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FOURIER EXPANSIONS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

BY

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Abstract. Let μ be a finite positive Boral measure on a compact Jordan region $E \subset C^2$ and $L^2_{(\mu)}$, the Hilbert space of functions of two complex variables holomorphic in E with inner product is defined as surface measure integral over E. The relations connection the growth of an entire function of two complex variables $f(z_1,z_2) \in L^2_{(\mu)}$ with its Fourier Coefficients with respect to an orthonormal sequence of polynomials in $L^2_{(\mu)}$, have been obtained. The necessary and sufficient conditions in terms of Fourier Coefficents have been obtained for $f(z_1,z_2) \in L^2_{(\mu)}$ to be of finite order and finite type.

1. Introduction. Let $f(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ be a function of two complex variables z_1 and z_2, regular for $|z_t| \le r_t$, $t = 1,2$. If r_1 and r_2 are arbitrary large then $f(z_1,z_2)$ is an entire function of two complex variables.

Let $\lceil$ denote the class of all entire functions of two complex variables in C^2. The growth of a $f(z_1,z_2) \in \lceil$ is studied in terms of its order ρ and if $0 < \rho < \infty$, in terms of its type T also, where

$$\limsup_{r_1,r_2\to\infty} \frac{\log\log M(r_1,r_2)}{\log(r_1 r_2)} = \rho, \tag{1.1}$$

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$$\limsup_{r_1,r_2\to\infty} \frac{\log M(r_1,r_2)}{r_1^\rho + r_2^\rho} \quad = \quad T, \tag{1.2}$$

where $M(r_1,r_2) = \max_{|z_t|\le r_t} |f(z_1,z_2)|$, $t = 1,2$.

The coefficents characterizations of above growth constants are known [1]. Thus

$$\rho \quad = \quad \limsup_{m_1,m_1\to\infty} \frac{\log m_1^{m_1} m_2^{m_2}}{\log |a_{m_1,m_2}|^{-1}}; \tag{1.3}$$

$$T \quad = \quad \limsup_{m_1,m_2\to\infty} \{m_1^{m_1} m_2^{m_2} |a_{m_1,m_2}|^\rho\}^{1/(m_1+m_2)}. \tag{1.4}$$

Let μ be a finite positive Borel measure on a compact jordan region $E \subset C^2$ of transfinite diameter $d_t > 0$, $t = 1,2$, and $L^2_{(\mu)}$, the Hilbert space of functions of two complex variables holomorphic in E with inner product

$$(f,g) = \int_E f(z_1,z_2)\overline{g(z_1,z_2)}d\mu, \quad f,g \in L^2_{(\mu)},$$

where $\|f\|_{L^2_{(\mu)}} = [\int_E |f|^2 d\mu]^{1/2} < \infty$.

We will assure that $E = \mathrm{supp}(\mu)$ is not contained in any (proper) algebraic subset of C^2. This is equivalent to the following property of E: If $P_{m_1,m_2}(z_1,z_2)$ is an (analytic) polynomial then

$$P_{m_1,m_2}(z_1,z_2)|_E \equiv 0 \Rightarrow P_{m_1,m_2}(z_1,z_2) \equiv 0 \text{ on } C^2. \tag{1.5}$$

Sets with this property are said unisolvent. In the case of one complex variable, E satisfies (1.5) if and only if E contains infinitely many points (see [3], p.2).

Proposition 1. *Let μ be a finite positive Borel measure with $E =$ $\mathrm{supp}(\mu)$ satisfying* (1.5)*. Let $P_{m_1,m_2}(z_1,z_2)$ be an* (*analytic*) *polynomial such* *that*

$$\|P_{m_1,m_2}(z_1,z_2)\|_{L^2_{(\mu)}} = 0. \quad \textit{Then} \quad P_{m_1,m_2}(z_1,z_2) \equiv 0 \quad \textit{on} \quad C^2.$$

Proof. We will show that if $P_{m_1,m_2}(z_1,z_2)|_E \neq 0$, then $\|P_{m_2,m_2}(z_1,z_2)\|_{L^2_{(\mu)}} > 0$. Suppose $P_{m_1,m_2}(z_1,z_2)|_E \neq 0$ and let $z_{0_t} \in E = E_1 \times E_2$, $t = 1,2$, be such that $|P_{m_1,m_2}(z_1,z_2)| > 0$. Then for some $r_t > 0$, $|P_{m_1,m_2}(z_1,z_2)| \geq (|P_{m_1,m_2}|/2)$ for all $z_t \in \triangle(z_{0_t}, r_t)$, where $\triangle(z_{0_t}, r_t)$ denotes the closed balls of centre z_{0_t} and radius r_t. Since $z_{0_t} \in \text{supp}(\mu)$, we have $\mu(\triangle(z_{0_t}, z_t)) > 0$. Hence

$$\begin{aligned}
\|P_{m_1,m_2}(z_1,z_2)\|^2_{L^2_{(\mu)}} &= \int_E |P_{m_1,m_2}(z_1,z_2)|^2 d\mu \\
&\geq \int_{E\cap\triangle(z_{0_t},r_t)} |P_{m_1,m_2}(z_1,z_2)|^2 d\mu \\
&\geq (|P_{m_1,m_2}(z_{0_1},z_{0_2})|/2)^2 \mu(\triangle(z_{0_t},r_t)) > 0.
\end{aligned}$$

Hence the proof is completed.

Here we consider the monomials $\{z_1^{m_1} z_2^{m_2}\}$ to be ordered lexicographically. By Proposition 1, we may apply the Gram-schmidt orthogonalization procedure to the monomials and one obtains orthonormal polynomials denoted $p_{m_1,m_2}(z_1,z_2) \equiv p_{m_1,m_2}(z_1,z_2,\mu)$ for each m_1 and $m_2 \cdot p_{m_1,m_2}(z_1,z_2,\mu)$ denotes the orthonomal polynomial which is a linear combination of $z_1^{m_1} z_2^{m_2}$ and monomials of lower lexicographic order. Thus $A_{m_1,m_2}(E) \equiv \{P_{m_1-1,m_2-1}(z_1,z_2)\}_{m_1,m_2=1}^{\infty}$, $P_{m_1,m_2}(z_1,z_2)$ being a polynomial of degree $\leq m_1 + m_2$, is a complete orthonormal sequence in $L^2_{(\mu)}$.

The Fourier expansion of $f(z_1,z_2) \in L^2_{(\mu)}$ is

$$f(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} b_{m_1,m_2} p_{m_1,m_2}(z_1,z_2),$$

where

$$b_{m_1 m_2} = \int_E f(z_1, z_2)\overline{p_{m_1,m_2}(z_1, z_2)}d\mu. \tag{1.6}$$

A question arises that "Do the relations (1.3) and (1.4) continue to hold if a_{m_1,m_2} is replaced by Fourier coefficient b_{m_1,m_2} of $f(z_1, z_2) \in \lceil\subset L^2_{(\mu)}$ with respect to $L^2_{(\mu)}$. In this paper we attempt to solve this question.

2. Auxiliary results. In this section we prove some lemmas which are required in proving the main theorems.

Let E_{r_t} be the largest equipotential curve of $E = E_1 \times E_2$ such that $E_{r_t} = \{z_t \in C^2 : d_t \exp V_\mu(z_t) = r_t\}$, $r_t/d_t > 1$, $t = 1, 2$ and $V_\mu(z_t)$ is the minimal Carrier Green function of the measure μ and $C^2 \setminus \hat{E}$ is simply connected [2], $\hat{E}$ denote the convex hull of E. Let D_{r_t} be the domain interiar to E_{r_t}.

Lemma 2.1. *If a polynomial $P_{m_1,m_2}(z_1, z_2)$ of degree $m_1 + m_2$ satisfies the inequality $|P_{m_1,m_2}(z_1, z_2)| \leq L$ for $z_t \in E$, then we have*

$$|P_{m_1,m_2}(z_1, z_2)| \leq LR_1^{m_1}R_2^{m_2} \quad \text{for} \quad z_t \in E_{R_t},\ R_t > 1,\ t = 1, 2. \tag{2.10}$$

Lemma 2.2. *If $f(z_1, z_2)$ is analytic on E and we have*

$$\int_E |P_{m_1,m_2}|^2 d\mu \leq L,$$

if E' is an arbitrary closed jordan region interior to E, then we have

$$|P_{m_1,m_2}(z_1, z_2)| \leq LL' \quad \text{for} \quad z_t \in E',$$

where L' depends on E' but not on $P_{m_1,m_2}(z_1, z_2)$ nor on L.

These lemmas can be proved in the same way as in single complex variable (see [4]).

Lemma 2.3. *If $P_{m_1,m_2}(z_1,z_2)$ forms a complete orthonormal sequence in $L^2_{(\mu)}$ then for any $\varepsilon > 0$.*

$$|P_{m_1,m_2}(z_1,z_2)| < M_0 \left(\frac{r_1}{d_1}\right)^{m_1} \left(\frac{r_2}{d_2}\right)^{m_2} (1+\varepsilon)^{m_1+m_2}, \quad z_t \in E_{r_t},$$

where M_0 depends on ε but not on $m_1,\ m_2$.

Proof. Since we may assume

$$\int_E |P_{m_1,m_2}(z_1,z_2)|^2 d\mu \leq 1 \quad \text{for all} \quad m_1, m_2.$$

By Lemma 2.2, we have for any $E' \subset E$,

$$|P_{m_1,m_2}(z_1,z_2)| \leq M_0 \quad \text{for} \quad z_1 \in E',$$

where M_0 depends on E'. So for any $\varepsilon > 0$, applying Lemma 2.1, we get

$$|P_{m_1,m_2}(z_1,z_2)| < M_0(1+\varepsilon)^{m_1+m_2} \quad \text{for} \quad z_t \in E'_{1+\varepsilon}.$$

Now let $E'_{1+\varepsilon} \subset E$, so that

$$|P_{m_1,m_2}(z_1,z_2)| < M_0(1+\varepsilon)^{m_1+m_2} \quad \text{holds on } E \text{ also.}$$

Again applying Lemma 2.1, proof is completed.

Lemma 2.4. *Let $f(z_1,z_2)$ be analytic in the domain D_{R_t} and have a singularity on E_{R_t}, then*

$$\limsup_{m_1,m_2\to\infty} |b_{m_1,m_2}|^{1/(m_1+m_2)} \leq \frac{1}{R_t}, \quad R_t > 1,\ t = 1,2. \tag{2.11}$$

Proof. Since $\|f(z_1,z_2)\|_{L^2_{(\mu)}} \leq 1$, we have

$$|b_{m_1,m_2}| < \int_E |\overline{P_{m_1,m_2}(z_1,z_2)}| d\mu,$$

using Cauchy-Schwarz inequality, we get

$$|b_{m_1,m_2}| \leq (\mu(E))^{1/2}$$

or

$$\limsup_{m_1,m_2} |b_{m_1,m_2}|^{1/(m_1+m_2)} \leq \frac{1}{R_t}, \quad R_t > 1. \tag{2.12}$$

However, strict inequality in (2.11) is equivalent to the analyticity of $f(z_1,z_2)$ in $D_{R'_t}$ for some R'_t with $R_t < R'_t$. Thus if $f(z_1,z_2)$ has a singularity on E_{R_t} then equality holds in (2.12).

Lemma 2.5. *Let $f(z_1,z_2) \in L^2_{(\mu)}$ and b_{m_1,m_2} satisfies* (2.11). *Then $f(z_1,z_2)$ can be continued analytically to the domain D_{R_t}, $t=1,2$.*

Proof. To see that the series $\sum_{m_1 m_2=0}^{\infty} b_{m_1,m_2} p_{m_1,m_2}(z_1,z_2)$ converges uniformly on compact subsets of D_{R_t}, choosing a number R_t^*, $1 < R_t^* < R_t$. Let $\varepsilon > 0$ and $\varepsilon < \frac{R_t - R_t^*}{R_t^*}$, so that $R_t^*(1+\varepsilon) < R_t$. Let R_t^{**} be such that $R_t^*(1+\varepsilon) < R_t^{**} < R_t$. (2.11) gives that there exists $m_{1_0} = m_{1_0}(R_t^{**})$, $m_{2_0} = m_{2_0}(R_2^{**})$ such that

$$|b_{m_1,m_2}| < \frac{1}{(R_1^{**})^{m_1}(R_2^{**})^{m_2}} \quad \text{for} \quad m_1 \geq m_{1_0},\ m_2 \geq m_{2_0}. \tag{2.13}$$

Applying Lemma 2.3, it gives

$$|P_{m_1,m_2}(z_1,z_2)| < M\left(\frac{R_1^*}{d_1}\right)^{m_1}\left(\frac{R_2^*}{d_2}\right)^{m_2}(1+\varepsilon)^{m_1+m_2} \text{ for } z_t \in E_{R_t^*},\ t=1,2. \tag{2.14}$$

Combining (2.13) and (2.14) implies that

$$|b_{m_1,m_2}P_{m_1,m_2}(z_1,z_2)| < M\left(\frac{R_1^*}{d_1R_1^{**}}\right)^{m_1}\left(\frac{R_2^*}{d_2R_2^{**}}\right)(1+\varepsilon)^{m_1+m_2} \quad \text{for} \quad z_t\in E_{R_1^*}.$$

Using above inequalities and Weirstrass M-test we conclude that $\sum_{m_1,m_2=0}^{\infty} b_{m_1,m_2}P_{m_1,m_2}(z_1,z_2)$ coverges uniformely on $E_{R_t^*}$. Since $R_t^* < R_t$ it implies that the series converges uniformly on compact subsets of D_{R_t}. But

$$\int_E \left\{f(z_1,z_2) - \sum_{m_1,m_2=0}^{\infty} b_{m_1,m_2}P_{m_1,m_2}(z_1,z_2)\right\}\overline{P_{m_1,m_2}(z_1,z_2)}d\mu = 0.$$

Since $P_{m_1,m_2}(z_1,z_2)$ forms a complete orthonormal sequence in $L^2_{(\mu)}$, so

$$f(z_z,z_2) = \sum_{m_1m_2=0}^{\infty} b_{m_1,m_2}P_{m_1,m_2}(z_1,z_2) \quad \text{on} \quad E\subset C^2.$$

Hence $f(z_1,z_2)$ can be continued analytically on D_{R_t}.

Corollary. *$f(z_1,z_2)\in L^2_{(\mu)}$ is an entire function of two complex variables if and only if*

$$\lim_{m_1,m_2\to\infty} |b_{m_1,m_2}|^{1/(m_1+m_2)} = 0.$$

Lemma 2.6. *Let $f(z_1,z_2)\in L^2_{(\mu)}$. For any $\varepsilon>0$, there exists two integers $N_1(\varepsilon,E_1)$ and $N_2(\varepsilon,E_2)$ such that*

$$|b_{m_1+1,m_2+1}| < K\overline{M}(r_1,r_2)\left(\frac{d_1e^{\varepsilon}}{r_1}\right)^{m_1}\left(\frac{d_2e^{\varepsilon}}{r_2}\right)^{m_2},$$

for all $R_1>r_1\ge r_{1_0}=r_{1_0}(\varepsilon)$, $R_2>r_2\ge r_{2_0}(\varepsilon)$ and $m_1>N_1$, $m_2>N_2$. Where $\overline{M}(r_1,r_2)=\max_{z_t\in E_{r_t}}|f(z_1,z_2)|$, K is independent of m_1, m_2 and r_1, r_2.

Proof. We construct a sequence $\{Q_{m_1,m_2}(z_1,z_2)\}_{m_1,m_2=0}^{\infty}$ of polynomials by induction. Such that

$$|f(z_1,z_2) - Q_{m_1,m_2}(z_1,z_2)| \le A\overline{M}(r_1,r_2)\left(\frac{d_1e^{\varepsilon}}{r_1}\right)^{m_1}\left(\frac{d_2e^{\varepsilon}}{r_2}\right)^{m_2},$$

for $z_t \in E_{r_t}$, $m_1 > N_{1_0} = N_{1_0}(\varepsilon, E_1)$, $m_2 > N_{2_0} = N_{2_0}(\varepsilon, E_2)$ and for every r_1, r_2, $R_1 > r_1 > R_{1_0} = R_{1_0}(\varepsilon, E_1)$, $R_2 > r_2 > R_{2_0} = R_{2_0}(\varepsilon, E_2)$. Thus

$$\begin{aligned} & \left(\int_E |f(z_1,z_2) - Q_{m_1,m_2}(z_1,z_2)|^2 d\mu\right)^{1/2} \\ \le\ & K\overline{M}(r_1,r_2)\left(\frac{d_1e^{\varepsilon}}{r_1}\right)^{m_1}\left(\frac{d_2e^{\varepsilon}}{r_2}\right)^{m_2}. \end{aligned} \tag{2.15}$$

Now by (1.6), we have

$$\begin{aligned} b_{m_1+1,m_2+1} &= \int_E f(z_1,z_2)\overline{P_{m_1+1,m_2+1}(z_1,z_2)d\mu} \\ &= \int_E \left\{ f(z_1,z_2) - \sum_{j_1,j_2}^{m_1,m_2} b_{j_1:j_2}P_{j_1:j_2}(z_1,z_2)\right\}\overline{P_{m_1+1,m_2+1}(z_1,z_2)d\mu}. \end{aligned}$$

By Schwarz'a inequality, we have

$$\begin{aligned} |b_{m_1+1,m_2+1}|^2 &\le \left(\int_E \left| f(z_1,z_2) - \sum_{j_1,j_2=0}^{m_1,m_2} b_{j_1,j_2}P_{j_1,j_2}\right|^2 d\mu\right)\left(\int_E |P_{m_1+1,m_2+1}|^2 d\mu\right) \\ &= \int_E \left| f(z_1,z_2) - \sum_{j_1,j_2=0}^{m_1,m_2} b_{j_1,j_2}P_{j_1,j_2}(z_1,z_2)\right|^2 d\mu \\ &\le \int_E |f(z_1,z_2) - Q_{m_1,m_2}(z_1,z_2)|^2 d\mu, \end{aligned}$$

since Fourier sums give the best $L^2_{(\mu)}$ approximation. So (2.15) gives $|b_{m_1+1,m_2+1}|^2 \le K^2\left[\overline{M}(r_1,r_2)\left(\frac{d_1e^{\varepsilon}}{r_1}\right)^{m_1}\left(\frac{d_2e^{\varepsilon}}{r_2}\right)^{m_2}\right]^2$, which gives required result.

Lemma 2.7. *Let $f(z_1,z_2) \in \lceil$ is of order $\rho(0 < \rho < \infty)$ and type T.*

Then

$$\rho = \limsup_{r_1,r_2\to\infty} \frac{\log\log \overline{M}(r_1,r_2)}{\log(r_1r_2)}, \tag{2.16}$$

$$T = \limsup_{r_1,r_2\to\infty} \frac{\log \overline{M}(r_1,r_2)}{r_1^\rho + r_2^\rho}. \tag{2.17}$$

Proof. Let $(z_{1_0} z_{2_0})$ be a fixed point of the set E, and $r_1 > 1$, $r_2 > 1$. For every point $z_t \in E_{r_t}$ there exists a $z_t^* = z_t^*(z_t) \in E$, $t = 1, 2$ such that

$$|z_t - z_t^*| = \text{ dist } (z_t, E).$$

By the triangle inequality and by

$$\text{dist } (z_t, E) \le d_t(E) \exp V_\mu(z_t) \le \text{dist } (z_t, E) + |E| \quad \text{for} \quad z_t \in C^2 \setminus E.$$

We have

$$|z_t - z_{t_0}| \le |z_t - z_t^*| + |z_t^* - z_{t_0}| \le r_t + |E| \quad \text{for} \quad z_t \in E_{r_t},\ r_t > 1.$$

and

$$r_t - |E| \le |z_t - z_t^*|, \quad |E| \ge |z_t^* - z_{t_0}|.$$

We see that

$$r_t - 2|E| - |z_{t_0}| \le |z_t| \le r_t + |E| + |z_{t_0}| \quad \text{for} \quad z \in E_{r_t},\ r_t > 1.$$

Let $R_t > 1$ be such that

$$r_t - 2|E| - |z_{t_0}| \ge \frac{r_1}{2} \quad \text{and} \quad r_t + |E| + |z_{t_0}| \le 2r_t \quad \text{for} \quad r_t > R_t.$$

Hence for $r_t > R_t$ we have

$$\frac{\log\log M(\frac{r_1}{2},\frac{r_2}{2})}{\log(r_1r_2)} \leq \frac{\log\log \overline{M}(r_1,r_2)}{\log(r_1r_2)} < \frac{\log\log M(2r_1,2r_2)}{\log(r_1r_2)}$$

and if $0<\rho<\infty$,

$$\frac{\log M(r_1-a_1,r_2-a_2)}{r_1^\rho+r_2^\rho} \leq \frac{\log \overline{M}(r_1,r_2)}{r_1^\rho+r_2^\rho} \leq \frac{\log M(r_1+b_1,r_2+b_2)}{r_1^\rho+r_2^\rho},$$

where

$$a_1{=}2|E_1|{+}|z_{1_0}|,\ a_2{=}2|E_2|{+}|E_{2_0}|,\ b_1{=}|E_1|{+}|z_{1_0}|,\ b_2{=}|E_2|{+}|z_{2_0}|,\ E{=}E_1{\times}E_2.$$

Passing to limit superior the proof is completed.

3. Main results.

Theorem 3.1. *The entire function $f(z_1,z_2)\in L^2_{(\mu)}$ is of finite order ρ, if and only if*

$$\partial = \limsup_{m_1,m_2\to\infty} \frac{\log(m_1^{m_1}m_2^{m_2})}{\log|b_{m_1,m_2}|^{-1}} < \infty; \tag{3.10}$$

and then $\partial=\rho$.

Proof. Let $\partial<\infty$. Then for any $\varepsilon>0$ there exists $m_{1_0}=m_{1_0}(\varepsilon)$, $m_{2_0}=m_{2_0}(\varepsilon)$ such that

$$\frac{\log m_1^{m_1}m_2^{m_2}}{\log|b_{m_1,m_2}|^{-1}} \leq \partial+\varepsilon \quad \text{for} \quad m_1>m_{1_0},\ m_2>m_{2_0}$$

or

$$|b_{m_1,m_2}| \leq m_1^{-m_1/(\partial+\varepsilon)} m_2^{-m_2/(\partial+\varepsilon)},$$

which implies that

$$\lim_{m_1,m_2\to\infty} |b_{m_1,m_2}|^{1/(m_1+m_2)} = 0. \tag{3.11}$$

By corollary to Lemma 2.5, $f(z_1,z_2)\in\lceil$. Let its order by ρ. Since the Fourier expansions of $f(z_1,z_2)$ in $L^2_{(\mu)}$ is

$$f(z_1,z_2)=\sum_{m_1,m_2=0}^{\infty} b_{m_1,m_2}p_{m_1,m_2}(z_1,z_2)$$

and

$$\|f(z_1,z_2)\|_{L^2_{(\mu)}}\le 1,\quad |b_{m_1,m_2}|\le(\mu(E))^{1/2}.$$

Thus

$$|f(z_1,z_2)|\le(\mu(E))^{1/2}(m_1+1)(m_2+1)M_0\Big(\frac{r_1}{d_1}\Big)^{m_1}\Big(\frac{r_2}{d_2}\Big)^{m_2}(1+\varepsilon)^{m_1+m_2} \text{ for } z_t\in E_{r_t}.$$

So

$$\begin{aligned}\overline{M}(r_1,r_2) &\le M_0'g\left(\left(\frac{r_1(1+\varepsilon)}{d_1}\right),\left(\frac{r_2(1+\varepsilon)}{d_2}\right)\right)\\ &= M_0'M\left(\frac{r_1(1+\varepsilon)}{d_1},\frac{r_2(1+\varepsilon)}{d_2}\right),\end{aligned} \tag{3.12}$$

where

$$g(z_1,z_2)=\sum_{m_1,m_2=0}^{\infty} b_{m_1,m_2}z_1^{m_1}z_1^{m_2}\quad\text{for}\quad z_t\in E_{r_t} \tag{3.13}$$

and

$$M(r_1,r_2)=\max_{|z_t|=r_t}|g(z_1,z_2)|.$$

Hence by (3.11), $g(z_1,z_2)\in\lceil$ and (1.3) implies that it is of order ρ and (3.12) gives

$$\frac{\log\log\overline{M}(r_1,r_2)}{\log(r_1r_2)}\le\frac{\log\log M\left(\frac{r_1(1+\varepsilon)}{d_1},\frac{r_2(1+\varepsilon)}{d_2}\right)}{\log(r_1r_2)}+o(1),\quad\text{for large } r_1 \text{ and } r_2.$$

So we get

$$\rho\le\partial, \tag{3.14}$$

which show that $f(z_1,z_2)$ is of finite order ρ. Now let $f(z_1,z_2)\in\lceil$ of order $\rho<\infty$. By (2.16) for any $\varepsilon>0$ there exists $r_{1_0}=r_{1_0}(\varepsilon)$, $r_{2_0}=r_{2_0}(\varepsilon)$ such that

$$\overline{M}(r_1,r_2)<\exp(r_1^{(\rho+\varepsilon)}r_2^{(\rho+\varepsilon)})\quad\text{for}\quad r_1>r_{1_0}(\varepsilon),\ r_2>r_{2_0}(\varepsilon),$$

using Lemma 2.6, we have

$$|b_{m_1,m_2}|\le K\frac{\exp(r_1^{(\rho+\varepsilon)}r_2^{(\rho+\varepsilon)})}{r_1^{m_1-1}r_2^{m_2-1}}d_1^{m_1-1}d_2^{m_2-1}e^{(m_1+m_2-2)\varepsilon}\ \text{for large } K \text{ and } r_1,\ r_2.$$

Choosing a sequence $r_{m_1}\to\infty$, $r_{m_2}\to\infty$ as m_1, $m_2\to\infty$ defined as

$$r_{m_1}=\left(\frac{m_1-1}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)},\quad r_{m_2}=\left(\frac{m_2-1}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)}$$

in above expression, we get

$$|b_{m_1,m_2}|\le K\exp\left\{\frac{(m_1-1)(m_2-1)}{(\rho+\varepsilon)^2}\right\}\frac{d_1^{m_1-1}d_2^{m_2-1}e^{(m_1+m_2-2)\varepsilon}}{(\frac{m_1-1}{\rho+\varepsilon})^{(m_1-1)/(\rho+\varepsilon)}\left(\frac{m_2-1}{\rho+\varepsilon}\right)^{(m_2-1)/(\rho+\varepsilon)}}$$

or

$$\frac{\log|b_{m_1,m_2}|^{-1}}{\log m_1^{m_1}m_2^{m_2}}\ge\frac{\frac{m_1-1}{\rho+\varepsilon}\log\left(\frac{m_1-1}{\rho+\varepsilon}\right)+\frac{(m_2-1)}{\rho+\varepsilon}\log\left(\frac{m_2-1}{\rho+\varepsilon}\right)}{\log m_1^{m_1}m_2^{m_2}}+o(1)\ \text{as}\ m_1\to\infty,\ m_2\to\infty$$

or

$$\liminf_{m_1,m_2\to\infty}\frac{\log|b_{m_1m_2}|^{-1}}{\log m_1^{m_1}m_2^{m_2}}\ge\frac{1}{\rho+\varepsilon},$$

or

$$\limsup_{m_1,m_2\to\infty}\frac{\log m_1^{m_1}m_2^{m_2}}{\log|b_{m_1m_2}|^{-1}}\le\rho+\varepsilon,$$

which gives

$$\partial\le\rho+\varepsilon.$$

Since ε is arbitrary, so we get

$$\partial \leq \rho. \tag{3.15}$$

Which prove that (3.10) holds. Taking (3.14) and (3.15) together in to account, we get $\partial = \rho$. Hence the proof is completed.

Theorem 3.2. *Let $f(z_1, z_2) \in L^2_{(\mu)}$ and for $0 < \rho < \infty$, then $f(z_1, z_2)$ can be extended to an entire function of order $\rho (0 < \rho < \infty)$ and type $T(0 < T < \infty)$ if and only if*

$$d^\rho e \rho T = \limsup_{m_1, m_2 \to \infty} \{m_1^{m_1} m_2^{m_2} |b_{m_1,m_2}|^\rho\}^{1/(m_1+m_2)}. \tag{3.16}$$

Proof. Let (3.16) be holds, then we have to show that $f(z_1, z_2)$ can be extended to an entire function of order ρ and type T.

By (3.16) it can be easily seen that

$$\rho = \limsup_{m_1, m_1 \to \infty} \frac{\log m_1^{m_1} m_2^{m_2}}{\log |b_{m_1,m_2}|^{-1}}.$$

Using Theorem 3.1, we see that $f(z_1, z_2)$ is an entire function of finite order $\rho \neq 0$. Suppose $f(z_1, z_2)$ has type T, then using Lemma 2.7,

$$T = \limsup_{r_1, r_2 \to \infty} \frac{\log \overline{M}(r_1, r_2)}{r_1^\rho + r_2^\rho}.$$

Let $T < \infty$. For any $\varepsilon > 0$, there exists $r_1^0 = r_1^0(\varepsilon)$, $r_2^0 = r_2^0(\varepsilon)$ such that $\log \overline{M}(r_1, r_2) < (T + \varepsilon)(r_1^\rho + r_2^\rho)$ for $r_1 > r_1^0$, $r_2 > r_2^0$.

By Lemma 2.6, we obtain

$$\begin{aligned} \log |b_{m_1,m_2}| \leq{} & (T+\varepsilon)(r_1^\rho + r_2^\rho) + (m_1+m_2-2)\varepsilon - (m_1-1)\log(r_1/d_1) \\ & -(m_2-1)\log(r_2/d_2) + \log K \\ & \text{for } r_1 > r_1^0,\ r_2 > r_2^0 \text{ and } m_1 > m_1^0(\varepsilon),\ m_2 > m_2^0(\varepsilon). \end{aligned} \tag{3.17}$$

Choosing $r_{m_1} = \left(\frac{m_1}{\rho(T+\varepsilon)}\right)^{1/\rho}$, $r_{m_2} = \left(\frac{m_2}{\rho(T+\varepsilon)}\right)^{1/\rho}$, then for $r_1 = r_{m_1}$, $r_2 = r_{m_2}$, we get

$$\log|b_{m_1,m_2}| \leq \left(\frac{m_1+m_2}{\rho}\right) + (m_1+m_2-2)\varepsilon - \left(\frac{m_1-1}{\rho}\right)\log\left(\frac{m_1}{d_1^\rho \rho(T+\varepsilon)}\right) - \frac{m_2-1}{\rho}\log\left(\frac{m_2}{d_2^\rho \rho(T+\varepsilon)}\right) + \log K,$$

which gives

$$\limsup_{m_1,m_2\to\infty}\{m_1^{m_1}m_2^{m_2}|b_{m_1,m_2}|^\rho\}^{1/(m_1,+m_2)} \leq e\rho(T+\varepsilon)d_1^\rho d_2^\rho e^{\rho\varepsilon},$$

since this is true for every $\varepsilon > 0$, we have

$$e\rho T d^\rho \geq \limsup_{m_1,m_2\to\infty}\{m_1^{m_1}m_2^{m_2}|b_{m_1,m_2}|^\rho\}^{1/m_1+m_2}, \tag{3.18}$$

By (3.12), we obtain

$$\limsup_{r_1,r_2\to\infty}\frac{\log\overline{M}(r_1,r_2)}{r_1^\rho + r_2^\rho} \leq \left(\frac{1+\varepsilon}{d_1}\right)^\rho\left(\frac{1+\varepsilon}{d_2}\right)^\rho \quad \text{type of } g(z_1,z_2).$$

Using Lemma 2.7 and (1.4), leads to

$$Te\rho d^\rho \leq \limsup_{m_1,m_2\to\infty}\{m_1^{m_1}m_2^{m_2}|b_{m_1,m_2}|^\rho\}^{1/(m_1+m_2)}. \tag{3.19}$$

Combinding (3.18) and (3.19) we get the required result.

The converse part is left to the reader.

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