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COMPLEMENTS OF COMPLETE INTERSECTIONS IN INFINITE-DIMENSIONAL COMPLEX PROJECTIVE SPACES

BY

E. BALLICO

Abstract. Let V be a Banach space with countable unconditional basis. Let $f_1, \dots, f_s$, $s > 0$, be finitely many continuous homogeneous polynomials on V. Set $X(f_i) := \{P \in \boldsymbol{P}(V) : f_i(P) \neq 0\}$ and $X := X(f_1) \cup \cdots \cup X(f_s)$. Here we prove that $H^i(X, \boldsymbol{O}_X) = 0$ for every $i \geq s$ and we give a condition assuring that $H^{s-1}(X, \boldsymbol{O}_X)$ is infinite-dimensional.

1. Introduction. Let V be a complex Banach space and $\boldsymbol{P}(V)$ the projective space of all one-dimensional linear subspaces of V. In this paper we will use the fundamental vanishing theorems proved in [5] to study the cohomology groups $H^i(X, \boldsymbol{O}_X)$ when X is a suitable open subset of $\boldsymbol{P}(V)$. In section three we will prove the following result.

Theorem 1.1. *Let V be a Banach space will countable unconditional basis. Let $f_1, \dots, f_s$, $s > 0$, be finitely many continuous homogeneous polynomials on V. Set $X(f_i) := \{P \in \boldsymbol{P}(V) : f_i(P) \neq 0\}$ and $X := X(f_1) \cup \cdots \cup X(f_s)$. Then $H^i(X, \boldsymbol{O}_X) = 0$ for every integer $i \geq s$. If there is at least one*

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$P \in \{f_1 = \cdots = f_s = 0\}$ *such that the differentials* $df_1, \ldots, df_s$ *are linearly independent at* P, *then* $H^{s-1}(X, \boldsymbol{O}_X)$ *is infinite-dimensional.*

All locally closed subsets of $\boldsymbol{P}(V)$ arising in this paper will be considered with the topology induced by the quotient topology of $\boldsymbol{P}(V)$ induced by the topology of $V \setminus \{0\}$. We stress that in Theorem 1.1 we assume $s > 0$ not only because $H^0(\boldsymbol{P}(V), \boldsymbol{O}_{\boldsymbol{P}(V)}) \cong \boldsymbol{C}$, but also for the vanishing of the higher cohomology groups: as far as we know the vanishing of $H^2(\boldsymbol{P}(V), \boldsymbol{O}_{\boldsymbol{P}(V)})$ is still an open question even when V is a separable Hilbert space because the fundamental vanishing theorem proved in [4], §7 and §8, are stated and proved for the Dolbeaut cohomology, not for the sheaf cohomology.

In section two we will study the cohomology of complete intersection closed subvarieties of a pseudoconvex open domain $U \subseteq V$.

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2. Complete intersections in pseudoconvex domains. Let U be a complex Banach manifold and $f_1, \ldots, f_k$ finitely many holomorphic functions on U. Set $Z(f_1, \ldots, f_k) := \{P \in U : f_1(P) = \cdots = f_k(P) = 0\}$. $Z(f_1, \ldots, f_k)$ is a Banach analytic subset of finite definition of U in the sense of [6]. There is a natural sheaf $\boldsymbol{A}$ on $Z = Z(f_1, \ldots, f_k)$: set $\boldsymbol{A}_Z := \boldsymbol{O}_U/(f_1, \ldots, f_k)\boldsymbol{O}_U|Z(f_1, \ldots, f_k)$.

Remark 2.1. If $(f_1, \ldots, f_k)$ is a radical ideal, then $\boldsymbol{A}_Z = \boldsymbol{O}_Z$ by the Nullstellensatz for geometric ideals ([6], Th. II.2.4.6). If Z is smooth and for every $P \in Z$ the rank at P of the differential of the map $F = (f_1, \ldots, f_k) : U \to \boldsymbol{C}^k$ is equal to the codimension of Z in X at P, then $\boldsymbol{A}_Z = \boldsymbol{O}_Z$ by the inverse mapping theorem.

Proposition 2.2. *Let* V *be a complex Banach space will countable unconditional basis,* $U \subseteq V$ *an open pseudoconvex domain and* $f_1, \ldots, f_k$

holomorphic functions on U *which form a regular sequence at each point of their common zero set* $Z := \{P \in U : f_1(P) = \cdots = f_k(P) = 0\}$. *Then* $H^i(Z, \boldsymbol{A}_Z) = 0$ *for every* $i > 0$.

Proof. Let $\boldsymbol{A}^*$ be the Koszul complex of the holomorphic functions $f_1, \ldots, f_k$ ([1], Ch.17; the definition part does not use the Noetherianity assumption of the local ring and it may be used to define depth-related concepts in non-Noetherian rings). Hence $\boldsymbol{A}^*$ is a length $k+1$ complex of free $\boldsymbol{O}_U$-modules of finite rank whose j-th piece is $\Lambda^j(\boldsymbol{O}_{U^{\oplus k}})$ and such that the last map u: $\boldsymbol{O}_{U^{\oplus k}} \to \boldsymbol{O}_U$ is defined by $u((g_1, \ldots, g_k)) = f_1 g_1 + \cdots + f_k g_k$. Let $\boldsymbol{B}^*$ be the complex obtained from $\boldsymbol{A}^*$ taking $\mathrm{Im}(u) = (f_1, \ldots, f_k)\boldsymbol{O}_U$ instead of $\boldsymbol{O}_U$ as the last brick of the complex. Both $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ are exact at each point of $V \setminus Z$. By the very definition of a regular sequence the complex $\boldsymbol{B}^*$ is exact at each point of Z. Thus $\boldsymbol{B}^*$ is exact. By [5], Th. 0.1, we have $H^i(U, \boldsymbol{O}_U) = 0$ for every $i > 0$. Hence $H^i(U, \Lambda^j \boldsymbol{O}_{U^{\oplus k}})) = 0$ for all i, j with $i > 0$ and $0 \le j \le k$. Splicing $\boldsymbol{B}^*$ into short exact sequences we obtain $H^i(U, \mathrm{Im}(u)) = 0$ for every $i > 0$. From the exact sequence

$$0 \to \mathrm{Im}(u) \to \boldsymbol{O}_U \to \boldsymbol{A}_Z \to 0 \tag{1}$$

we obtain $H^i(U, \boldsymbol{A}_Z) = 0$ for every $i > 0$. Since $H^i(U, \boldsymbol{A}_Z) = H^i(Z, \boldsymbol{A}_Z)$, we conclude.

Remark 2.3. Let U be an open connected subset of a Banach space and f_1 a nonidentically zero holomorphic function on U. Since for every $Q \in U$ the local ring $\boldsymbol{O}_{U,Q}$ is factorial ([6], Th. I.1.4.4), the sequence f_1 is regular unless f_1 is identically zero (see [1], p.424). Hence the regularity condition in Proposition 2.2 is always satisfied when $k = 1$ and f_1 is not identically zero.

3. Proof of Theorem 1.1.

Proposition 3.1. *Let V be a Banach space with countable unconditional basis. Let f be a continuous homogeneous polynomial on V. Set $X := \{P \in \boldsymbol{P}(V) : f(P) \neq 0\}$. Then $H^i(X, \boldsymbol{O}_X) = 0$ for every integer $i > 0$.*

Proof. Set $Y := \{P \in V : f(P) = 1\}$. By Remarks 2.1 and 2.3 we have $\boldsymbol{A}_Y = \boldsymbol{O}_Y$ and hence we may apply Proposition 2.2 for $k = 1$ and $f_1 := f - 1$ to obtain $H^i(Y, \boldsymbol{O}_Y) = 0$ for every $i > 0$. Since f is homogeneous, we have $f(0) = 0$. Thus $0 \notin Y$. Thus the natural map $\pi : V \setminus \{0\} \to \boldsymbol{P}(V)$ induces a holomorphic map $g = \pi | Y : Y \to \boldsymbol{P}(Y)$. It is easy to check that $\pi(Y) = X$ and that $g(P) = g(Q)$ for some P, $Q \in Y$ if and only if there is $\lambda \in \boldsymbol{C}$ such that $\lambda^d = 1$ and $\lambda P = Q$, where d is the degree of f. Thus the map g is étale of degree d, the sheaf $g_*(\boldsymbol{O}_Y)$ is a holomorphic rank d vector bundle on X and $R^k g_*(\boldsymbol{O}_Y) = 0$ for every $k > 0$ ([3], Satz 5 at p.48). Thus the Leray spectral sequence of g gives $H^i(X, g_*(\boldsymbol{O}_Y)) = H^i(Y, \boldsymbol{O}_Y)$. Thus $H^i(X, g_*(\boldsymbol{O}_Y)) = 0$ for every $i > 0$. The trace map shows that $\boldsymbol{O}_X$ is a direct summand of $g_*(\boldsymbol{O}_Y)$. Hence $H^i(X, \boldsymbol{O}_X)$ is a direct factor of $H^i(X, g_*(\boldsymbol{O}_Y))$. Thus $H^i(X, \boldsymbol{O}_X) = 0$.

Proof of Theorem 1.1. (a) Since the case $s = 1$ of the first assertion is covered by Proposition 3.1, to check the first part we may assume $s \geq 2$. Notice that $X(f) \cap X(g) = X(fg)$ for all homogeneous polynomials. Hence we may apply Proposition 3.1 to any finite intersection of open subsets $X(f_i)$, $1 \leq i \leq s$, of X. Thus $X(f_1)$, $1 \leq i \leq s$, is a Leray covering of X for the sheaf $\boldsymbol{O}_X$ and we may compute all cohomology groups $H^k(X, \boldsymbol{O}_X)$ using the Cech cohomology of the covering $\{X(f_1), \ldots, X(f_s)\}$ ([2], Cor. at p.213). Since $\{X(f_1), \ldots, X(f_s)\}$ has only s members, the i-th alternating Cech cohomology group of this covering vanishes for all $i \geq s$.

(b) Now we check the last assertion, i.e. the infinite-dimensionality of $H^{s-1}(X, \boldsymbol{O}_X)$. Since Y is smooth and of codimension s at P, there is a dimension $s+1$ linear subspace W of V such that $P \in \boldsymbol{P}(W)$ and $\boldsymbol{P}(W)$ intersects transversally Y and P. Hence there is an open neighborhood Δ of P in $\boldsymbol{P}(W)$ with Δ biholomorphic to an s-dimensional polydisc with P as center such that $\Delta \cap Y = \{P\}$ and Δ intersects transversally Y at P. Since the differentials of $f_1, \ldots, f_s$ are linearly independent at P, we may take the restrictions of $f_1, \ldots, f_s$ to Δ as coordinates $z_1, \ldots, z_s$ of $\Delta \subset \boldsymbol{C}^s$. As in part (a) we may use the Leray covering $\{X(f_1), \ldots, X(f_s)\}$ to compute $H^{s-1}(X, \boldsymbol{O}_X)$. Choose a closed hyperplane $\{z = 0\}$ of $\boldsymbol{P}(V)$ such that $\Delta \cap \{z = 0\} = \emptyset$. Notice that for all integers $k \geq 1$ the symbol $z^d/(f_1 \cdots f_s)^k$ defines a holomorphic function on X if $d = k(\deg(f_1) + \cdots + \deg(f_s))$. Hence in this way we define infinitely many elements of $H^{s-1}(X, \boldsymbol{O}_X)$ using the covering $\{X(f_1), \ldots, X(f_s)\}$. Let E be the linear subspace of $H^s(\Delta \setminus \{P\}, \boldsymbol{O}_{\Delta\setminus\{P\}})$ spanned by the pull-backs of these elements where the pull-back is with respect to the inclusion $\Delta\setminus\{P\} \rightarrow X$. To show that these elements of $H^{s-1}(X, \boldsymbol{O}_X)$ are linearly independent and hence to conclude the proof, it is sufficient to show that $\dim(E) = +\infty$. This may be done using Laurent expansions with respect to $z_1, \ldots, z_s$. The case $s = 2$ is done in details in [3], pp.135-136, and the general case is similar.

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Dept. of Mathematics, Università di Trento, 38050 Povo (TN), Italy.

E-mail: ballico@science.unitn.it