.

Moreover, $x_{rg}^p = 1 + r^p g^p (1 - g_p)^{p^{t+1} - p} = 1$ since $(1 - g_p)^{2(p^t - 1)} = 0$, so every x_{rg} is in $S(RG)[p]$ because it is a normed element. We claim that:

$$(*) \qquad x_{rg} S(RG^p)[p]G[p] = x_{fh} S(RG^p)[p]G[p] \Longleftrightarrow r = f \quad \text{and} \quad g = h.$$

In fact, to substantiate our claim, foremost observe that $x_{fh}^{-1} = x_{fh}^{p-1} = [1 + fh(1 - g_p)^{p^t - 1}]^{p-1} = 1 + (p-1)fh(1 - g_p)^{p^t - 1} = 1 - fh(1 - g_p)^{p^t - 1}$, using the standard binom Newton formula and the simple facts that $(1 - g_p)^{2(p^t - 1)} = 0$ and $pf = 0$.

We assume now that $x_{rg} x_{fh}^{-1} \in S(RG^p)[p]G[p]$. It is easily seen that $x_{rg} x_{fh}^{-1} = [1 + rg(1 - g_p)^{p^t - 1}][1 - fh(1 - g_p)^{p^t - 1}] = 1 + (rg - fh)(1 - g_p)^{p^t - 1}$. Thus we obtain $1 + (rg - fh)(1 - g_p)^{p^t - 1} \in S(RG^p)[p]G[p]$, i.e. equivalently:

$1+(rg-fh)(1-g_p)^{p^t-1}=a_p(r_1c_1+\cdots+r_kc_k)$, where $a_p\in G[p]$; $r_i\in R$ with $\sum_i r_i=1$; $c_i\in G^p$; $1\le i\le k$. If (a) holds, then $g=h=1$, and by looking at the coefficients of g_p and g_p^2 in the canonical form of the left hand-side of the above equality, we deduce immediately that $r=f$. Really, if not, i.e. $r\neq f$, applying the Newton's binom formula we must conclude that $g_p\in a_pG^p$ and $g_p^2\in a_pG^p$. That is why, $g_p\in G^p$, i.e. $g_p\in G_p^p$, that is wrong. If (b) is valid, then $r=f=1$ and $g,h\in T$. Hence, if $g\neq h$, the support of the right hand-side of the above equality contains only those elements of $g\langle g_p\rangle$ which belong to the support of $g(1-g_p)^{p^t-1}$. But therefore $gg_p^s\in a_pG^p$ for all $s\in\{0,1,\dots,p^t-1\}$ and thus $g_p\in G^p$ whence $g_p\in G_p^p$, a contradiction. Well, (*) is indeed sustained and finally it follows automatically that $f_0(S(RG)/G_p)=\max(|R|,|G|)$, as stated. The proof is end.

Remark. Using the same technique, the Ulm-Kaplansky functions of $V(RG)/G$ can be obtained when G is an abelian p-group and R is a perfect commutative ring with identily of prime characteristic p.

The applied results are selected in the next hot section.

3. The description of $S(RG)$ and invariants for group algebras. A fundamental consequence to our main theorem is the following

Theorem 7. *Suppose that R is with trivial nilradical and the GDFP holds. Then the structure of $S(RG)$ is completely determined.*

Proof. Since $S(RG)\cong G_p\times S(RG)/G_p$ whenever $S(RG)/G_p$ is totally projective (simply presented), we may employ the result on the description of the maximal divisible subgroup of the quotient group $S(RG)/G_p$ given by us in [11] along with the above Theorem 6 and [13] to complete the proof.

As an immediate consequence, we deduce

Corollary 8. *Assume that F is a perfect field of characteristic p and G is an abelian p-group so that it is a C_λ-group of countable length, or is a summable group of countable length, or is with cardinality $\aleph_1$. Then the isomorphic class of $V(FG)$ is fully obtained.*

Proof. In view of facts proved by us in [7, 10] or by a result of Hill-Ullery [14] (see also [12]), we can write $V(FG) \cong G \times V(FG)/G$, where $V(FG)/G$ is totally projective. Thus our Theorem 7 is applicable to finish the proof.

We begin now with claims of another type, namely on the isomorphism of commutative group algebras.

Proposition 9. *Let R be with elementary nilradical. Then the Ulm-Kaplansky functions of $S(RG)/G_p$ may be retrieved from the R-group algebra RG, i.e. they are its invariants.*

Proof. Follows directly conforming with Theorem 6 since the powers of the objects from the formula for equality of the Ulm-Kaplansky functions for $S(RG)/G_p$ combined with $f_\sigma(G_p)$ can be deduced from RG (see cf. [16]). The claim is verified.

Now, we can attack

Proposition 10. *Assume that the GDFP is true and that $FG \cong FH$ as F-algebras for a p-mixed abelian group G. Then there exists a totally projective p-group T with the property $G \times T \cong H \times T$.*

Proof. It is a routine matter to establish that H_t is p-primary too, i.e. $H_t = H_p$. By hypothesis, $S(FG) \cong G_p \times S(FG)/G_p$ and $S(FH) \cong H_p \times S(FH)/H_p$. But on the other hand, it is apparent that $V(FG) = GS(FG)$ and $V(FH) = HS(FH)$ where $V(FG)/G \cong S(FG)/G_p$ and $V(FH)/H \cong S(FH)/H_p$ are both totally projective (see for example [16] plus [2]). Thus $V(FG) \cong G \times V(FG)/G$ and $V(FH) \cong H \times V(FH)/H$. But Proposition 9 leads us to $V(FG)/G \cong V(FH)/H$ and because $V(FG) \cong V(FH)$, we have $G \times V(FG)/G \cong H \times V(FG)/G$. Putting $T = V(FG)/G$, we are done.

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