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TRAVELING WAVE SOLUTIONS OF A BISTABLE FAST DIFFUSION EQUATION

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Abstract. We study traveling wave solutions of a bistable fast diffusion equation. Using a method of Fife and McLeod, we obtain the existence and uniqueness (up to translations) of traveling wave solution of this equation. Then the stability of this traveling wave solution is also derived.

1. Introduction. In this paper, we consider the following quasilinear parabolic equation:

$$u_t = (u^m)_{xx} + f(u), \quad x \in \mathbf{R},\ \mathbf{t} > \mathbf{0}, \tag{1.1}$$

where $0 < m < 1$ and $f(u) = u(1-u)(u-a)$ with $0 < a < 1$. This equation is of fast diffusion type with reaction term $f(u)$.

We say that u is a traveling wave solution of (1.1) if $u(x,t) = \varphi(x-ct)$ for some constant c such that $\varphi(-\infty) = 0$ and $\varphi(+\infty) = 1$. Here c is called the wave speed.

When $m = 1$, i.e., the case of standard heat operator, there have been many works on the existence and stability of traveling wave solutions of (1.1) with various source terms (see, e.g., [1], [2], [4], [5], [7], [8], etc.). The case $m > 1$, i.e., the slow diffusion case, we refer the readers to [3] and the

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references cited therein. We are concerned with the existence, uniqueness and stability of traveling wave solutions of (1.1) for $0 < m < 1$. We recall that for each $a \in (0,1)$ there is a unique wave speed c when $m = 1$. While there is a continuum of wave speed c for a given $a \in (0,1)$ when $m > 1$. In the case $0 < m < 1$, we find that there is a unique wave speed c for each given $a \in (0,1)$. Note that for $h(u) := u^m$ we have $h'(0^+) = 0$ if $m > 1$; $h'(0^+) = 1$ if $m = 1$; and $h'(0^+) = \infty$ if $0 < m < 1$. The singularity of diffusion term at $u = 0$ makes the case when $0 < m < 1$ interesting.

We shall apply a method used in [2] to obtain the existence and uniqueness (up to translations) of traveling wave solution of (1.1) in Section 2. Then we study its stability in Section 3 by using a super-sub-solution method as in [2] and [3].

2. Existence and uniqueness. In this section, we shall study the existence and uniqueness of traveling wave solution of (1.1). Although the proof is almost the same as that of Fife and McLeod [2], for reader's convenience we present the detail here.

Given a real constant c. Let $z = x - ct$. A function $U(z)$ is said to be a traveling wave solution of (1.1) with wave speed c if U satisfies

$$\begin{cases} (U^m)'' + cU' + f(U) = 0, & z \in \mathbf{R}, \\ U(-\infty) = 0,\ U(+\infty) = 1, & 0 \le U \le 1. \end{cases} \tag{2.1}$$

By a phase plane analysis as in [2], $U' > 0$ in $\mathbf{R}$ for any solution U of (2.1).

Let $v = U^m$, $\alpha = 1/m - 1$, and $g(v) = f(v^{\alpha+1})$. Note that $\alpha > 0$, since $m \in (0,1)$. Then (U, c) satisfies (2.1) if and only if (v, c) satisfies

$$\begin{cases} v'' + c(\alpha+1)v^\alpha v' + g(v) = 0, & z \in \mathbf{R}, \\ v(-\infty) = 0,\ v(+\infty) = 1, & 0 \le v \le 1. \end{cases} \tag{2.2}$$

Since $v' > 0$, we can invert $v = v(z)$ to obtain $z = z(v)$ and so $w = v'$ can be defined as $w = w(v)$ for $0 \le v \le 1$.

If (v, c) is a solution of (2.2), then (w, c) will satisfy

$$\begin{cases} w'(v) = -c(\alpha+1)v^\alpha - g(v)/w, & 0 < v < 1, \\ w(0) = 0, \ w(1) = 0, & w > 0. \end{cases} \tag{2.3}$$

Notice that (2.2) is a second order ordinary differential equation, whereas (2.3) is a first order equation. Furthermore, the sign of the speed c is determined by the sign of $-\int_0^1 g(v)dv$, since from the equation

$$ww' = -c(\alpha+1)v^\alpha w - g(v), \quad 0 < v < 1,$$

we obtain

$$0 = -c(\alpha+1)\int_0^1 v^\alpha w(v)dv - \int_0^1 g(v)dv$$

and so

$$c = -\frac{\displaystyle\int_0^1 g(v)dv}{(\alpha+1)\displaystyle\int_0^1 v^\alpha w(v)dv}. \tag{2.4}$$

Conversely, if (2.3) has a solution (w, c), then by solving the initial value problem:

$$v'(z) = w(v(z)), \quad v(0) = 1/2,$$

we obtain a solution v defined in a neighborhood of $z = 0$. We claim that v exists in the whole real line $\mathbf{R}$. Suppose that the maximal existence interval of v is (z_0, z_1). It is clear that

$$v(z) \to 0 \ \text{ as } \ z \to z_0,$$
$$v(z) \to 1 \ \text{ as } \ z \to z_1.$$

Recall that $g(0) = g(1) = 0$ and $g \in C^1[0,1]$. Then there exists a positive constant β such that

$$|g(v)| \leq \beta v, \quad v \in [0,1].$$

On the line $\{w = \gamma v\}$, we have

$$\frac{dw}{dv} < \gamma,$$

if $\gamma > 0$ is chosen so that

$$\gamma > -c(\alpha+1) + \beta/\gamma.$$

It follows that $w(v) < \gamma v$ for $v \in (0,\delta)$ for some $\delta > 0$. Therefore,

$$-z_0 = \int_{z_0}^{0} dz = \int_0^{1/2} \frac{dv}{w(v)} \geq \int_0^{\delta} \frac{dv}{\gamma v} = +\infty.$$

This shows that $z_0 = -\infty$. Similarly, $z_1 = +\infty$. Clearly, v is a solution of (2.2) with the same speed c.

In summary, we have shown that there is a one-to-one correspondence between solutions of (2.2) with $v(0) = 1/2$ and solutions of (2.3). We now focus our attention on the existence of solutions of (2.3).

Note that there is a unique $\sigma \in (0,1)$ such that $g(v) < 0$ for $v \in (0,\sigma)$ and $g(v) > 0$ for $v \in (\sigma,1)$.

Lemma 2.1. *Let w_i be a solution of the initial value problem:*

$$w' = -c(\alpha+1)v^{\alpha} - \frac{g(v)}{w}, \quad v > 0, \quad w(0) = 0, \tag{2.5}$$

with $c = c_i$, $i = 1,2$. Then $c_1 > c_2$ if and only if $w_1(v) < w_2(v)$ for $v \in (0,\tau]$ for some $\tau \in (0,1)$.

Proof. From the equality

$$(w_1 - w_2)' - \frac{g(v)}{w_1 w_2}(w_1 - w_2) = (c_2 - c_1)(\alpha + 1)v^\alpha,$$

by integration we obtain the conclusion.

Similarly, we obtain

Lemma 2.2. *Let $\tilde{w}_i$ be a solution of the initial value problem:*

$$w' = -c(\alpha + 1)v^\alpha - \frac{g(v)}{w}, \quad v < 1,\ w(1) = 0, \tag{2.6}$$

with $c = c_i$, $i = 1, 2$. Then $c_1 > c_2$ if and only if $\tilde{w}_1(v) > \tilde{w}_2(v)$ for $v \in [\tau, 1)$ for some $\tau \in (0, 1)$.

Next, we study the existence of solutions of (2.5).

Lemma 2.3. *For each given $c \in \mathbf{R}$, the problem (2.5) has a unique solution w which is defined in an interval containing $[0, \sigma)$ and is positive in $(0, \sigma)$.*

Proof. Given $\epsilon > 0$. Consider the ϵ-problem:

$$w' = -c(\alpha + 1)v^\alpha - \frac{g(v)}{w}, \quad v > 0,\ w(0) = \epsilon. \tag{2.7}$$

By the standard theory of ordinary differential equations, (2.7) has a unique positive solution w_ϵ defined in a right neighborhood of 0. If there is a $v_0 < \sigma$ such that $w_\epsilon(v_0) = 0$ and $w_\epsilon > 0$ in $(0, v_0)$, then we have

$$w_\epsilon'(v_0) \le 0, \quad \frac{-g(v)}{w_\epsilon(v)} \to +\infty \ \text{ as } \ v \to v_0^-,$$

a contradiction. Therefore, w_ϵ is defined and positive in $(0, \sigma)$.

Notice that from the equality

$$(w_{\epsilon_1} - w_{\epsilon_2})' - \frac{g(v)}{w_{\epsilon_1} w_{\epsilon_2}}(w_{\epsilon_1} - w_{\epsilon_2}) = 0,$$

it follows that $w_{\epsilon_1} < w_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$. Hence

$$w(v) = \lim_{\epsilon \to 0^+} w_\epsilon(v), \quad v \in (0, \sigma),$$

is well-defined.

Claim that $w > 0$ in $(0, \sigma)$. Clearly, $w \geq 0$. From

$$w_\epsilon w_\epsilon' = -c(\alpha+1)v^\alpha w_\epsilon - g(v), \quad w_\epsilon(0) = \epsilon,$$

we obtain

$$\frac{1}{2} w_\epsilon^2(v) = \frac{1}{2}\epsilon^2 - c(\alpha+1)\int_0^v s^\alpha w_\epsilon(s) ds - \int_0^v g(s) ds, \quad v \in (0, \sigma).$$

Then Monotone Convergence Theorem implies that

$$\frac{1}{2} w^2(v) = -c(\alpha+1)\int_0^v s^\alpha w(s) ds - \int_0^v g(s) ds, \quad v \in (0, \sigma). \tag{2.8}$$

Differentiating (2.8) we obtain that

$$w(v)w'(v) = -c(\alpha+1)v^\alpha w(v) - g(v), \quad v \in (0, \sigma). \tag{2.9}$$

Since $g < 0$ in $(0, \sigma)$, it follows that $w(v) > 0$ for $v \in (0, \sigma)$. Also, from (2.9) w is a solution of (2.5) in $[0, \sigma)$.

Finally, the uniqueness of positive solutions follows from the identity

$$(w_1 - w_2)' - \frac{g(v)}{w_1 w_2}(w_1 - w_2) = 0.$$

Hence the lemma is proved.

Similarly, the following result can be proved.

Lemma 2.4. *For each given* $c \in \mathbf{R}$, *the problem* (2.6) *has a unique solution* $\tilde{w}$ *which is defined in an interval containing* $(\sigma, 1]$ *and is positive in* $(\sigma, 1)$.

Note that the solution $\tilde{w}$ of (2.6) satisfies the identity

$$\frac{1}{2}\tilde{w}^2(v) = c(\alpha+1)\int_v^1 s^\alpha w(s)ds + \int_v^1 g(s)ds, \quad v \in (\sigma, 1). \tag{2.10}$$

Given $c \in \mathbf{R}$, let w_c and $\tilde{w}_c$ be the solutions of (2.5) and (2.6), respectively. Since

$$w_c(v) = -c(\alpha+1)\int_0^v s^\alpha ds - \int_0^v \frac{g(s)}{w_c(s)}ds, \quad v \in (0, \sigma),$$

it follows that

$$w_c(v) \to \infty \ \text{ as } \ c \to -\infty, \quad \forall v \in (0, \sigma). \tag{2.11}$$

Similarly, we have

$$\tilde{w}_c(v) = c(\alpha+1)\int_v^1 s^\alpha ds + \int_v^1 \frac{g(s)}{\tilde{w}_c(s)}ds, \quad v \in (\sigma, 1),$$

and so

$$\tilde{w}_c(v) \to \infty \ \text{ as } \ c \to \infty, \quad \forall v \in (\sigma, 1). \tag{2.12}$$

We are ready to state and prove the main theorem of this section as follows.

Theorem 2.5. *There is a unique* (*up to translations*) *traveling wave solution of* (1.1).

Proof. Let

$$A=\int_0^\sigma g(s)ds,\quad B=\int_\sigma^1 g(s)ds,\quad K=\int_0^1 g(s)ds.$$

Note that $A<0$ and $B>0$. First, if $K=0$, then $B=-A$. Hence it follows from (2.4), (2.8) and (2.10) that $c=0$ and the corresponding solution w_0 is given by

$$w_0(v)=\{-2\int_0^v g(s)ds\}^{1/2},\quad v\in(0,1).$$

Suppose that $K>0$, i.e., $B>-A$. From (2.8) it follows that

$$\frac{1}{2}[w_c(\sigma^-)]^2>-A,\quad \forall c<0. \tag{2.13}$$

Similarly, by (2.10) we have

$$\frac{1}{2}[\tilde{w}_c(\sigma^+)]^2<B,\quad \forall c<0. \tag{2.14}$$

Recall (2.11). Then from Lemmas 2.1 and 2.2 we conclude that there is a unique $c<0$ such that $w_c(\sigma^-)=\tilde{w}_c(\sigma^+)$.

Similarly, if $K<0$, then there is a unique $c>0$ such that $w_c(\sigma^-)=\tilde{w}_c(\sigma^+)$. Therefore, in any case the problem (2.3) has a unique solution. Hence the theorem follows.

3. Stability. Now we turn to study the stability of the traveling wave solution of (1.1) for the case $c\le 0$. Let $u(x,t)$ be the solution of (1.1) with the initial value $u(x,0)=\phi(x)$. We assume that $\phi(x)$ satisfies

$$\begin{cases} 0\le\phi(x)\le 1,\\ \liminf_{x\to+\infty}\phi(x)>\max\{a,(1-a_1^m)^{\frac{1}{m}}\},\\ \limsup_{x\to-\infty}\phi(x)<a, \end{cases}$$

where a_1 is the unique minimal point of $f(u)$ in $(0, a)$.

Suppose that U is the traveling wave solution of (1.1) with the speed c (≤ 0) derived in Section 2. Then (U, c) satisfies

$$\begin{cases} (U^m)'' + cU' + f(U) = 0, & \xi \in \mathbf{R}, \\ U(-\infty) = 0,\ U(+\infty) = 1, & U'(\xi) > 0,\ 0 < U < 1. \end{cases}$$

For convenience, we let $N[u] = u_t - (u^m)_{xx} - f(u)$ and $z = x - ct$.

First we prove the following lemma. Although the comparison functions constructed below for our case $(0 < m < 1)$ are the same type as those constructed in [3] for the case $m > 1$, the condition $h'(0^+) = \infty$ for $0 < m < 1$ $(h(u) := u^m)$ gives some difficulties to verify the following lemma. Therefore, we shall give a detailed proof for the following lemma.

Lemma 3.1. *Suppose that $c \leq 0$. Then there are constants α_1, α_2, r_1, r_2 and μ (the last three are positive) such that*

$$(\max\{0, U^m(x-ct-\alpha_1)-r_1e^{-\mu t}\})^{\frac{1}{m}} \leq u(x,t)$$
$$\leq (\min\{1, U^m(x-ct-\alpha_2)+r_2e^{-\mu t}\})^{\frac{1}{m}}.$$

Proof. First we prove the left-hand inequality. Let

$$\underline{u}(x,t) = (\max\{0, U^m(x-ct-\xi(t)) - r(t)\})^{\frac{1}{m}},$$

where the functions $\xi(t)$ and $r(t)$ with $\xi'(t) \geq 0$ and $r(t) > 0$ are to be chosen.

Since $\liminf_{z\to+\infty} \phi(z) > \max\{a, (1-a_1^m)^{\frac{1}{m}}\}$, $\liminf_{z\to+\infty} \phi^m(z) > \max\{a^m, 1-a_1^m\}$. Choose a $r_1 > 0$ such that $\liminf_{z\to+\infty} \phi^m(z) > 1 - r_1 > \max\{a^m, 1-a_1^m\}$. Then there exists y_1 such that $1 - r_1 < \phi^m(z)$, $\forall z > y_1$. Since $U \in [0,1]$, $U' > 0$ and $0 < m < 1$, $U^m \in [0,1]$ and U^m is increasing. There exists y_2 such that $U^m(z) \leq r_1$, $\forall z \leq y_2$. We take z^* with $y_1 - y_2 \leq z^*$, i.e., $y_1 - z^* \le y_2$. Then $\phi^m(z) > 1 - r_1 \ge U^m(z - z^*) - r_1$, $\forall z > y_1$, and $U^m(z - z^*) \le r_1$, $\forall z \le y_1$. So $U^m(z - z^*) - r_1 \le \phi^m(z)$, $\forall z$.

If $\underline{u}(x,t) > 0$, then $N[\underline{u}] = N[(U^m - r)^{\frac{1}{m}}]$ and

$$\begin{aligned}
&N[(U^m - r)^{\frac{1}{m}}]\\
&= \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}[mU^{m-1}(-cU' - \xi' U') - r'] - (U^m - r)'' - f((U^m - r)^{\frac{1}{m}})\\
&= \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}[mU^{m-1}(-cU' - \xi' U') - r'] + cU' + f(U) - f((U^m - r)^{\frac{1}{m}}) \qquad (3.1)\\
&= cU'\Big[1 - \Big(\frac{U^m - r}{U^m}\Big)^{\frac{1}{m}-1}\Big] + \Big(\frac{U^m - r}{U^m}\Big)^{\frac{1}{m}-1}(-\xi' U') + \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}(-r')\\
&\quad + [f(U) - f((U^m - r)^{\frac{1}{m}})].
\end{aligned}$$

Define Φ by

$$\Phi(u,r) = \begin{cases} \dfrac{f((u^m - r)^{\frac{1}{m}}) - f(u)}{u - (u^m - r)^{\frac{1}{m}}}, & r > 0,\\ -f'(u), & r = 0, \end{cases}$$

where $0 \le r \le r_1$ and $0 \le (u^m - r)^{\frac{1}{m}} \le u \le 1$. For $0 < r \le r_1$, $f((1-r)^{\frac{1}{m}}) > 0$, since $a < (1 - r_1)^{\frac{1}{m}} \le (1 - r)^{\frac{1}{m}}$. Thus we have

$$\Phi(1,r) = \frac{f((1-r)^{\frac{1}{m}}) - f(1)}{1 - (1-r)^{\frac{1}{m}}} = \frac{f((1-r)^{\frac{1}{m}})}{1 - (1-r)^{\frac{1}{m}}} > 0$$

for $0 < r \le r_1$. Also we have $\Phi(1,0) = -f'(1) > 0$. Then $\Phi(1,r) \ge 2\mu_1$ on $0 \le r \le r_1$ for some $\mu_1 > 0$. We set $\Gamma = \{(u,r) | 0 \le (u^m - r)^{\frac{1}{m}} \le u \le 1$ and $0 \le r \le r_1\}$. Clearly, $\Phi(u,r)$ is continuous in (u,r) for $r > 0$ in Γ. Next we fix a point $(u_0, 0)$. We know that

$$|\Phi(u,r) - \Phi(u_0,0)| = \begin{cases} |f'(u_0) - f'(u)|, & \text{if } r = 0,\\ \left| f'(u_0) + \dfrac{f((u^m - r)^{\frac{1}{m}}) - f(u)}{u - (u^m - r)^{\frac{1}{m}}} \right|, & \text{if } r > 0, \end{cases}$$

$$= \begin{cases} |f'(u) - f'(u_0)|, & \text{if } r = 0, \\ |f'(v) - f'(u_0)|, & \text{if } r > 0, \end{cases}$$

where $(u^m - r)^{\frac{1}{m}} < v < u$. Since $f \in C^1[0,1]$, $\Phi(u,r)$ is continuous at $(u_0, 0)$. By the above argument, $\Phi(u,r)$ is continuous on Γ. So there exists $\delta_1 > 0$ such that $\Phi(u,r) \geq \mu_1$ for $1 - \delta_1 \leq u \leq 1$ and $0 \leq r \leq r_1$. Thus

$$f((u^m - r)^{\frac{1}{m}}) - f(u) \geq \mu_1[u - (u^m - r)^{\frac{1}{m}}]$$

for $1 - \delta_1 \leq u \leq 1$ and $0 \leq r \leq r_1$. Now we suppose that $\xi' \geq 0$. For $1 - \delta_1 \leq U \leq 1$ and $0 \leq r \leq r_1$, from (3.1), we have

$$\begin{aligned}
& N[(U^m - r)^{\frac{1}{m}}] \\
\leq\ & \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}(-r') + [f(U) - f((U^m - r)^{\frac{1}{m}})] \\
\leq\ & \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}(-r') - \mu_1[U - (U^m - r)^{\frac{1}{m}}] \\
=\ & \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}(-r') - \mu_1[U^m(U^m)^{\frac{1}{m}-1} - U^m(U^m - r)^{\frac{1}{m}-1} + (U^m - r)^{\frac{1}{m}-1}r] \\
\leq\ & \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}(-r') - \mu_1(U^m - r)^{\frac{1}{m}-1}r \\
=\ & \frac{1}{m}(U^m - r)^{\frac{1}{m}-1}[-r' - m\mu_1 r].
\end{aligned}$$

Since $\Phi(u,0) = -f'(u) > 0$ for $0 \leq u \leq r_1^{\frac{1}{m}} < a_1$ and

$$\Phi(u,r) = \frac{f((u^m - r)^{\frac{1}{m}}) - f(u)}{u - (u^m - r)^{\frac{1}{m}}} > 0$$

for $r > 0$, $u^m - r \geq 0$ and $0 \leq u \leq r_1^{\frac{1}{m}}$, $\Phi(u,r) \geq 2\mu_2$ on $u^m - r \geq 0$, $r \geq 0$ and $0 \leq u \leq r_1^{\frac{1}{m}}$ for some $\mu_2 > 0$. Since $\Phi(u,r)$ is continuous in (u,r) on Γ, there exists $\eta_1 > 0$ with $r_1^{\frac{1}{m}} < \eta_1 < a_1$ such that $\Phi(u,r) \geq \mu_2$ on the set

$$\{(u,r) | 0 \leq r \leq r_1, 0 \leq u \leq \eta_1 \text{ and } u^m - r \geq 0\}.$$

Hence

$$f((u^m-r)^{\frac{1}{m}})-f(u)\geq \mu_2(u-(u^m-r)^{\frac{1}{m}})$$

on the set

$$\{(u,r)|0\leq r\leq r_1, 0\leq u\leq \ \eta_1 \text{ and } u^m-r\geq 0\}.$$

For $0\leq U\leq \eta_1$, $0\leq r\leq r_1$ and $U^m-r\geq 0$, by the same argument as above, we have

$$\begin{aligned}
& N[(U^m-r)^{\frac{1}{m}}]\\
\leq\ & \frac{1}{m}(U^m-r)^{\frac{1}{m}-1}(-r')-\mu_2[U-(U^m-r)^{\frac{1}{m}}]\\
\leq\ & \frac{1}{m}(U^m-r)^{\frac{1}{m}-1}(-r')-\mu_2(U^m-r)^{\frac{1}{m}-1}r\\
=\ & \frac{1}{m}(U^m-r)^{\frac{1}{m}-1}[-r'-m\mu_2 r].
\end{aligned}$$

Let $0<\mu<\min\{m\mu_1, m\mu_2\}$. Solving the problem: $r'+\mu r=0$ with $r(0)=r_1$, we find that $r(t)=r_1e^{-\mu t}$. Hence $N[(U^m-r)^{\frac{1}{m}}]\leq 0$ for the above two cases.

For $\eta_1\leq U\leq 1-\delta_1$, there exists a $\beta>0$ such that $U'\geq\beta$. There exists $M=\max_{[0,1]}|f'|$, $0<\theta_7<1$, and $\kappa_0=\beta(\eta_1^m-r_1)^{\frac{1}{m}-1}>0$ such that

$$\begin{aligned}
& N[(U^m-r)^{\frac{1}{m}}]\\
\leq\ & \Big(\frac{U^m-r}{U^m}\Big)^{\frac{1}{m}-1}(-\xi'U')+\frac{1}{m}(U^m-r)^{\frac{1}{m}-1}(-r')+[f(U)-f((U^m-r)^{\frac{1}{m}})]\\
\leq\ & (\eta_1^m-r_1)^{\frac{1}{m}-1}\cdot 1\cdot(-\xi'\beta)+\frac{1}{m}\cdot 1\cdot \mu r+M[U-(U^m-r)^{\frac{1}{m}}]\\
=\ & \kappa_0(-\xi')+\frac{\mu r}{m}+\frac{M}{m}(U^m-r+\theta_7 r)^{\frac{1}{m}-1}r\\
\leq\ & -\kappa_0\xi'+\frac{\mu r}{m}+\frac{M}{m}r\\
=\ & \kappa_0\Big(-\xi'+\frac{(\mu+M)r}{m\kappa_0}\Big)\\
\leq\ & 0
\end{aligned}$$

if $-\xi' + \frac{(\mu+M)r}{m\kappa_0} \le 0$. In particular, we may take $\xi' = \frac{(\mu+M)r}{m\kappa_0}$ with $\xi(0) = z^*$. Then $\xi(t) = \alpha_1 + \beta_1 e^{-\mu t}$, where $\beta_1 = -\frac{(\mu+M)r_1}{m\mu\kappa_0}$, $\alpha_1 = z^* - \beta_1$. Hence $N[\underline{u}] \le 0$ whenever $\underline{u} > 0$. By our condition on z^*,

$$\begin{aligned}\underline{u}(x,0) &= (\max\{0, U^m(x-\xi(0)) - r(0)\})^{\frac{1}{m}} \\ &= (\max\{0, U^m(z-z^*) - r_1\})^{\frac{1}{m}} \\ &\le \phi(z) \\ &= \phi(x).\end{aligned}$$

Thus by the maximum principle (cf. [6])

$$\underline{u}(x,t) \le u(x,t).$$

Note that $\xi(t)$ is increasing and $\lim_{t\to+\infty} \xi(t) = \alpha_1$. Therefore, we obtain that

$$(\max\{0, U^m(x-ct-\alpha_1) - r_1 e^{-\mu t}\})^{\frac{1}{m}} \le \underline{u}(x,t).$$

This proves the left-hand inequality for the case $c \le 0$.

Next, we prove the right-hand inequality. Let

$$\overline{u}(x,t) = (\min\{1, U^m(x-ct-\overline{\xi}(t)) + \overline{r}(t)\})^{\frac{1}{m}},$$

where the functions $\overline{\xi}(t)$ and $\overline{r}(t)$ with $\overline{\xi}'(t) \le 0$ and $\overline{r}(t) > 0$ are to be chosen.

Since $\limsup_{z\to-\infty} \phi(z) < a$, $\limsup_{z\to-\infty} \phi^m(z) < a^m$. Choose a $r_2 > 0$ such that $\limsup_{z\to-\infty} \phi^m(z) < r_2 < a^m < 1$. Then there exists y_3 such that $\phi^m(z) < r_2, \forall z < y_3$. Since $U \in [0,1]$, $U' > 0$ and $0 < m < 1$, $U^m \in [0,1]$ and U^m is increasing. There exists y_4 such that $U^m(z) \ge 1 - r_2, \forall z \ge y_4$. We take $\overline{z^*}$ with $y_3 - y_4 \ge \overline{z^*}$, i.e., $y_3 - \overline{z^*} \ge y_4$. Then $\phi^m(z) < r_2 \le U^m(z - \overline{z^*}) + r_2$, $\forall z < y_3$, and $U^m(z-\overline{z^*}) \ge 1 - r_2$, $\forall z \ge y_3$. So $U^m(z-\overline{z^*}) + r_2 \ge \phi^m(z)$, $\forall z$.

If $\overline{u}(x,t)<1$, then $N[\overline{u}]=N[(U^m+\overline{r})^{\frac{1}{m}}]$ and

$$
\begin{aligned}
&N[(U^m+\overline{r})^{\frac{1}{m}}]\\
&=\frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}[mU^{m-1}(-cU'-\overline{\xi}'U')+\overline{r}']-(U^m+\overline{r})''-f((U^m+\overline{r})^{\frac{1}{m}})\\
&=\frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}[mU^{m-1}(-cU'-\overline{\xi}'U')+\overline{r}']+cU'+f(U)-f((U^m+\overline{r})^{\frac{1}{m}}) \qquad (3.2)\\
&=cU'\Big[1-\Big(\frac{U^m+\overline{r}}{U^m}\Big)^{\frac{1}{m}-1}\Big]+\Big(\frac{U^m+\overline{r}}{U^m}\Big)^{\frac{1}{m}-1}(-\overline{\xi}'U')+\frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r}'\\
&\quad+[f(U)-f((U^m+\overline{r})^{\frac{1}{m}})].
\end{aligned}
$$

Define Ψ by

$$
\Psi(u,\overline{r})=\begin{cases}\dfrac{f(u)-f((u^m+\overline{r})^{\frac{1}{m}})}{(u^m+\overline{r})^{\frac{1}{m}}-u}, & \overline{r}>0,\\ -f'(u), & \overline{r}=0,\end{cases}
$$

where $0\le\overline{r}\le r_2$ and $0\le u\le(u^m+\overline{r})^{\frac{1}{m}}\le 1$. For $0<\overline{r}\le r_2$, $f(\overline{r}^{\frac{1}{m}})<0$, since $a>r_2^{\frac{1}{m}}\ge\overline{r}^{\frac{1}{m}}$. Thus we have

$$
\Psi(0,\overline{r})=\frac{f(0)-f(\overline{r}^{\frac{1}{m}})}{\overline{r}^{\frac{1}{m}}}=\frac{-f(\overline{r}^{\frac{1}{m}})}{\overline{r}^{\frac{1}{m}}}>0
$$

for $0<\overline{r}\le r_2$. Also we have $\Psi(0,0)=-f'(0)>0$. Then $\Psi(0,\overline{r})\ge 2\mu_3$ on $0\le\overline{r}\le r_2$ for some $\mu_3>0$. We set $\overline{\Gamma}=\{(u,\overline{r})|0\le u\le(u^m+\overline{r})^{\frac{1}{m}}\le 1$ and $0\le\overline{r}\le r_2\}$. Clearly, $\Psi(u,\overline{r})$ is continuous in $(u,\overline{r})$ for $\overline{r}>0$ in $\overline{\Gamma}$. Next we fix a point $(u_0,0)$. We have

$$
\begin{aligned}
|\Psi(u,\overline{r})-\Psi(u_0,0)| &= \begin{cases}|f'(u_0)-f'(u)|, & \text{if } \overline{r}=0,\\ \left|f'(u_0)+\dfrac{f(u)-f((u^m+\overline{r})^{\frac{1}{m}})}{(u^m+\overline{r})^{\frac{1}{m}}-u}\right|, & \text{if } \overline{r}>0,\end{cases}\\
&= \begin{cases}|f'(u)-f'(u_0)|, & \text{if } \overline{r}=0,\\ |f'(v)-f'(u_0)|, & \text{if } \overline{r}>0,\end{cases}
\end{aligned}
$$

where $u<v<(u^m+\overline{r})^{\frac{1}{m}}$. Since $f\in C^1[0,1]$, $\Psi(u,\overline{r})$ is continuous at

$(u_0, 0)$. By the above argument, $\Psi(u,\overline{r})$ is continuous on $\overline{\Gamma}$. So there exists $\eta_2 > 0$ such that $\Psi(u,\overline{r}) \geq \mu_3$ for $0 \leq u \leq \eta_2$ and $0 \leq \overline{r} \leq r_2$. Thus

$$f(u) - f((u^m + \overline{r})^{\frac{1}{m}}) \geq \mu_3[(u^m + \overline{r})^{\frac{1}{m}} - u]$$

for $0 \leq u \leq \eta_2$ and $0 \leq \overline{r} \leq r_2$. Now we suppose that $\overline{\xi}' \leq 0$. For $0 \leq U \leq \eta_2$ and $0 \leq \overline{r} \leq r_2$, from (3.2) we have

$$\begin{aligned}
& N[(U^m + \overline{r})^{\frac{1}{m}}] \\
\geq\; & \frac{1}{m}(U^m + \overline{r})^{\frac{1}{m}-1}\overline{r}' + [f(U) - f((U^m + \overline{r})^{\frac{1}{m}})] \\
\geq\; & \frac{1}{m}(U^m + \overline{r})^{\frac{1}{m}-1}\overline{r}' + \mu_3[(U^m + \overline{r})^{\frac{1}{m}} - U] \\
=\; & \frac{1}{m}(U^m + \overline{r})^{\frac{1}{m}-1}\overline{r}' + \mu_3[U^m(U^m + \overline{r})^{\frac{1}{m}-1} + (U^m + \overline{r})^{\frac{1}{m}-1}\overline{r} - U^m(U^m)^{\frac{1}{m}-1}] \\
\geq\; & \frac{1}{m}(U^m + \overline{r})^{\frac{1}{m}-1}\overline{r}' + \mu_3(U^m + \overline{r})^{\frac{1}{m}-1}\overline{r} \\
=\; & \frac{1}{m}(U^m + \overline{r})^{\frac{1}{m}-1}[\overline{r}' + m\mu_3\overline{r}].
\end{aligned}$$

Since $\Psi(1,0) = -f'(1) > 0$ and $\Psi(u,\overline{r})$ is continuous in $(u,\overline{r})$ on $\overline{\Gamma}$, there exists $\delta_2 > 0$ such that $\Psi(u,\overline{r}) \geq \mu_4$ on the set

$$\{(u,\overline{r})|0 \leq \overline{r} \leq r_2, 1 - \delta_2 \leq u \leq 1 \text{ and } u^m + \overline{r} \leq 1\}$$

for some $\mu_4 > 0$. So

$$f(u) - f((u^m + \overline{r})^{\frac{1}{m}}) \geq \mu_4((u^m + \overline{r})^{\frac{1}{m}} - u)$$

on the set

$$\{(u,\overline{r})|0 \leq \overline{r} \leq r_2, 1 - \delta_2 \leq u \leq 1 \text{ and } u^m + \overline{r} \leq 1\}.$$

For $1 - \delta_2 \leq U \leq 1$, $0 \leq \overline{r} \leq r_2$ and $U^m + \overline{r} \leq 1$, by the same argument as above, we have

$$\begin{aligned} & N[(U^m+\overline{r})^{\frac{1}{m}}] \\ \geq\ & \frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r}'+\mu_4[(U^m+\overline{r})^{\frac{1}{m}}-U] \\ \geq\ & \frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r}'+\mu_4(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r} \\ =\ & \frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}[\overline{r}'+m\mu_4\overline{r}]. \end{aligned}$$

Let $0<\overline{\mu}<\min\{m\mu_3, m\mu_4\}$. Note that we may take $\overline{\mu}=\mu$. Solving the problem: $\overline{r}'+\mu\overline{r}=0$ with $\overline{r}(0)=r_2$, we find that $\overline{r}(t)=r_2e^{-\mu t}$. Hence $N[(U^m+\overline{r})^{\frac{1}{m}}]\geq 0$ for the above two cases.

For $\eta_2\leq U\leq 1-\delta_2$, there exists a $\overline{\beta}>0$ such that $U'\geq\overline{\beta}$. Note that we may take $\overline{\beta}=\beta$. There exists $M=\max_{[0,1]}|f'|$ and $0<\theta_8<1$ such that

$$\begin{aligned} & N[(U^m+\overline{r})^{\frac{1}{m}}] \\ \geq\ & \Big(\frac{U^m+\overline{r}}{U^m}\Big)^{\frac{1}{m}-1}(-\overline{\xi}'U')+\frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r}'+[f(U)-f((U^m+\overline{r})^{\frac{1}{m}})] \\ \geq\ & (U^m+\overline{r})^{\frac{1}{m}-1}\cdot 1\cdot(-\overline{\xi}'\beta)+\frac{1}{m}(U^m+\overline{r})^{\frac{1}{m}-1}\overline{r}'-M[(U^m+\overline{r})^{\frac{1}{m}}-U] \\ =\ & (U^m+\overline{r})^{\frac{1}{m}-1}(-\overline{\xi}'\beta)+\frac{-\mu\overline{r}}{m}(U^m+\overline{r})^{\frac{1}{m}-1}-\frac{M\overline{r}}{m}(U^m+\theta_8\overline{r})^{\frac{1}{m}-1} \\ \geq\ & (U^m+\overline{r})^{\frac{1}{m}-1}(-\overline{\xi}'\beta)-\frac{\mu\overline{r}}{m}(U^m+\overline{r})^{\frac{1}{m}-1}-\frac{M\overline{r}}{m}(U^m+\overline{r})^{\frac{1}{m}-1} \\ =\ & \beta(U^m+\overline{r})^{\frac{1}{m}-1}\Big(-\overline{\xi}'-\frac{(\mu+M)\overline{r}}{m\beta}\Big) \\ \geq\ & 0 \end{aligned}$$

if $-\overline{\xi}'-\frac{(\mu+M)\overline{r}}{m\beta}\geq 0$. In particular, we may take $\overline{\xi}'=-\frac{(\mu+M)\overline{r}}{m\beta}$ with $\overline{\xi}(0)=\overline{z^*}$. Then $\overline{\xi}(t)=\alpha_2+\beta_2e^{-\mu t}$, where $\beta_2=\frac{(\mu+M)r_2}{m\mu\beta}$, $\alpha_2=\overline{z^*}-\beta_2$. Hence $N[\overline{u}]\geq 0$ whenever $\overline{u}<1$. By our condition on $\overline{z^*}$,

$$\begin{aligned} \overline{u}(x,0) &= (\min\{1, U^m(x-\overline{\xi}(0))+\overline{r}(0)\})^{\frac{1}{m}} \\ &= (\min\{1, U^m(z-\overline{z^*})+r_2\})^{\frac{1}{m}} \\ &\geq \phi(z) \end{aligned}$$

$$= \phi(x).$$

Thus by the maximum principle

$$\overline{u}(x,t) \geq u(x,t).$$

Note that $\overline{\xi}(t)$ is decreasing and $\lim_{t\to+\infty}\overline{\xi}(t) = \alpha_2$. Therefore, we have

$$(\min\{1, U^m(x-ct-\alpha_2) + r_2e^{-\mu t}\})^{\frac{1}{m}} \geq \overline{u}(x,t).$$

This proves the right-hand inequality for the case $c \leq 0$.

The following theorem yields the stability of traveling wave solutions for the case $c \leq 0$.

Theorem 3.2. *Suppose that $c \leq 0$. Then there exists a function $\omega_0(\delta)$, defined for small positive δ, with $\lim_{\delta\to0^+}\omega_0(\delta) = 0$ and the property that if $0 \leq \phi \leq 1$ and $|\phi(x) - U(x-z_0)| < \delta$ for some z_0, then*

$$|u(x,t) - U(x-ct-z_0)| < \omega_0(\delta).$$

for all x and all $t > 0$.

Proof. In the proof of Lemma 3.1, the constants r_1, r_2, z^* and $\bar{z^*}$ are defined so that

$$U^m(z-z^*) - r_1 \leq \phi^m(z) \leq U^m(z-\bar{z^*}) + r_2, \quad \forall z. \tag{3.3}$$

Fix $\delta > 0$ small. Since

$$|\phi(z) - U(z-z_0)| < \delta, \quad \forall z,$$

we have

$$U(z-z_0)-\delta<\phi(z)<U(z-z_0)+\delta,\quad \forall z.$$

Since

$$(t+\delta)^m\le t^m+\delta^m,\quad \forall t\in[0,1],$$

we may choose $\bar{z^*}=z_0$ and $r_2=r_2(\delta)=\delta^m$ in (3.3). Note that $\lim_{\delta\to 0^+} r_2(\delta)$ $=0$.

Since $U'>0$ and $U\in[0,1]$, there exists z_5 such that $U(z-z_0)\ge\delta$, $\forall z\ge z_5$, and $U(z-z_0)<\delta$, $\forall z<z_5$. Since

$$(t-\delta)^m\ge t^m-\delta^m,\quad \forall t\in[\delta,1],$$

we have

$$\phi^m(z)\ge U^m(z-z_0)-\delta^m,\quad \forall z\ge z_5.$$

Since $U(z-z_0)<\delta$, $\forall z<z_5$, and $\phi\ge 0$, we obtain

$$\phi^m(z)>U^m(z-z_0)-\delta^m,\quad \forall z<z_5.$$

Thus we may choose $z^*=z_0$ and $r_1=r_1(\delta)=\delta^m$ in (3.3). Note that $\lim_{\delta\to 0^+} r_1(\delta)=0$.

Recall from the proof of Lemma 3.1 that

$$\begin{aligned}
&r(t)=r_1e^{-\mu t}\\
&\xi(t)=\alpha_1+\beta_1e^{-\mu t},\quad \text{where}\ \ \beta_1=-\frac{(\mu+M)r_1}{m\mu\kappa_0}\quad \text{and}\quad \alpha_1=z_0-\beta_1\\
&\bar r(t)=r_2e^{-\mu t}\\
&\bar\xi(t)=\alpha_2+\beta_2e^{-\mu t},\quad \text{where}\ \ \beta_2=\frac{(\mu+M)r_2}{m\mu\beta}\quad \text{and}\quad \alpha_2=z_0-\beta_2
\end{aligned}$$

for some positive constants μ, β, M, and κ_0.

By Lemma 3.1, we have

$$(\max\{0, U^m(x-ct-\alpha_1)-r_1e^{-\mu t}\})^{\frac{1}{m}} \le u(x,t)$$
$$\le (\min\{1, U^m(x-ct-\alpha_2)+r_2e^{-\mu t}\})^{\frac{1}{m}}.$$

So we have

$$U^m(x-ct-\alpha_1)-r_1 \le u^m(x,t) \le U^m(x-ct-\alpha_2)+r_2.$$

Let $d_2=(1+r_2)^{1/m}-1$. Then

$$(t^m+r_2)^{1/m} \le t+d_2, \quad \forall t \in [0,1].$$

Thus

$$u(x,t) \le (U^m(x-ct-\alpha_2)+r_2)^{\frac{1}{m}} \le U(x-ct-\alpha_2)+d_2, \quad \forall x, \ \ \forall t>0.$$

Since $U'>0$ and $U \in [0,1]$, there exists z_6 such that $U(z-\alpha_1) \ge \delta$, $\forall z \ge z_6$, and $U(z-\alpha_1)<\delta$, $\forall z<z_6$. Since $(t^m-r_1)^{\frac{1}{m}} \ge t-b_1$, $\forall t \in [\delta,1]$, where $b_1=1-(1-r_1)^{\frac{1}{m}}$, we have

$$u(x,t) \ge (U^m(x-ct-\alpha_1)-r_1)^{\frac{1}{m}} \ge U(x-ct-\alpha_1)-b_1, \quad \forall x-ct \ge z_6.$$

Since $U(x-ct-\alpha_1)<\delta$, $\forall x-ct<z_6$, and $u \ge 0$, we obtain that

$$u(x,t) > U(x-ct-\alpha_1)-\delta, \quad \forall x-ct<z_6.$$

Let $d_1=\max\{1-(1-r_1)^{\frac{1}{m}}, \delta\}$. Then

$$u(x,t) \ge U(x-ct-\alpha_1)-d_1, \quad \forall x, \ \ \forall t>0.$$

We conclude that

$$(3.4)\ U(x-ct-\alpha_1)-d_1 \le u(x,t) \le U(x-ct-\alpha_2)+d_2, \quad \forall x, \quad \forall t>0,$$

where $d_i = d_i(\delta)$ and $\lim_{\delta\to 0^+} d_i(\delta)=0$, $i=1,2$.

Since

$$\lim_{z\to\pm\infty}(U^m(z))'=0, \quad \text{i.e.,} \quad \lim_{z\to\pm\infty} mU^{m-1}(z)U'(z)=0,$$

and

$$\lim_{z\to+\infty} U^{1-m}(z)=1, \quad \lim_{z\to-\infty} U^{1-m}(z)=0,$$

we obtain that $\lim_{z\to\pm\infty} U'(z)=0$. Also, U' is continuous. Thus there exists a $\widetilde{M}>0$ such that $|U'(z)|\le\widetilde{M}$, $\forall z$. The mean value theorem implies that

$$|U(x-ct-\alpha_1)-U(x-ct-z_0)| \le \widetilde{M}\ |\beta_1|,$$
$$|U(x-ct-\alpha_2)-U(x-ct-z_0)| \le \widetilde{M}\ |\beta_2|.$$

It follows from (3.4) that

$$|u(x,t)-U(x-ct-z_0)| \le \max\left\{\frac{\widetilde{M}\ (\mu+M)r_1}{m\mu\kappa_0}+d_1, \frac{\widetilde{M}\ (\mu+M)r_2}{m\mu\beta}+d_2\right\}.$$

Then the function $\omega_0=\omega_0(\delta)$ defined by

$$\omega_0(\delta)=\max\left\{\frac{\widetilde{M}\ (\mu+M)r_1(\delta)}{m\mu\kappa_0}+d_1(\delta), \frac{\widetilde{M}\ (\mu+M)r_2(\delta)}{m\mu\beta}+d_2(\delta)\right\}$$

is the desired function. This proves the theorem.

Remark. Unfortunately, we are unable to verify the stability for the case $c>0$. Note that the function $g(v):=f(v^{1/m})$ defined in Section 2 has the properties that $g'(1)<0$ and $g'(0)=0$. Then $u=0$ is a degenerate fixed point. Hence as it expected that $u = 0$ is *less stable* than $u = 1$. If Lemma 3.1 were true, then $\lim_{t\to\infty} u(x,t) = 0$ for each $x \in \mathbf{R}$. We are not sure whether this is possible or not. On the other hand, from Lemma 3.1 it follows that $\lim_{t\to\infty} u(x,t) = 1$ for each $x \in \mathbf{R}$, if $c < 0$.

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