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THE PRODUCT FORMULA OF MULTIPLE LÉVY-ITÔ INTEGRALS

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Abstract. In this paper, we apply the stochastic representation of exponential vector functionals to derive the product formula of multiple Lévy-Itô integrals. It unifies the Wiener multiplication formula and the Poisson multiplication formula.

1. Introduction. Let $\mathcal{F}_s(L^2_c(\mathbb{R}, dt))$ be the boson Fock space over the space $L^2_c(\mathbb{R}, dt)$ of the complex-valued square integrable functions with respect to the Lebesgue measure dt. The Wiener-Itô theorem asserts that every square integrable functionals of the Brownian motion W can be written as the orthogonal direct sum of the multiple stochastic integrals with respect to W, the same is also true for the Possion process P. It asserts that $\mathcal{F}_s(L^2_c(\mathbb{R}, dt))$ is unitarily isomorphic to $L^2(\Omega_W)$ and $L^2(\Omega_P)$, where $L^2(\Omega_W)$ and $L^2(\Omega_P)$ denote respectively the L^2-spaces of square integrable functions with respect to the Wiener space Ω_W and the Poisson space Ω_P. Though this sets up a 1-1 correspondence between $L^2(\Omega_W)$ and $L^2(\Omega_P)$, yet the calculus of their functionals remains different, for example, the product formula which establish the rule of multiplication for two random variables in $L^2(\Omega_W)$ is different from the formula in $L^2(\Omega_P)$. Each random variable f in $L^2(\Omega_W)$ (or in $L^2(\Omega_P)$) defines a self-adjoint operator on $L^2(\Omega_W)$ (or

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in $L^2(\Omega_P))$ by the multiplication by f. For instance, the Brownian motion $W = \{W(t) : t \geq 0\}$ can be regarded as a family of self-adjoint operators on $\mathcal{F}_s(L^2_c(\mathbb{R}, dt))$ defined by

$$W(t) = \boldsymbol{a}_t^* + \boldsymbol{a}_t,$$

and the Poisson process $P = \{P(t) : t \geq 0\}$ by

$$P(t) = \boldsymbol{a}_t^* + \boldsymbol{a}_t + \boldsymbol{\lambda}_t + t, \qquad t \geq 0,$$

where $\boldsymbol{a}_t^*$, $\boldsymbol{a}_t$, and $\boldsymbol{\lambda}_t$ respectively denote the creation, annihilation, and conservation operators which are densely defined on $\mathcal{F}_s(L^2_c(\mathbb{R}, dt))$.

Note that both the Brownian motion and the Poisson processes are Lévy processes. It is desirable to establish the product formula of Lévy functionals which unifies both the Wiener and the Poisson multiplication formulas, and this would be helpful for us to construct the quantum Lévy process on a multiple symmetric Fock space.

As the role of the Wiener multiplication formula in white noise analysis, the product formula of multiple Lévy-Itô integrals also provides us a tool in developing the Lévy white noise analysis, for instance, we can use the product formula to define the additive renormalization and the multiplication of generalized Lévy white noise functionals. The related results can be found in our next paper [14]

The contents of this paper is organized as follows. In Section 2, we first review the elements of square integrable Lévy white noise functionals, the details of which are referred to [12] and the references cited there. Then we reprove the Wiener-Itô [7] decomposition theorem by using the reproducing kernel Hilbert space theory. The main product formula of multiple Lévy-Itô integrals will be established in Section 3. We will present this formula in the combinatorial form and in the modified Guichardet's notations concerning the symmetric Fock spaces.

The following notations will be often used in this paper. Let μ be a positive measure on a σ-field $\mathcal{B}(E)$ in a set $E \subseteq \mathbb{R}^2$ (or $\mathbb{R}$). Denote by $\hat{L_c^2}(E^n, \mu^{\otimes n})$, $n \in \mathbb{N}$, the closed subspace containing all symmetric elements of the space $L_c^2(E^n, \mu^{\otimes n})$ of square integrable complex-valued functions on E^n, and by $f_1 \hat{\otimes} \cdots \hat{\otimes} f_n$ the symmetrization of $f_1 \otimes \cdots \otimes f_n$ for $f_i \in L_c^2(E, \mu)$, $i = 1, \ldots, n$, where $f_1 \otimes \cdots \otimes f_n \in L_c^2(E^n, \mu^{\otimes n})$ is defined by $f_1 \otimes \cdots \otimes f_n(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$. For $A \in \mathcal{B}(E)$, χ_A denotes the indicator function of A. For a complex-valued function f on E^n, $\hat{f}$ stands for the symmetrization of f, and $f^*((t_1, u_1), \ldots, (t_n, u_n))$ means that $u_1 \cdots u_n f((t_1, u_1), \ldots, (t_n, u_n))$ for $(t_i, u_i) \in \mathbb{R}^2$, $i = 1, \ldots, n$.

2. Square integrable Lévy white noise functionals. We first review the basic facts of Lévy white noise functionals. For the details, we refer the reader to [12]. Let $X = \{X(t) : t \in \mathbb{R}\}$ be a Lévy process on $\mathbb{R}$ with $X(0) = 0$. That is, X is a right continuous additive process with left limits; and the characteristic function of $X(t) - X(s)$ for $s < t$ is given by

$$\mathbb{E}[\exp[i\, r(X(t) - X(s))]] = \exp[(t - s)\, f_X(r)]. \tag{1}$$

Here, f_X is a $\mathbb{C}$-valued function on $\mathbb{R}$ by

$$f_X(r) = i\, \mu\, r - \frac{\sigma^2 r^2}{2} + \int_{|u|>0} \left(e^{iru} - 1 - \frac{iru}{1+u^2} \right) \frac{1+u^2}{u^2}\, d\beta(u), \tag{2}$$

where β is a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\sigma^2 = \beta(\{0\})$; and μ is a real constant. Let $\mathcal{S}$ be the Schwartz space with dual space $\mathcal{S}'$ consisting of tempered distributions on $\mathbb{R}$. It can be shown that if the first absolute moment $\int_{-\infty}^{+\infty} |u|\, d\beta(u)$ is finite, the functional $\mathcal{C}$ on $\mathcal{S}$, defined by $\mathcal{C}(\eta) = \exp[\int_{-\infty}^{+\infty} f_X(\eta(t))\, dt]$, is positive definite with $\mathcal{C}(0) = 1$ so that, by the Minlos theorem, there exists a unique probability measure Λ on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, called the Lévy white noise measure, defining on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, such that $\mathcal{C}$ is the characteristic functional of Λ. That is, $\mathcal{C}(\eta) = \int_{\mathcal{S}'} \exp[i\,(x,\,\eta)]\,\Lambda(dx)$, where $(\cdot,\cdot)$ is the $\mathcal{S}'$-$\mathcal{S}$ pairing.

Throughout this paper, we devote our attention to the class of the measure β satisfying the absolute moment condition:

$$\int_{-\infty}^{+\infty} |u|^n \, d\beta(u) < +\infty \qquad \text{for all } n \in \mathbb{N}. \tag{3}$$

For example, when $\mu = 0$ and $\beta = \delta_0$, we obtain the Gaussian white noise measure Λ_W; when $\mu = 1/2$ and $\beta = (1/2)\delta_1$, we obtain the Poisson white noise measure Λ_P; when $\mu = \int_0^\infty e^{-u}/(1+u^2)\,du$ and $\beta(E) = \int_{E\cap[0,+\infty)} u\,e^{-u}/(1+u^2)\,du \quad (E \in \mathcal{B}(\mathbb{R}))$, we obtain the Gamma white noise measure Λ_G . It is necessary and sufficient that the condition in (3) holds in order that the space $L^2(\mathcal{S}', \Lambda)$ contains all cylinder polynomial functionals (see [12, Theorem 3.3]).

For any $\eta \in \mathcal{S}$, it is easy to see that the mean and variance of the random variables $(\cdot,\,\eta)$ are separately given by

$$\mathbb{E}[(\cdot,\,\eta)] = \tau_1 \int_{-\infty}^{+\infty} \eta(t)\,dt \quad \text{and} \quad \mathbb{V}[(\cdot,\,\eta)] = \tau_2 \int_{-\infty}^{+\infty} \eta(t)^2\,dt, \tag{4}$$

where $\tau_1 = \mu + \int_{-\infty}^{+\infty} u\,d\beta(u)$ and $\tau_2 = \int_{-\infty}^{+\infty} (1+u^2)\,d\beta(u)$. For an arbitrary $\rho \in L^1 \cap L^2(\mathbb{R}, dt)$, choose a sequence $\{\eta_n\} \subset \mathcal{S}$ so that $\eta_n \to \rho$ in $L^1 \cap L^2(\mathbb{R}, dt)$ under the norm $|\cdot|_{L^1(\mathbb{R},dt)} + |\cdot|_{L^2(\mathbb{R},dt)}$. Then $\{(\cdot,\,\eta_n)\}$ forms a Cauchy sequence in $L^2(\mathcal{S}', \Lambda)$. Denote by $\tilde{\rho}$ the L^2-limit of $\{(\cdot,\,\eta_n)\}$. In particular, when $\rho = \chi_{(s,\,t]}$, $\mathbb{E}[\exp[i\,\tilde{\rho}]]$ is exactly the same as one in (1). This implies that the Lévy process X on the probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \Lambda)$ has a representation by

$$X(t\,;\,x) = \begin{cases} \tilde{\chi}_{[0,\,t]}(x), & \text{if } t \geq 0 \\ -\tilde{\chi}_{[t,\,0]}(x), & \text{if } t < 0,\ x \in \mathcal{S}'. \end{cases}$$

Formally, $\dot{X}(t\,;\,x)(= (d/dt)\,X(t\,;\,x)) = x(t)$ for $x \in \mathcal{S}'$ and $t \in \mathbb{R}$. In other words, the elements of $\mathcal{S}'$ are regarded as the sample paths of Lévy white noises. For this reason, members of $L^2(\mathcal{S}', \Lambda)$ are called square integrable Lévy white noise functionals.

For a real locally convex space V, V_c denotes the complexification of V. For $\eta = \eta_1 + i\,\eta_2 \in L^1_c \cap L^2_c(\mathbb{R}, dt)$ with $\eta_1,\ \eta_2 \in L^1 \cap L^2(\mathbb{R}, dt)$, $\tilde{\eta}$ is defined as $\tilde{\eta}_1 + i\,\tilde{\eta}_2$. It is easy to check that the equalities in (4) still hold.

Let $\mathcal{B}_b(\mathbb{R}^2_*)$ be the class of all bounded Borel sets $E \subset \mathbb{R}^2_* (= \mathbb{R}^2 \setminus \{(t,0) : t \in \mathbb{R}\})$ away from the t-axis. Let β_0 be the Lévy measure and define the product measure $d\nu(t,u) = d\beta_0(u)\,dt$ on $\mathcal{B}(\mathbb{R}^2_*)$, where $d\beta_0 = (1+u^2)/u^2\,d\beta(u)$. For $E \in \mathcal{B}_b(\mathbb{R}^2_*)$, let

$$N(E\,;\,x) = \ |\{(t,u) \in E : X(t\,;\,x) - X(t-\,;\,x) = u\}|.$$

Then $N(E\,;\cdot)$ is Poisson distributed with the intensity measure ν. The system of $\{N_0(E\,;\,x) \equiv N(E\,;\,x) - \nu(E) : E \in \mathcal{B}_b(\mathbb{R}^2_*)\,;\, x \in \mathcal{S}'\}$ forms an independent random measure with zero mean. Let $f = \sum_{i=1}^m a_i \cdot \chi_{E_i}$, a_i's $\in \mathbb{C}$ and E_i's $\in \mathcal{B}_b(\mathbb{R}^2_*)$ being disjoint, be a complex-valued simple function on $\mathbb{R}^2_*$. The stochastic integral $\int_{\mathbb{R}^2_*} f(s)\,dN_0(s)$ defined by $\sum_{i=1}^m a_i\,N_0(E_i)$ haves the isometry

$$\left\| \int_{\mathbb{R}^2_*} f(s)\,dN_0(s) \right\|^2 = \int_{\mathbb{R}^2_*} |f(s)|^2\,d\nu(s).$$

Here and in what follows, $\|\cdot\|$ will denote the norm of $L^2(\mathcal{S}', \Lambda)$ associated with the inner product $\ll \cdot, \cdot \gg$. Extend the above isometry by continuity the integral $\int_{\mathbb{R}^2_*} f(s)\,dN_0(s)$ is then defined for all $f \in L^2_c(\mathbb{R}^2_*, \nu)$.

By the well-known Lévy-Itô decomposition theorem (see [6]), there exists a Wiener process $B = \{B(t) : t \in \mathbb{R}\}$, independent to the system of $\{N(E) : E \in \mathcal{B}_b(\mathbb{R}^2_*)\}$, so that for $b > a$,

$$(5)\quad X(b) - X(a) = \tau_1(b-a) + \sigma(B(b) - B(a)) + \int_{\mathbb{R}^2_*} u\chi_{(a,b]\times\mathbb{R}_*}(t,u)\,dN_0(t,u).$$

Let λ be a positive measure on $\mathcal{B}(\mathbb{R}^2)$ given by $d\lambda(t,u) = (1+u^2)\,d\beta(u)\,dt$. Define a $L^2(\mathcal{S}',\Lambda)$-valued function M on $\{E \in \mathcal{B}(\mathbb{R}^2) : \lambda(E) < +\infty\}$ by

$$M(E) = \sigma \int_{-\infty}^{+\infty} \chi_E(t,0)\,dB(t) + \int_{\mathbb{R}^2_*} u\,\chi_E(t,u)\,dN_0(t,u).$$

Then the system of $\{M(E\,;\,x) : E \in \mathcal{B}(\mathbb{R}^2)$ with $\lambda(E) < +\infty;\ x \in \mathcal{S}'\}$ forms an independent random measure with zero mean.

Next we briefly introduce the multiple stochastic integrals with respect to M, initiated by Itô [7]. Let V_n denote the class of all complex-valued symmetric simple functions on $(\mathbb{R}^2)^n$ of the form

$$\sum_{i=1}^{m} a_i \cdot \hat{\chi}_{E_1^i \times \cdots \times E_n^i} \tag{6}$$

in which $\lambda(E_j^i) < +\infty$ for $j = 1, \ldots, n$ and $E_1^i < \cdots < E_n^i$. Here, "$A < B$" means that $t_1 < t_2$ for any $(t_1,u_1) \in A$ and $(t_2,u_2) \in B$. For any $g \in V_n$ of the form (6), the multiple stochastic integral $I_n(g)$ of g with respect to M is defined by $\sum_{i=1}^m a_i \prod_{j=1}^n M(E_j^i)$. It satisfies the isometry

$$\|I_n(g)\|^2 = n! \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} |g(s_1,\ldots,s_n)|^2\,d\lambda(s_1)\cdots d\lambda(s_n). \tag{7}$$

By the denseness of V_n in $\hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$, $I_n(g)$ is extended by continuity to all $g \in \hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$, called the multiple Lévy-Itô integral of order n with respect to M. In notation, we write

$$I_n(g) = \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} g(s_1,\ldots,s_n)\,dM(s_1)\cdots dM(s_n).$$

From the definition, it follows that $I_n(g)$ and $I_m(h)$ are orthogonal to each other as $n \neq m$ for $g \in \hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$ and $h \in \hat{L_c^2}((\mathbb{R}^2)^m, \lambda^{\otimes m})$. Regard $L_c^2(\mathbb{R}, dt)$ as a subset of $L_c^2(\mathbb{R}^2, \lambda)$ by identifying $\eta \in L_c^2(\mathbb{R}, dt)$ with $\eta \otimes 1$, where $\eta \otimes h(t,u) = \eta(t)\, h(u)$ for any $(t,u) \in \mathbb{R}^2$. Then

$$\tilde{\eta} = \tau_1 \int_{-\infty}^{+\infty} \eta(t)\, dt + I_1(\eta). \tag{8}$$

For more details, we refer the reader to [12, Proposition 3.9].

The following formula is important in the developments of Lévy white noise analysis.

Theorem 2.1.([12, Theorem 5.3]) *Let* $h \in \hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$ *and let* $g \in L_c^2(\mathbb{R}^2, \lambda)$ *satisfy*

$$\int_{\mathbb{R}_*^2} (e^{2|u\, g(t,u)|} - 1 - 2|u\, g(t,u)|)\, d\nu(t,u) < +\infty.$$

Then

$$\begin{aligned}\int_{\mathcal{S}'} I_n(h)(x)\, e^{I_1(g)(x)}\, \Lambda(dx) &= \mathbb{E}[e^{I_1(g)}] \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} h((t_1,u_1),\ldots,(t_n,u_n)) \\ &\quad \times \prod_{j=1}^{n} \frac{e^{u_j g(t_j,u_j)}-1}{u_j} d\lambda(t_1,u_1)\cdots d\lambda(t_n,u_n),\end{aligned}$$

where the value of the function $(e^{ux}-1)/u$ *is defined to be* x *at* $u=0$.

In general, the Lévy processes X do not possess the chaotic representation property (CRP) which means that, for each f in $L^2(\mathcal{S}', \Lambda)$, there is a sequence $\{f_n\}_{n=0}^{\infty}$ such that $f_n \in \hat{L_c^2}(\mathbb{R}^n, (dt)^{\otimes n})$ and $f = \sum_{n=0}^{\infty} I_n^X(f_n)$, where $I_n^X(f_n)$ is the multiple stochastic integral with respect to the martingale $\{X(t) - \tau_1 t : t \in \mathbb{R}\}$. For instance, the Gamma process has no CRP (see [9, 14]). Itô in [7] showed that every member of $L^2(\mathcal{S}', \Lambda)$ admits an orthogonal expansion with respect to M. Further, we show that this expansion may be linked with the theory of reproducing kernel Hilbert space.

Define an integral transform $\mathcal{U}$ on $L^2(\mathcal{S}', \Lambda)$ by

$$\mathcal{U}\varphi(\eta) = \frac{1}{\mathcal{C}(\eta)} \int_{\mathcal{S}'} \varphi(x)\, e^{i(x,\eta)}\, \Lambda(dx),$$

for $\varphi \in L^2(\mathcal{S}', \Lambda)$ and $\eta \in \mathcal{S}$. Since $\{e^{i(\cdot,\eta)} : \eta \in \mathcal{S}\}$ is dense in $L^2(\mathcal{S}', \Lambda)$, $\mathcal{U}$ is one to one. Let $\mathcal{W} = \{\mathcal{U}\varphi : \varphi \in L^2(\mathcal{S}', \Lambda)\}$. Then $\mathcal{W}$ is a linear space. Endow $\mathcal{W}$ with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ by

$$\langle \mathcal{U}\varphi, \mathcal{U}\psi \rangle_{\mathcal{W}} = \int_{\mathcal{S}'} \varphi(x)\, \overline{\psi(x)}\, \Lambda(dx).$$

Then $(\mathcal{W}, \langle \cdot, \cdot \rangle_{\mathcal{W}})$ is obviously a complex Hilbert space unitarily isomorphic to $L^2(\mathcal{S}', \Lambda)$.

For a fixed $\eta \in \mathcal{S}$, the point evaluation $\varphi \mapsto \mathcal{U}\varphi(\eta)$ is a bounded linear functional. By Riesz representation theorem, there exists a unique functional $\mathbf{h}_\eta$ in $L^2(\mathcal{S}', \Lambda)$ so that

$$\mathcal{U}\varphi(\eta) = \int_{\mathcal{S}'} \varphi(x)\, \overline{\mathbf{h}_\eta(x)}\, \Lambda(dx) = \langle \mathcal{U}\varphi, \mathcal{U}(\mathbf{h}_\eta) \rangle_{\mathcal{W}}.$$

In fact $\mathbf{h}_\eta(x) = e^{-i(x,\eta)}/\mathcal{C}(-\eta)$ for any $\eta \in \mathcal{S}$ and $\mathcal{W}$ is a reproducing kernel Hilbert space with the reproducing kernel $K(\eta, \zeta)$ for any $\eta, \zeta \in \mathcal{S}$, given by

$$K(\eta, \zeta) = \int_{\mathcal{S}'} \mathbf{h}_\zeta(x)\, \overline{\mathbf{h}_\eta(x)}\, \Lambda(dx).$$

By a direct computation, we see that

$$K(\eta, \zeta) = \exp\Big[\int_{\mathbb{R}^2} \Big(\frac{e^{-iu\zeta(t)} - 1}{u} \Big) \Big(\frac{e^{iu\eta(t)} - 1}{u} \Big)\, d\lambda(t, u) \Big], \tag{9}$$

for any $\eta, \zeta \in \mathcal{S}$, where the value of the function $(e^{iux} - 1)/u$ is defined to be ix at $u = 0$.

For $n \in \mathbb{N}$, let

$$K_n(\eta, \zeta) = \frac{1}{n!} \Big(\int_{\mathbb{R}^2} \Big(\frac{e^{-iu\zeta(t)} - 1}{u} \Big) \Big(\frac{e^{iu\eta(t)} - 1}{u} \Big)\, d\lambda(t, u) \Big)^n,$$

and let $K_0(\eta, \zeta) \equiv 1$ for any $(\eta, \zeta) \in \mathcal{S} \times \mathcal{S}$. By Theorem 2.1, we see that

$$\mathcal{U}(I_n(f_{n,\zeta})) = K_n(\cdot, \zeta), \tag{10}$$

where

$$f_{n,\zeta}((t_1,u_1),\ldots,(t_n,u_n)) = \frac{1}{n!}\prod_{j=1}^{n}\frac{e^{-iu_j\zeta(t_j)}-1}{u_j}. \tag{11}$$

For $n \neq m$, $K_m(\cdot,\eta)$ is orthogonal to $K_n(\cdot,\zeta)$ in $\mathcal{W}$ for any $\eta,\zeta \in \mathcal{S}$. Now, let $\mathcal{W}^{(n)}$ be the closed subspace of $\mathcal{W}$ generated by $\{K_n(\cdot,\zeta) : \zeta \in \mathcal{S}\}$, if $n \in \mathbb{N}$; and $\mathcal{W}^{(0)} = \mathbb{C}$. Since $\{e^{i(\cdot,\eta)} : \eta \in \mathcal{S}\}$ is total in $L^2(\mathcal{S}',\Lambda)$, $\mathcal{W}$ is the completion of the pre-Hilbert space spanned by $\{K(\cdot,\zeta) : \zeta \in \mathcal{S}\}$. Also $K(\cdot,\zeta) = \sum_{n=0}^{\infty} K_n(\cdot,\zeta)$. Thus,

$$\mathcal{W} = \sum_{n=0}^{\infty} \oplus \mathcal{W}^{(n)},$$

where $\sum\oplus$ denotes the orthogonal direct sum. From (10) it follows that $\mathcal{W}^{(n)}$ is contained in $\mathcal{N} = \{\mathcal{U}(I_n(f)) : f \in \hat{L}^2_c((\mathbb{R}^2)^n,\lambda^{\otimes n})\}$. On the other hand, taking $f \in \hat{L}^2_c((\mathbb{R}^2)^n,\lambda^{\otimes n})$ and $\eta \in \mathcal{S}$, we see that for $m \neq n$,

$$\langle \mathcal{U}(I_n(f)),\, K_m(\cdot,\eta)\rangle_{\mathcal{W}} = \langle \mathcal{U}(I_n(f)), \mathcal{U}(I_m(f_{m,\eta}))\rangle_{\mathcal{W}} = 0,$$

where $f_{m,\eta}$ is defined as in (11). It implies that $\mathcal{U}(I_n(f)) \in \mathcal{W}^{(n)}$. Therefore, $\mathcal{N} = \mathcal{W}^{(n)}$. As a result, we have established the following orthogonal decomposition theorem.

Theorem 2.2.(Itô, [7]) *For $\varphi \in L^2(\mathcal{S}',\Lambda)$, φ can be expressed as an orthogonal direct sum*

$$\varphi = \sum_{n=0}^{\infty} \oplus I_n(\phi_n),$$

where $\phi_n \in \hat{L}^2_c((\mathbb{R}^2)^n,\lambda^{\otimes n})$ for any $n \in \mathbb{N}$. In notation, we write $\varphi \sim (\,\phi_n\,)$.

Theorem 2.2 implies that $L^2(\mathcal{S}',\Lambda)$ is a probabilistic realization of the boson Fock space $\mathcal{F}_s(L^2_c(\mathbb{R}^2,\lambda))$ over $L^2_c(\mathbb{R}^2,\lambda)$ by a unitary isomorphism Ψ

given by

$$\Psi(\varphi) = \bigoplus_{n=0}^{\infty} \sqrt{n!}\,\phi_n \quad \text{for } \varphi \sim (\,\phi_n\,) \text{ in } L^2(\mathcal{S}', \Lambda). \tag{12}$$

3. Lévy product formula. In this section, we shall derive the product formula of multiple Lévy-Itô integrals. It will unify the Wiener and Poisson multiplication formula (see [15]). To begin with we introduce the stochastic representation of exponential vector (or coherent state) functionals (see [12] or Theorem 3.1 below) which plays a key role. Recall that for any $g \in L_c^2(\mathbb{R}^2, \lambda)$, the exponential vector functional $\mathcal{E}_M(g)$ associated with g is defined by

$$\mathcal{E}_M(g) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g^{\otimes n}).$$

Theorem 3.1.([12, Theorem 5.8]) *For any $g \in L_c^2(\mathbb{R}^2, \lambda)$, define*

$$\vartheta_g(x) = \exp\Big[-\frac{\sigma^2}{2}\int_{-\infty}^{+\infty} g(t,0)^2\,dt + \sigma\int_{-\infty}^{+\infty} g(t,0)\,dB(t;\,x)\Big]; \tag{13}$$

and let

$$\gamma_g(x) = \exp\Big[-\int_{\mathbb{R}^2_*} g^*(x)\,d\nu(x)\Big]\prod_{t\in J_X(x)}(1+g^*(t, X(t;x)-X(t-;x))), \tag{14}$$

if g^* lies in $L_c^2 \cap L_c^\infty(\mathbb{R}^2_*, \nu)$; otherwise, γ_g is defined to be the L^2-limit of $\{\gamma_{g_m}\}$ for any sequence $\{g_m\} \subset \mathcal{O}$ such that $g_m \to g$ in $L_c^2(\mathbb{R}^2, \lambda)$ for Λ-almost all $x \in \mathcal{S}'$, where $\mathcal{O} = \{g \in L_c^2(\mathbb{R}^2, \lambda) : g^* \in L_c^1 \cap L_c^\infty(\mathbb{R}^2_*, \nu)\}$ and $J_X(x)$ is the set $\{t \in \mathbb{R} : X(t;\,x) - X(t-;\,x) \neq 0\}$. Then

$$\mathcal{E}_M(g) = \gamma_g \cdot \vartheta_g.$$

By applying Theorem 3.1, we get the following formula for the pointwise multiplication of two exponential vector functionals.

Proposition 3.2. *Let $f, g \in L_c^2(\mathbb{R}^2, \lambda)$ such that $(fg)^* \in L_c^2(\mathbb{R}^2, \lambda)$. Then, for Λ-almost all $x \in \mathcal{S}'$,*

$$\mathcal{E}_M(f)(x) \cdot \mathcal{E}_M(g)(x) = \exp\left[\int_{\mathbb{R}^2} f(s)g(s)d\lambda(s)\right] \cdot \mathcal{E}_M(f+g+(fg)^*)(x). \tag{15}$$

Proof. First, assume that both f^* and g^* are in $L_c^1 \cap L_c^\infty(\mathbb{R}_*^2, \nu)$. Then $(f+g+(fg)^*)^*$ is also in $L_c^1 \cap L_c^\infty(\mathbb{R}_*^2, \nu)$. By Theorem 3.1, the formula (15) follows immediately. For the general case, we put

$$E_k = A_k \cap \{s \in \mathbb{R}^2 : |f(s)| \leq k\} \quad \text{and} \quad F_k = A_k \cap \{s \in \mathbb{R}^2 : |g(s)| \leq k\},$$

where

$$A_k = \{(t, u) \in \mathbb{R}^2 : \ |u| \geq 1/k \text{ or } u = 0\}.$$

Let $f_k = f \cdot \chi_{E_k}$ and $g_k = g \cdot \chi_{F_k}$ for any $k \in \mathbb{N}$. Then $f_k \to f$, $g_k \to g$, and $(f_k\, g_k)^* \to (fg)^*$ in $L_c^2(\mathbb{R}^2, \lambda)$. This implies that $\mathcal{E}_M(f_k) \to \mathcal{E}_M(f)$, $\mathcal{E}_M(g_k) \to \mathcal{E}_M(g)$, and $\mathcal{E}_M(f_k+g_k+(f_k\, g_k)^*) \to \mathcal{E}_M(f+g+(fg)^*)$ in $L^2(\mathcal{S}', \Lambda)$. Note that the family $\{f_k^*, g_k^* : k \in \mathbb{N}\}$ is contained in $L_c^1 \cap L_c^\infty(\mathbb{R}_*^2, \nu)$. Thus the formula (15) hold when $f = f_k$ and $g = g_k$ for any $k \in \mathbb{N}$. Then, we complete the proof by taking limit as $k \to \infty$.

Lemma 3.3. *Let $f, g \in L_c^2(\mathbb{R}^2, \lambda)$. Then the $L^2(\mathcal{S}', \Lambda)$-valued function $\mathcal{E}_M(z\, f + g)$ with one complex variable z is analytic on $\mathbb{C}$. Moreover, for any $k \in \mathbb{N}$,*

$$\left.\frac{d^k}{dz^k}\right|_{z=0} \mathcal{E}_M(z\, f + g) = \ k! \sum_{n=k}^{\infty} \frac{1}{n!} \binom{n}{k} I_n(f^{\otimes k} \,\hat{\otimes}\, g^{\otimes(n-k)}). \tag{16}$$

Proof. Let $\varphi \sim (\, \phi_n \,)$ be arbitrarily chosen in $L^2(\mathcal{S}', \Lambda)$. Then

$$\begin{aligned} \ll \mathcal{E}_M(z\, f + g),\, \varphi \gg \ &= \ \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} (z\, f + g)^{\otimes n}(s_1, \ldots, s_n) \\ &\quad \times \phi_n(s_1, \ldots, s_n)\, d\lambda(s_1) \cdots d\lambda(s_n). \end{aligned} \tag{17}$$

Denote by $S_\varphi(z)$ the sum in the right hand side of (17). It is clear that S_φ is analytic on $\mathbb{C}$, and thus the function $z \to \mathcal{E}_M(z\,f+g)$ is analytic on $\mathbb{C}$. Applying the binomial theorem to $(z\,f+g)^{\otimes n}$, (17) becomes that

$$\begin{aligned} S_\varphi(z) &= \sum_{k=0}^\infty z^k \left\{ \sum_{n=k}^\infty \frac{1}{n!}\binom{n}{k}, \ll I_n(f^{\otimes k}\,\hat{\otimes}\,g^{\otimes(n-k)},\,\varphi \gg \right\} \\ &= \sum_{k=0}^\infty \frac{z^k}{k!}\left\langle\!\!\left\langle \left.\frac{d^k}{dz^k}\right|_{z=0} \mathcal{E}_M(z\,f+g),\ \varphi \right\rangle\!\!\right\rangle. \end{aligned}$$

Then the formula (16) is verified.

Replace f by $z\,f$, $z\in\mathbb{C}$, in (15), and differentiate n times in the variable z by using Leibniz rule. Then, letting $z=0$ and applying Lemma 3.3, it leads to

$$\begin{aligned} & I_n(f^{\otimes n})\cdot\mathcal{E}_M(g) \\ = & \sum_{k=0}^n \binom{n}{k}\langle f,\,g\rangle^k \\ & \times\left\{ \sum_{m=n-k}^\infty \frac{(n-k)!}{m!}\binom{m}{n-k} I_m((f+(fg)^*)^{\otimes(n-k)}\hat{\otimes} g^{\otimes(m-n+k)}) \right\} \\ = & \sum_{k=0}^n \binom{n}{k}\langle f,g\rangle^k \left\{ \sum_{m'=0}^\infty \frac{1}{(m')!} I_{m'+n-k}((f+(fg)^*)^{\otimes(n-k)}\hat{\otimes} g^{\otimes(m')}) \right\}, \end{aligned} \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes the bilinear scalar product on $L^2_c(\mathbb{R}^2,\lambda)$. We note that the sum in the brace of (18) is convergent in $L^2(\mathcal{S}',\Lambda)$.

Continuing in this way by replacing g by $z\,g$ ($z\in\mathbb{C}$) in (18), we have

$$\begin{aligned} I_n(f^{\otimes n})\cdot\mathcal{E}_M(z\,g) &= \sum_{k=0}^n \binom{n}{k} z^k\,\langle f,\,g\rangle^k \left\{ \sum_{m'=0}^\infty \sum_{r=0}^{n-k} z^{m'+r} \right. \\ &\qquad \left. \times \frac{\binom{n-k}{r}}{(m')!}\, I_{m'+n-k}(f^{\otimes(n-k-r)}\,\hat{\otimes}\,((f\,g)^*)^{\otimes r}\,\hat{\otimes}\,g^{\otimes(m')}) \right\}. \end{aligned} \tag{19}$$

Assume that $n\le m$. Differentiating both sides of the equation (19) m times in the variable z by using the Leibniz rule and letting $z=0$ we obtain

$$\begin{aligned} I_n(f^{\otimes n})\cdot I_m(g^{\otimes m}) &= m!\sum_{k=0}^{n}\binom{n}{k}\langle f,\,g\rangle^k\left\{\sum_{r=0}^{n-k}\frac{\binom{n-k}{r}}{(m-k-r)!}\right.\\ &\qquad\left.\times I_{m+n-2k-r}(f^{\otimes(n-k-r)}\hat{\otimes}((fg)^*)^{\otimes r}\hat{\otimes}g^{\otimes(m-k-r)})\right\}\\ &= \sum_{k=0}^{n}k!\binom{n}{k}\binom{m}{k}\langle f,\,g\rangle^k\left\{\sum_{r=0}^{n-k}r!\binom{m-k}{r}\binom{n-k}{r}\right.\\ &\qquad\left.\times I_{m+n-2k-r}(f^{\otimes(n-k-r)}\hat{\otimes}((fg)^*)^{\otimes r}\hat{\otimes}g^{\otimes(m-k-r)})\right\}. \end{aligned} \tag{20}$$

For any $(t,u)\in\mathbb{R}^2$, let $p_2(t,u)\equiv u$. For any $n\in\mathbb{N}$, let $A=\{s_1,\ldots,s_n\}\subset\mathbb{R}^2$ with $|A|=n$, where $|A|$ is the cardinality of A. In what follows, for a symmetric measurable function ξ on $(\mathbb{R}^2)^n$, we denote $\xi(A)=\xi(s_1,\ldots,s_n)$ and $d\lambda^{\otimes n}(A)=d\lambda(s_1)\cdots d\lambda(s_n)$.

Let $\phi\in\hat{L}^2_c((\mathbb{R}^2)^n,\lambda^{\otimes n})$ and $\psi\in\hat{L}^2_c((\mathbb{R}^2)^m,\lambda^{\otimes m})$. For $0\le k\le n\wedge m$ and $0\le r\le(n\wedge m)-k$, we define $\phi\,\hat{\otimes}^r_k\,\psi$ to be the symmetrization of the function $\phi\otimes^r_k\psi$, where $\phi\otimes^r_k\psi$ is the function defined on $(\mathbb{R}^2)^{n-k-r}\times(\mathbb{R}^2)^{m-k-r}\times(\mathbb{R}^2)^r$ by

$$\phi\otimes^r_k\psi(X,Y,Z)=\left(\prod_{s\in Z}p_2(s)\right)\int_{(\mathbb{R}^2)^k}\phi(X,Z,A)\psi(A,Z,Y)d\lambda^{\otimes k}(A), \tag{21}$$

for $(X,Y,Z)\in(\mathbb{R}^2)^{n-k-r}\times(\mathbb{R}^2)^{m-k-r}\times(\mathbb{R}^2)^r$, where $\prod_{s\in Z}p_2(s):=u_1\cdot u_2\cdots u_r$ when $Z=((t_1,u_1),\ldots,(t_r,u_r))$. For instance, as $C\subset\mathbb{R}^2$ with $|C|=m+n-2k-r$,

$$\begin{aligned} \phi\,\hat{\otimes}^r_k\,\psi(C) &= \frac{r!\,(m-k-r)!\,(n-k-r)!}{(m+n-2k-r)!}\sum_{\substack{X+Y+Z=C\\ |X|=n-k-r,|Y|=m-k-r}}\\ &\qquad\times\left(\prod_{s\in Z}p_2(s)\right)\int_{(\mathbb{R}^2)^k}\phi(X,Z,A)\,\psi(A,Z,Y)\,d\lambda^{\otimes k}(A), \end{aligned}$$

$X+Y+Z=C$ meaning that C is the disjoint union of X, Y, and Z. In general, $\phi\,\hat{\otimes}^r_k\,\psi$ is not square integrable unless some extra-conditions are posted.

Lemma 3.4. *Let* $\phi \in \hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$ *and* $\psi \in \hat{L_c^2}((\mathbb{R}^2)^m, \lambda^{\otimes m})$.

(i) *For any* $0 \le k \le n \wedge m$ *and* $r = 0$, $\phi \hat{\otimes}_k^0 \psi$ *is in* $\hat{L_c^2}((\mathbb{R}^2)^{m+n-2k}, \lambda^{\otimes(m+n-2k)})$. *Moreover,*

$$|\phi \,\hat{\otimes}_k^0\, \psi|_2 \le\, |\phi \otimes_k^0 \psi|_2 \le\, |\phi|_2\, |\psi|_2.$$

(ii) *Assume that both* ϕ *and* ψ *are bounded functions with compact support. Then, for* $0 \le k \le n \wedge m$ *and* $1 \le r \le (n \wedge m) - k$, $\phi \hat{\otimes}_k^r \psi$ *is in* $\hat{L_c^2}((\mathbb{R}^2)^{m+n-2k-r}, \lambda^{\otimes(m+n-2k-r)})$. *Further, let* $S_M = \{(t,u) : |(t,u)| \le M\}$, $M > 0$, *so that the compact supports* $\operatorname{supp}\phi$ *and* $\operatorname{supp}\psi$ *of* ϕ *and* ψ *are contained in* $(S_M)^n$ *and* $(S_M)^m$ *respectively. Then*

$$|\phi \,\hat{\otimes}_k^r\, \psi|_2^2 \le\, M^{2r} \sqrt{\lambda(S_M)^{m+n-2r}} \cdot |\phi|_\infty\, |\psi|_\infty\, |\phi|_2\, |\psi|_2. \tag{22}$$

Here $|\cdot|_2$ *and* $|\cdot|_\infty$ *stand for the* L^2*- and* L^∞*- norms respectively.*

Proof. The assertion (i) follows immediately by a direct computation. For the assertion (ii), we observe that that

$$\begin{aligned} |\phi \,\hat{\otimes}_k^r\, \psi|_2^2 \;\le\; & \int_{(\mathbb{R}^2)^{m+n-2k-r}} M^{2r} \left| \int_{(\mathbb{R}^2)^k} \phi(X,Z,A)\, \psi(A,Z,Y)\, d\lambda^{\otimes k}(A) \right|^2 \\ & \times d\lambda^{\otimes(n-k-r)}(X)\, d\lambda^{\otimes(m-k-r)}(Y)\, d\lambda^{\otimes r}(Z). \end{aligned} \tag{23}$$

by formula (21). Next applying the Schwarz inequality to the integral in the right-hand of (23), we obtain

$$\begin{aligned} |\phi \,\hat{\otimes}_k^r\, \psi|_2^2 \;\le\; & M^{2r} \int_{(\mathbb{R}^2)^{m+n-2k-r}} \\ & \left\{ \int_{(\mathbb{R}^2)^{2k}} |\phi(X,Z,A)|^2\, |\psi(B,Z,Y)|^2\, d\lambda^{\otimes k}(A)\, d\lambda^{\otimes k}(B) \right\} \\ & d\lambda^{\otimes r}(Z)\, d\lambda^{\otimes(m-k-r)}(X)\, d\lambda^{\otimes(n-k-r)}(Y). \end{aligned} \tag{24}$$

By the Fubini's theorem, the estimation (24) becomes that

$$
\begin{aligned}
(24) \quad \le\ & M^{2r} \cdot |\phi|_\infty\, |\psi|_\infty \int_{(S_M)^{m+n-2r}} \\
& \left\{ \int_{(\mathbb{R}^2)^r} |\phi(X,Z,A)| |\psi(B,Z,Y)|\, d\lambda^{\otimes r}(Z) \right\} \\
& \qquad \times d\lambda^{\otimes (m-k-r)}(X)\, d\lambda^{\otimes (n-k-r)}(Y)\, d\lambda^{\otimes k}(A)\, d\lambda^{\otimes k}(B) \\
\le\ & M^{2r} \cdot |\phi|_\infty |\psi|_\infty \int_{(S_M)^{m+n-2r}} \\
& \left\{ \int_{(\mathbb{R}^2)^{2r}} |\phi(X,Z,A)|^2\, |\psi(B,Z',Y)|^2 d\lambda^{\otimes r}(Z)\, d\lambda^{\otimes r}(Z') \right\}^{1/2} \\
& \qquad \times d\lambda^{\otimes (m-k-r)}(X)\, d\lambda^{\otimes (n-k-r)}(Y)\, d\lambda^{\otimes k}(A)\, d\lambda^{\otimes k}(B) \\
\le\ & M^{2r} \sqrt{\lambda(S_M)^{m+n-2r}} \cdot |\phi|_\infty\, |\psi|_\infty\, |\phi|_2\, |\psi|_2,
\end{aligned}
$$

where the last inequality follows by the Schwarz inequality. Then the assertion (ii) is verified.

Employing the preceding notations, we can rewrite (21) in the manner:

$$
\begin{aligned}
& I_n(f^{\otimes n}) \cdot I_m(g^{\otimes m}) \\
= & \sum_{k=0}^{n} \sum_{r=0}^{n-k} k!\, r! \binom{n}{k} \binom{m}{k} \binom{m-k}{r} \binom{n-k}{r} I_{m+n-2k-r}(f^{\otimes n} \hat{\otimes}_k^r g^{\otimes m}).
\end{aligned}
$$

Applying the polarization identity, we have the following

Theorem 3.5. *Let* $f_1, \ldots, f_n;\ g_1, \ldots, g_m \in L_c^2(\mathbb{R}^2, \lambda)$ *for* $n, m \in \mathbb{N}$. *Assume that* $(f_i\, g_j)^* \in L_c^2(\mathbb{R}^2, \lambda)$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$. *Then*

$$
\begin{aligned}
& I_n(f_1 \hat{\otimes} \cdots \hat{\otimes} f_n) \cdot I_m(g_1 \hat{\otimes} \cdots \hat{\otimes} g_m) \\
= & \sum_{k=0}^{m \wedge n} \sum_{r=0}^{(m \wedge n)-k} k!\, r! \binom{n}{k} \binom{m}{k} \binom{m-k}{r} \binom{n-k}{r} \\
& \times I_{m+n-2k-r}((f_1 \hat{\otimes} \cdots \hat{\otimes} f_n) \hat{\otimes}_k^r (g_1 \hat{\otimes} \cdots \hat{\otimes} g_m)).
\end{aligned}
$$

We are now ready to prove the main product formula.

Theorem 3.6. *Let* $\phi \in \hat{L_c^2}((\mathbb{R}^2)^n, \lambda^{\otimes n})$ *and* $\psi \in \hat{L_c^2}((\mathbb{R}^2)^m, \lambda^{\otimes m})$ *with* $n, m \in \mathbb{N}$. *Assume that* $|\phi| \otimes_k^r |\psi|$ *is in* $L_c^2((\mathbb{R}^2)^{m+n-2k-r}, \lambda^{\otimes(m+n-2k-r)})$ *for* $0 \le k \le n \wedge m$ *and* $0 \le r \le (n \wedge m) - k$. *Then*

$$\begin{aligned} & I_n(\phi) \cdot I_m(\psi) \\ = & \sum_{k=0}^{m\wedge n} \sum_{r=0}^{(m\wedge n)-k} k!r! \binom{n}{k}\binom{m}{k}\binom{m-k}{r}\binom{n-k}{r} I_{m+n-2k-r}(\phi \hat{\otimes}_k^r \psi). \end{aligned} \tag{25}$$

Proof. We divide the proof into two parts. Assume first that both ϕ and ψ are bounded functions with compact support. Let S_M be defined as in Lemma 3.4(ii) so that $\operatorname{supp}\phi \subset (S_M)^n$ and $\operatorname{supp}\psi \subset (S_M)^m$. By Lusin's theorem [17, 2.24], there are two sequences $\{\phi_j\}$ and $\{\psi_j\}$ of complex-valued simple functions on $(\mathbb{R}^2)^n$ and $(\mathbb{R}^2)^m$, given by

$$\phi_j = n! \sum_{i=1}^{\ell_j} a_{i,j} \cdot \chi_{A_1^{i,j}} \hat{\otimes} \cdots \hat{\otimes} \chi_{A_n^{i,j}} \quad \text{and} \quad \psi_j = m! \sum_{i=1}^{\ell_j} b_{i,j} \cdot \chi_{B_1^{i,j}} \hat{\otimes} \cdots \hat{\otimes} \chi_{B_m^{i,j}},$$

in which $A_1^{i,j} < \cdots < A_n^{i,j}$; $B_1^{i,j} < \cdots < B_m^{i,j}$ so that $\operatorname{supp}\phi_j \subset (S_M)^n$ and $\operatorname{supp}\psi_j \subset (S_M)^m$; $|a_{i,j}| \le |\phi|_\infty$ and $|b_{i,j}| \le |\psi|_\infty$; $|\phi_j - \phi|_2 \to 0$ and $|\psi_j - \psi|_2 \to 0$ as $j \to \infty$. Then $I_n(\phi_j) \to I_n(\phi)$ and $I_m(\psi_j) \to I_m(\psi)$ in $L^2(\mathcal{S}', \Lambda)$. Choose subsequences, still denoted by $\{I_n(\phi_j)\}$ and $\{I_m(\psi_j)\}$, such that $\{I_n(\phi_j)\} \to I_n(\phi)$ and $\{I_m(\psi_j)\} \to I_m(\psi)$ pointwise a.e $[\Lambda]$ and then apply Lemma 3.4, we see that

$$\begin{aligned} & |\phi_j \hat{\otimes}_k^r \psi_j - \phi \hat{\otimes}_k^r \psi|_2 \\ \le\ & |\phi_j \hat{\otimes}_k^r (\psi_j - \psi)|_2 + |(\phi_j - \phi) \hat{\otimes}_k^r \psi|_2 \\ \le\ & M^r \cdot \sqrt[4]{\lambda(S_M)^{m+n-2r}} (|\phi_j|_\infty^{1/2} |\psi_j - \psi|_\infty^{1/2} |\phi_j|_2^{1/2} |\psi_j - \psi|_2^{1/2} \\ & + |\phi_j - \phi|_\infty^{1/2} |\psi|_\infty^{1/2} |\phi_j - \phi|_2^{1/2} |\psi|_2^{1/2}) \\ \le\ & \sqrt{2} M^r \sqrt[4]{\lambda(S_M)^{m+n-2r}} |\phi|_\infty^{1/2} |\psi|_\infty^{1/2} (|\phi_j|_2^{1/2} |\psi_j - \psi|_2^{1/2} + |\phi_j - \phi|_2^{1/2} |\psi|_2^{1/2}) \end{aligned}$$

$$\leq \sqrt{2}M^r\sqrt{\lambda(S_M)^{m+n-r}}|\phi|_\infty^{1/2}|\psi|_\infty^{1/2}(|\psi_j-\psi|_2^{1/2}+|\phi_j-\phi|_2^{1/2}),$$

which tends to 0 as $j\to\infty$. Consequently, $I_{m+n-2k-r}(\phi_j\hat{\otimes}_k^r\psi_j)\to I_{m+n-2k-r}(\phi\hat{\otimes}_k^r\psi)$ for any $0\leq k\leq m\wedge n$ and $0\leq r\leq (m\wedge n)-k$. Since

$$\left(\chi_{A_p^{i,j}}\cdot\chi_{B_q^{i',j}}\right)^* \quad \text{for any } i,i',j,p,q$$

is obviously in $L^2(\mathbb{R}^2,\lambda)$, it follows from Theorem 3.5 that the formula (25) hold for ϕ_j and ψ_j. Letting $j\to\infty$, (25) is true for ϕ and ψ. For the general case, let $\ell(k,r)=m+n-2k-r$ for any $0\leq k\leq m\wedge n$ and $0\leq r\leq (m\wedge n)-k$,

$$\begin{aligned} E_j &\equiv (S_j)^n\cap\{(s_1,\ldots,s_n)\in(\mathbb{R}^2)^n : |\phi(s_1,\ldots,s_n)|\leq j\}, \quad \text{and}\\ F_j &\equiv (S_j)^m\cap\{(s_1,\ldots,s_m)\in(\mathbb{R}^2)^m : |\psi(s_1,\ldots,s_m)|\leq j\} \quad \text{for each } j\in\mathbb{N}. \end{aligned}$$

Then $\phi\cdot\chi_{E_j}$'s and $\psi\cdot\chi_{F_j}$'s are all bounded functions with compact support. Also, $I_n(\phi\cdot\chi_{E_j})\to I_n(\phi)$ and $I_m(\psi\cdot\chi_{F_j})\to I_m(\psi)$ in $L^2(\mathcal{S}',\Lambda)$. Combining the conclusion in the first part, it remains to show that $I_{\ell(k,r)}((\phi\cdot\chi_{E_j})\,\hat{\otimes}_k^r\,(\psi\cdot\chi_{F_j}))\to I_{\ell(k,r)}(\phi\,\hat{\otimes}_k^r\,\psi)$ for any $0\leq k\leq m\wedge n$ and $0\leq r\leq (m\wedge n)-k$. Take a arbitrarily chosen $(X,Y,Z)\in(\mathbb{R}^2)^{n-k-r}\times(\mathbb{R}^2)^{m-k-r}\times(\mathbb{R}^2)^r$. We note that $\phi\cdot\chi_{E_j}(X,Z,A)\,\psi\cdot\chi_{F_j}(A,Z,Y)\to\phi(X,Z,A)\,\psi(A,Z,Y)$ for any $A\in(\mathbb{R}^2)^k$ as $j\to\infty$. Also, $|\phi\cdot\chi_{E_j}(X,Z,\cdot)\,\psi\cdot\chi_{F_j}(\cdot,Z,Y)|\leq|\phi|(X,Z,\cdot)\,|\psi|(\cdot,Z,Y)$, and from the assumption and by applying Fubini's theorem, $|\phi|(X,Z,\cdot)\,|\psi|(\cdot,Z,Y)$ is integrable on $(\mathbb{R}^2)^k$ with respect to $\lambda^{\otimes k}$ for $\lambda^{\otimes\,\ell(k,r)}$-almost all (X,Y,Z). By a dominated convergence argument, $(\phi\cdot\chi_{E_j})\otimes_k^r(\psi\cdot\chi_{F_j})$ converges pointwise to $\phi\otimes_k^r\psi$ almost everywhere. Since for any $j\in\mathbb{N}$,

$$\left|(\phi\cdot\chi_{E_j})\otimes_k^r(\psi\cdot\chi_{F_j})-\phi\otimes_k^r\psi\right|\leq 2\,|\,|\phi|\otimes_k^r|\psi|\,|$$

and $||\phi|\otimes_k^r|\psi||$ is square integrable on $(\mathbb{R}^2)^{\ell(k,r)}$ with respect to $\lambda^{\otimes(m+n-2k-r)}$, again by a dominated convergence argiment we see that $(\phi\cdot\chi_{E_j})\otimes_k^r(\psi\cdot$

$\chi_{F_j}) \to \phi \otimes_k^r \psi$ in $L_c^2((\mathbb{R}^2)^\ell, \lambda^{\otimes \ell})$, and so $(\phi \cdot \chi_{E_j}) \hat{\otimes}_k^r (\psi \cdot \chi_{F_j}) \to \phi \hat{\otimes}_k^r \psi$ in $\hat{L}_c^2((\mathbb{R}^2)^{\ell(k,r)}, \lambda^{\otimes \ell(k,r)})$. It implies that $I_{\ell(k,r)}((\phi \cdot 1_{E_j}) \hat{\otimes}_k^r (\psi \cdot 1_{F_j})) \to I_{\ell(k,r)}(\phi \hat{\otimes}_k^r \psi)$ in $L^2(\mathcal{S}', \Lambda)$. We complete the proof.

If the measure β is concentrated at one single point, $L^2(\mathbb{R}^2, \lambda)$ is identified with $L^2(\mathbb{R}, dt)$ (see [12]). Consider the Gaussian white noise measure Λ_W. Then $\beta = \delta_0$, and in the formula (25), only the terms with $r = 0$ contribute. Consequently, we obtain the Wiener multiplication formula.

Corollary 3.7. *Let Λ be the Gaussian white noise measure Λ_W. Then, for $\phi \in \hat{L}_c^2(\mathbb{R}^n, dt^{\otimes n})$ and $\psi \in \hat{L}_c^2(\mathbb{R}^m, dt^{\otimes m})$ with $n, m \in \mathbb{N}$,*

$$I_n(\phi) \cdot I_m(\psi) = \sum_{k=0}^{m \wedge n} k! \binom{n}{k} \binom{m}{k} I_{m+n-2k}(\phi \hat{\otimes}_k^0 \psi).$$

At the end of this paper we give an another form of the the product formula in terms of the Guichardet's notation in a simple symmetric Fock space (see [2]). First we redefine the Lévy-Itô integrals.

For $(t, u) \in \mathbb{R}^2$, let $p_1(t, u) = t$. For any $n \in \mathbb{N}$, denote by $\mathcal{P}_n$ the space of all finite subsets A of $\mathbb{R}^2$ with the conditions: (a) $|A| = n$ (b) $p_1(s)$'s are all distinct for any $s \in A$ (c) $\mathcal{P}_0 = \{\emptyset\}$. Identify $\mathcal{P}_n$, $n \in \mathbb{N}$, with the subset of $(\mathbb{R}^2)^n$ by strictly ordered n-tuples through

$$\mathcal{P}_n \ni A = \{s_1 < \cdots < s_n\} \mapsto (s_1, \ldots, s_n),$$

where $s < s'$ means that $p_1(s) < p_1(s')$. Let $\mathcal{P}$ be the sum of all $\mathcal{P}_n$. Define a measure $\vec{\boldsymbol{\lambda}}$ on $\mathcal{P}$ by putting the restriction of $\vec{\boldsymbol{\lambda}}$ to $\mathcal{P}_n$, denoted by $\vec{\boldsymbol{\lambda}}_n$, equal to the restriction of $\lambda^{\otimes n}$ to $\mathcal{P}_n$ and let $\vec{\boldsymbol{\lambda}}(\mathcal{P}_0) = 1$. Then $(\mathcal{P}, \vec{\boldsymbol{\lambda}})$ is a σ-finite measure space. We call $(\mathcal{P}, \vec{\boldsymbol{\lambda}})$ the symmetric measure space over $(\mathbb{R}^2, \lambda)$ in the sense of A. Guichardet [2]. The n-dimensional stochastic differential $dM(s_1) \cdots dM(s_n)$ is considered as the restriction to $\mathcal{P}_n$ of a single differential $dM(A)$. For $f \in L_c^2(\mathcal{P}_n, \vec{\boldsymbol{\lambda}}_n)$, let $\tilde{f}$ be a symmetric function on $(\mathbb{R}^2)^n$ by $\tilde{f}(s_{\pi(1)},\ldots,s_{\pi(n)}) \equiv f(\{s_1,\ldots,s_n\})$ if $\{s_1,\ldots,s_n\} \in \mathcal{P}_n$, where π runs over all permutations of $\{1,2,\ldots,n\}$; otherwise $\tilde{f}(s_1,\ldots,s_n) = 0$. Then $\tilde{f} \in \hat{L^2_c}((\mathbb{R}^2)^n, \lambda^{\otimes n})$ and we define $\int_{\mathcal{P}_n} f(A)\, dM(A)$ to be $(1/n!)\, I_n(\tilde{f})$.

For $\phi \in L^2_c(\mathcal{P}, \vec{\boldsymbol{\lambda}})$, define

$$\int_{\mathcal{P}} \phi(X)\, dM(X) = \sum_{n=0}^{\infty} \int_{\mathcal{P}_n} \phi(X)\, dM(X).$$

Then, we see that

$$\left\| \int_{\mathcal{P}} \phi(X)\, dM(X) \right\|^2 = \int_{\mathcal{P}} |\phi(X)|^2\, d\vec{\boldsymbol{\lambda}}(X) = \sum_{n=0}^{\infty} \int_{\mathcal{P}_n} |\phi(X)|^2\, d\vec{\boldsymbol{\lambda}}_n(X).$$

Conversely, for $\varphi \sim (\,\phi_n\,) \in L^2(\mathcal{S}', \Lambda)$, define a complex-valued function K_φ on $\mathcal{P}$ by $K_\varphi(X) = |X|!\, \phi_n(X)$ if $X \in \mathcal{P}_n$ for any $n \in \mathbb{N} \cup \{0\}$. Since

$$\lambda^{\otimes n}(\{(s_1,\ldots,s_n) \in (\mathbb{R}^2)^n : p_1(s_i)\text{'s are not all distinct}\}) = 0,$$

$\widetilde{g|_{\mathcal{P}_n}} = g$, $\lambda^{\otimes n}$-almost everywhere on $(\mathbb{R}^2)^n$ for $g \in \hat{L^2_c}((\mathbb{R}^2)^n, \lambda^{\otimes n})$, where $g|_{\mathcal{P}_n}$ is the restriction of g to $\mathcal{P}_n$. Thus

$$\varphi = \int_{\mathcal{P}} K_\varphi(X)\, dM(X). \tag{26}$$

Lemma 3.8. (Integral-Sum Identity) *For $n \in \mathbb{N}$, assume that the complex-valued function F on $\mathcal{P}^n = \mathcal{P} \times \cdots \times \mathcal{P}$ (n-fold) is integrable with respect to $\vec{\boldsymbol{\lambda}}^{\otimes n}$. Then*

$$\begin{aligned}&\int_{\mathcal{P}} \cdots \int_{\mathcal{P}} F(X_1,\ldots,X_n)\, d\vec{\boldsymbol{\lambda}}(X_1) \cdots d\vec{\boldsymbol{\lambda}}(X_n) \\ &= \int_{\mathcal{P}} \sum_{\omega_1+\cdots+\omega_n = X} F(\omega_1,\ldots,\omega_n)\, d\vec{\boldsymbol{\lambda}}(X).\end{aligned}$$

Proof. See [18, Proposition 2.1.2].

Theorem 3.9. *Let $f, g \in L_c^2(\mathcal{P}, \vec{\boldsymbol{\lambda}})$. Assume that the function F from $\mathcal{P}^3$ into $\mathbb{C}$ by*

$$K(X,Y,Z) \equiv \sqrt{3}^{|X|+|Y|+|Z|} \left(\prod_{s\in Z} p_2(s)\right) \int_{\mathcal{P}} |f|(X\cup Z\cup A)\, |g|(A\cup Z\cup Y)\, d\vec{\boldsymbol{\lambda}}(A)$$

is square integrable on $\mathcal{P}^3$ with respect to $\vec{\boldsymbol{\lambda}}^{\otimes 3}$. Then

$$\left(\int_{\mathcal{P}} f(C)\, dM(C)\right) \cdot \left(\int_{\mathcal{P}} g(C)\, dM(C)\right) = \int_{\mathcal{P}} h(C)\, dM(C),$$

where

$$h(C) = \sum_{X+Y+Z=C} \left(\prod_{s\in Z} p_2(s)\right) \int_{\mathcal{P}} f(X+Z+A)\, g(A+Z+Y)\, d\vec{\boldsymbol{\lambda}}(A).$$

Proof. Choose a subsequence $\{\alpha_1 < \alpha_2 < \cdots\}$ of $\{1, 2, 3, \ldots\}$ so that

$$f_i \equiv f \cdot \chi_{\mathcal{P}_0 + \cdots + \mathcal{P}_{\alpha_i}} \to f \quad \text{and} \quad g_i \equiv g \cdot \chi_{\mathcal{P}_0 + \cdots + \mathcal{P}_{\alpha_i}} \to g$$

pointwise $\vec{\boldsymbol{\lambda}}$-almost everywhere on $\mathcal{P}$, and

$$\begin{aligned} F_i &\equiv \int_{\mathcal{P}} f_i(C)\, dM(C) \to \int_{\mathcal{P}} f(C)\, dM(C) \quad \text{and} \\ G_i &\equiv \int_{\mathcal{P}} g_i(C)\, dM(C) \to \int_{\mathcal{P}} g(C)\, dM(C) \end{aligned}$$

pointwise Λ-almost everywhere on $\mathcal{S}'$ as $i \to +\infty$. Then, by the assumption on the function K and applying Theorem 3.6, we see that

$$F_i \cdot G_i = \int_{\mathcal{P}} h_i(C)\, dM(C),$$

where for any $i \in \mathbb{N}$,

$$h_i(C) = \sum_{X+Y+Z=C} \left(\prod_{s\in Z} p_2(s)\right) \int_{\mathcal{P}} f_i(X+Z+A)\, g_i(A+Z+Y)\, d\vec{\boldsymbol{\lambda}}(A).$$

For the remainder of the proof, we have to show that $h_i \to h$ in $L_c^2(\mathcal{P}, \vec{\boldsymbol{\lambda}})$. Observe that

$$\begin{aligned} |(h_i - h)(C)|^2 \;\le\; & \Bigg| \sum_{X+Y+Z=C} \left(\prod_{s\in Z} p_2(s) \right) \\ & \int_{\mathcal{P}} [|f_i - f|(X+Z+A)|g_i|(A+Z+Y) \\ & +|f|(X+Z+A)\,|g_i - g|(A+Z+Y)\,]\, d\vec{\boldsymbol{\lambda}}(A) \Bigg|^2 , \end{aligned}$$

which is bounded by the function

$$\begin{aligned} & 2 \Bigg\{ \sum_{X+Y+Z=C} 3^{|X|+|Y|+|Z|} \\ & \quad \left| \left(\prod_{s\in Z} p_2(s) \right) \int_{\mathcal{P}} |f_i - f|(X+Z+A)\,|g|(A+Z+Y)\, d\vec{\boldsymbol{\lambda}}(A) \right|^2 \\ & \quad + \sum_{X+Y+Z=C} 3^{|X|+|Y|+|Z|} \\ & \quad \left| \left(\prod_{s\in Z} p_2(s) \right) \int_{\mathcal{P}} |f|(X+Z+A)\,|g_i - g|(A+Z+Y)\, d\vec{\boldsymbol{\lambda}}(A) \right|^2 \Bigg\} \\ = \;& 2\left(\textstyle\sum_1(C) + \sum_2(C) \right) \end{aligned}$$

by the Schwarz inequality. Then $|h_i - h|^2 \to 0$ pointwise $\vec{\boldsymbol{\lambda}}$-almost everywhere on $\mathcal{P}$. Also, for any $C \in \mathcal{P}$,

$$\textstyle\sum_i(C) \le \displaystyle\sum_{X+Y+Z=C} |K(X,Y,Z)|^2, \quad i = 1, 2;$$

and by Lemma 3.8,

$$\begin{aligned} & \int_{\mathcal{P}} \sum_{X+Y+Z=C} |K(X,Y,Z)|^2 \, d\vec{\boldsymbol{\lambda}}(C) \\ = \;& \int_{\mathcal{P}} \int_{\mathcal{P}} \int_{\mathcal{P}} |K(X,Y,Z)|^2 \, d\vec{\boldsymbol{\lambda}}(X)\, d\vec{\boldsymbol{\lambda}}(Y)\, d\vec{\boldsymbol{\lambda}}(Z) < +\infty. \end{aligned}$$

Thus, a dominated convergence argument yields that $h_i \to h$ in $L_c^2(\mathcal{P}, \vec{\boldsymbol{\lambda}})$, and then the proof is complete.

Remark 3.10. When Λ is the Poisson white noise measure Λ_P, the product formula in Theorem 3.9 reduces to the Poisson multiplication formula. In this case, the kernel function $h(C)$ becomes that

$$\sum_{X+Y+Z=C} \int_{\mathcal{P}} f(X+Z+A)\, g(A+Z+Y)\, d\vec{\boldsymbol{\lambda}}(A).$$

See [15].

Example 3.11. For $f \in L_c^2(\mathbb{R}^2, \lambda)$, let $\varphi_1 = \mathcal{E}_M(f \cdot \chi_{\mathbb{R}\times\{0\}})$ and $\varphi_2 = \mathcal{E}_M(f \cdot \chi_{\mathbb{R}_*^2})$. Then, for $X \in \mathcal{P}_n$,

$$K_{\varphi_1}(X) = \left(f \cdot \chi_{\mathbb{R}\times\{0\}}\right)^{\otimes n}(X) \text{ and } K_{\varphi_2}(X) = \left(f \cdot \chi_{\mathbb{R}_*^2}\right)^{\otimes n}(X), \;\; n \in \mathbb{N}\cup\{0\},$$

where $K_{\varphi_1}, K_{\varphi_2}$ are defined as before in (26). Observe that for $X, Y, Z \in \mathcal{P}$,

$$\begin{aligned} K(X,Y,Z) &\equiv \sqrt{3}^{|X|+|Y|+|Z|}\left(\prod_{s\in Z} p_2(s)\right) \\ &\quad \int_{\mathcal{P}} |K_{\varphi_1}|(X\cup Z\cup A)\,|K_{\varphi_2}|(A\cup Z\cup Y)\, d\vec{\boldsymbol{\lambda}}(A) \\ &= \chi_{\mathcal{P}_0}(Z)\cdot\sqrt{3}^{|X|+|Y|}\cdot|K_{\varphi_1}|(X)\,|K_{\varphi_2}|(Y). \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathcal{P}}\int_{\mathcal{P}}\int_{\mathcal{P}} |K(X,Y,Z)|^2\, d\vec{\boldsymbol{\lambda}}(X)\, d\vec{\boldsymbol{\lambda}}(Y)\, d\vec{\boldsymbol{\lambda}}(Z) \\ = &\int_{\mathcal{P}}\int_{\mathcal{P}} 3^{|X|+|Y|}\cdot|K_{\varphi_1}|^2(X)\,|K_{\varphi_2}|^2(Y)\, d\vec{\boldsymbol{\lambda}}(X)\, d\vec{\boldsymbol{\lambda}}(Y) \\ = &\left(\int_{\mathcal{P}} 3^{|X|}\cdot|K_{\varphi_1}|^2(X)\, d\vec{\boldsymbol{\lambda}}(X)\right)\left(\int_{\mathcal{P}} 3^{|Y|}\cdot|K_{\varphi_2}|^2(Y)\, d\vec{\boldsymbol{\lambda}}(Y)\right) \\ = &\exp\left[3\int_{\mathbb{R}^2}|f\cdot\chi_{\mathbb{R}\times\{0\}}(t,u)|^2 d\lambda(t,u)\right]\exp\left[3\int_{\mathbb{R}^2}|f\cdot\chi_{\mathbb{R}_*^2}(t,u)|^2\, d\lambda(t,u)\right] \end{aligned}$$

$$= \exp\left[3\int_{\mathbb{R}^2} |f(t,u)|^2\, d\lambda(t,u)\right] < +\infty.$$

By Theorem 3.9, we see that

$$\begin{aligned}
\mathcal{E}_M(f\cdot\chi_{\mathbb{R}\times\{0\}})\cdot\mathcal{E}_M(f\cdot\chi_{\mathbb{R}^2_*}) &= \int_{\mathcal{P}}\left(\sum_{X+Y=C} K_{\varphi_1}(X)\,K_{\varphi_2}(Y)\right) dM(C)\\
&= \int_{\mathcal{P}} \sum_{\substack{X+Y=C\\ X\subset\mathbb{R}\times\{0\},\ Y\subset\mathbb{R}^2_*}} f^{\otimes|X|}(X)\, f^{\otimes|Y|}(Y)\, dM(C)\\
&= \int_{\mathcal{P}} f^{\otimes|C|}(C)\, dM(C)\\
&= \mathcal{E}_M(f).
\end{aligned}$$

For $h\in L^2_c(\mathbb{R},dt)$ and $g\in L^2_c(\mathbb{R}^2_*,\nu)$, let us put

$$\mathcal{E}_B(h) = \sum_{n=0}^{\infty}\frac{1}{n!}\,I^B_n(h^{\otimes n}) \quad\text{and}\quad \mathcal{E}_{N_0}(g) = \sum_{n=0}^{\infty}\frac{1}{n!}\,I^{N_0}_n(g^{\otimes n}),$$

where $I^B_n(h^{\otimes n})$ is referred to as n-fold Wiener-Itô integral of $h^{\otimes n}$, i.e.,

$$I^B_n(h^{\otimes n}) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h^{\otimes n}(t_1,\ldots,t_n)\, dB(t_1)\cdots dB(t_n),$$

and $I^{N_0}_n(g^{\otimes n})$ is the n-fold multiple stochastic integral

$$\int_{\mathbb{R}^2_*}\cdots\int_{\mathbb{R}^2_*} g^{\otimes n}((t_1,u_1),\ldots,(t_n,u_n))\, dN_0(t_1,u_1)\cdots dN_0(t_n,u_n)$$

of $g^{\otimes n}$ with respect to the independent random measure N_0, defined analogously to the multiple Lévy-Itô integrals. Here, we remark that for any $\phi\in \hat{L}^2_c((\mathbb{R}^2_*)^n,\nu^{\otimes n})$,

$$\|I^{N_0}_n(\phi)\|^2 = n!\int_{\mathbb{R}^2_*}\cdots\int_{\mathbb{R}^2_*} |\phi(s_2,\ldots,s_n)|^2\, d\nu(s_1)\cdots d\nu(s_n).$$

Proposition 3.12. *Let* $f \in L_c^2(\mathbb{R}^2, \lambda)$*. Then, for* Λ*-almost all* $x \in \mathcal{S}'$*,*

$$\mathcal{E}_M(f)(x) = \mathcal{E}_B(\sigma\, f|_{\mathbb{R}})(x) \cdot \mathcal{E}_{N_0}(f^*)(x),$$

where $f|_{\mathbb{R}}(t)$ *is defined to be* $f(t, 0)$*.*

Proof. From Example 3.11 it is sufficient to show that $\mathcal{E}_M(f \cdot \chi_{\mathbb{R}\times\{0\}}) = \mathcal{E}_B(\sigma f|_{\mathbb{R}})$ and $\mathcal{E}_M(f \cdot \chi_{\mathbb{R}^2_*}) = \mathcal{E}_{N_0}(f^*)$. By Theorem 3.1, it is clear that $\mathcal{E}_B(\sigma\, f|_{\mathbb{R}}) = \mathcal{E}_M(f \cdot \chi_{\mathbb{R}\times\{0\}})$. For any $n \in \mathbb{N}$, we choose a sequence $\{g_{n,k}\}$ of complex-valued simple functions on $(\mathbb{R}^2)^n$ of the form as in (6), say

$$g_{n,k} = \sum_{i=1}^{m_k} a_{k,i} \cdot \hat{\chi}_{E^i_{k,1}\times\cdots\times E^i_{k,n}},$$

such that $g_{n,k} \to f^{\otimes n}$ in $L_c^2((\mathbb{R}^2)^n, \lambda^{\otimes n})$. Then $g^*_{n,k} \to (f^*)^{\otimes n}$ in $L_c^2((\mathbb{R}^2_*)^n, \nu^{\otimes n})$. Observe that

$$\begin{aligned} I_n^{N_0}(g^*_{n,k}) &= \sum_{i=1}^{m_k} a_{k,i} \prod_{j=1}^{n} I_1^{N_0}(\chi^*_{E^i_{k,j}}) \\ &= \sum_{i=1}^{m_k} a_{k,i} \prod_{j=1}^{n} I_1(\chi_{E^i_{k,j}\cap\mathbb{R}^2_*}) \\ &= \sum_{i=1}^{m_k} a_{k,i}\, I_n(\hat{\chi}_{E^i_{k,1}\cap\mathbb{R}^2_*\times\cdots\times E^i_{k,n}\cap\mathbb{R}^2_*}) = I_n(g_{n,k} \cdot \chi_{(\mathbb{R}^2_*)^n}). \end{aligned}$$

Letting $k \to \infty$ we get that $I_n^{N_0}((f^*)^{\otimes n}) = I_n((f \cdot \chi_{\mathbb{R}^2_*})^{\otimes n}))$, Λ-almost everywhere on $\mathcal{S}'$. It implies that $\mathcal{E}_{N_0}(f^*) = \mathcal{E}_M(f \cdot \chi_{\mathbb{R}^2_*})$.

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