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# ON A FUNCTIONAL EQUATION CHARACTERIZING POLYNOMIALS OF DEGREE THREE

BY

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**Abstract**. In this paper, we determine the general solution of the functional equation $f(x+2y)+f(x-2y)+6f(x)=4\left[f(x+y)+f(x-y)\right]$ for all $x,y\in\mathbb{R}$ without assuming any regularity conditions on the unknown function $f$. The method used for solving this functional equation is elementary but exploits an important result due to M. Hosszu [2]. The solution of this functional equation is also determined in certain commutative groups using two important results due to L. Székelyhidi [4].

## 1. Introduction.

The following identities

$$(x+2y)+(x-2y)+6x=4(x+y)+4(x-y), \tag{1.1}$$

$$(x+2y)^2+(x-2y)^2+6x^2=4(x+y)^2+4(x-y)^2, \tag{1.2}$$

$$(x+2y)^3+(x-2y)^3+6x^3=4(x+y)^3+4(x-y)^3 \tag{1.3}$$

can be combined into $f(x+2y)+f(x-2y)+6f(x)=4\left[f(x+y)+f(x-y)\right]$ where $f(x)=x^n$ for $n=1,2,3$. In this paper, we determine the general

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solution of the functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4\,[f(x+y)+f(x-y)] \tag{1.4}$$

for all $x,y\in\mathbb{R}$ (the set of reals). We will solve this functional equation using an elementary technique but without using any regularity condition.

A function $A:\mathbb{R}\to\mathbb{R}$ is said to be *additive* if $A(x+y)=A(x)+A(y)$ for all $x$, $y\in\mathbb{R}$ (see [1]). Let $n\in\mathbb{N}$ (the set of natural numbers). A function $A_n:\mathbb{R}^n\to\mathbb{R}$ is called $n$-*additive* if it is additive in each of its variable. A function $A_n$ is called *symmetric* if $A_n(x_1,x_2,\dots,x_n)=A_n(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)})$ for every permutation $\{\pi(1),\pi(2),\dots,\pi(n)\}$ of $\{1,2,\dots,n\}$. If $A_n(x_1,x_2,\dots,x_n)$ is a $n$-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x,x,\dots,x)$. Further the resulting function after substitution $x_1=x_2=\cdots=x_\ell=x$ and $x_{\ell+1}=x_{\ell+2}=\cdots=x_n=y$ in $A_n(x_1,x_2,\dots,x_n)$ will be denoted by $A^{\ell,n-\ell}(x,y)$.

For $f:\mathbb{R}\to\mathbb{R}$, let $\Delta_h$ be the difference operator defined as follows:

$$\Delta_h f(x)=f(x+h)-f(x)\qquad \text{for } h\in\mathbb{R}.$$

Further, let $\Delta_h^0 f(x)=f(x)$, $\Delta_h^1 f(x)=\Delta_h f(x)$ and $\Delta_h\circ\Delta_h^n f(x)=\Delta_h^{n+1}f(x)$ for all $n\in\mathbb{N}$ and all $h\in\mathbb{R}$. Here $\Delta_h\circ\Delta_h^n$ denotes the composition of the operators $\Delta_h$ and $\Delta_h^n$. For any given $n\in\mathbb{N}\cup\{0\}$, the functional equation

$$\Delta_h^{n+1}f(x)=0$$

for all $x,h\in\mathbb{R}$ is well studied. In explicit form the last functional equation can be written as

$$\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}f(x+kh)=0.$$

It is known (see Kuczma [3]) that in the case where one deals with functions defined in $\mathbb{R}$ the last functional equation is equivalent to the Fréchet functional equation

$$\Delta_{h_1,\dots,h_{n+1}} f(x) = 0 \tag{1.5}$$

where $\Delta_{h_1,\dots,h_k} = \Delta_{h_k} \circ \cdots \circ \Delta_{h_1}$ for every $k \in \mathbb{N}$ and $x,\ h_1, \dots, h_k \in \mathbb{R}$.

The following lemma is a special case of a more general result due to Hosszu [2], and will be instrumental in determining the general solution of (1.4).

**Lemma 1.1.** *The map $F$ from $\mathbb{R}$ into $\mathbb{R}$ satisfies the functional equation*

$$\Delta_{x_1,\dots,x_4} F(x_0) = 0 \tag{1.6}$$

*for all $x_0,\ x_1,\ x_2,\ x_3,\ x_4 \in \mathbb{R}$ if and only if $F$ is given by*

$$F(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \qquad \forall\, x \in \mathbb{R}, \tag{1.7}$$

*where $A^0(x) = A^0$ is an arbitrary constant and $A^n(x)$ is the diagonal of a $n$-additive symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3$.*

**2. Solution of the equation (1.4) on reals.** Now we determine the general solution of the functional equation (1.4) by reducing it to the Fréchet functional equation (1.6).

**Theorem 2.1.** *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.4) for all $x, y \in \mathbb{R}$ if and only if $f$ is of the form*

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \qquad \forall\, x \in \mathbb{R},$$

*where $A^n(x)$ is the diagonal of the $n$-additive map $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3$, and $A^0(x) = A^0$ is an arbitrary constant.*

*Proof.* Substitute $x_0 = x + 2y$ and $y_1 = x - 2y$ that is $x = \frac{1}{2}(x_0 + y_1)$ and $y = \frac{1}{4}(x_0 - y_1)$ in (1.4) to get

$$f(x_0)+f(y_1)+6f\left(\frac{1}{2}x_0+\frac{1}{2}y_1\right)=4f\left(\frac{3}{4}x_0+\frac{1}{4}y_1\right)+4f\left(\frac{1}{4}x_0+\frac{3}{4}y_1\right). \tag{2.1}$$

Replacing $x_0$ by $x_0 + x_1$ in (2.1), we obtain

$$\begin{aligned} & f(x_0+x_1)+f(y_1)+6f\left(\frac{1}{2}(x_0+x_1)+\frac{1}{2}y_1\right) \\ = \; & 4f\left(\frac{3}{4}(x_0+x_1)+\frac{1}{4}y_1\right)+4f\left(\frac{1}{4}(x_0+x_1)+\frac{3}{4}y_1\right). \end{aligned} \tag{2.2}$$

Subtracting (2.1) from (2.2), we have

$$\begin{aligned} & f(x_0+x_1)-f(x_0)+6f\left(\frac{1}{2}(x_0+x_1)+\frac{1}{2}y_1\right)-6f\left(\frac{1}{2}x_0+\frac{1}{2}y_1\right) \\ & =4f\left(\frac{3}{4}(x_0+x_1)+\frac{1}{4}y_1\right)-4f\left(\frac{3}{4}x_0+\frac{1}{4}y_1\right) \\ & \quad +4f\left(\frac{1}{4}(x_0+x_1)+\frac{3}{4}y_1\right)-4f\left(\frac{1}{4}x_0+\frac{3}{4}y_1\right). \end{aligned} \tag{2.3}$$

Letting $y_2 = \frac{3}{4}x_0 + \frac{1}{4}y_1$ (that is, $y_1 = 4y_2 - 3x_0$) in (2.3), we see that

$$\begin{aligned} & f(x_0+x_1)-f(x_0)+6f\left(\frac{1}{2}x_1-x_0+2y_2\right)-6f\left(2y_2-x_0\right) \\ & =4f\left(y_2+\frac{3}{4}x_1\right)-4f\left(y_2\right)+4f\left(-2x_0+3y_2+\frac{1}{4}x_1\right)-4f\left(-2x_0+3y_2\right). \end{aligned} \tag{2.4}$$

Now replacing $x_0$ by $x_0+x_2$ in (2.4) and subtracting (2.4) from the resulting expression, we obtain

$$\begin{aligned} & f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0) \\ & +6f\left(2y_2-(x_0+x_2)+\frac{1}{2}x_1\right)-6f\left(2y_2-(x_0+x_2)\right) \\ & +6f\left(2y_2-x_0+\frac{1}{2}x_1\right)-6f\left(2y_2-x_0\right) \\ = \; & 4f\left(3y_2+\frac{1}{4}x_1-2(x_0+x_2)\right)-4f\left(3y_2-2(x_0+x_2)\right) \end{aligned} \tag{2.5}$$

$$-4f\left(3y_2-2x_0+\frac{1}{4}x_1\right)+4f\left(3y_2-2x_0\right).$$

Now we substitute $y_3=3y_2-2x_0$ in (2.5) to get

$$\begin{aligned}
&f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0)\\
&+6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0+\frac{1}{2}x_1-x_2\right)-6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0-x_2\right)\\
&+6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0+\frac{1}{2}x_1\right)-6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0\right)\\
=\ &4f\left(y_3+\frac{1}{4}x_1-2x_2\right)-4f\left(y_3-2x_2)\right)-4f\left(y_3+\frac{1}{4}x_1\right)+4f\left(y_3\right).
\end{aligned}\tag{2.6}$$

Again we replace $x_0$ by $x_0+x_3$ in (2.6) and then subtracting (2.6) from the resulting expression, we have

$$\begin{aligned}
&f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)\\
&-f(x_0+x_2+x_3)+f(x_0+x_1)+f(x_0+x_2)+f(x_0+x_3)-f(x_0)\\
&+6f\left(\frac{2}{3}y_3+\frac{1}{3}(x_0+x_3)+\frac{1}{2}x_1-x_2\right)-6f\left(\frac{2}{3}y_3+\frac{1}{3}(x_0+x_3)-x_2\right)\\
&-6f\left(\frac{2}{3}y_3+\frac{1}{3}(x_0+x_3)+\frac{1}{2}x_1\right)+6f\left(\frac{2}{3}y_3+\frac{1}{3}(x_0+x_3)\right)\\
&-6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0+\frac{1}{2}x_1-x_2\right)+6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0-x_2\right)\\
&+6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0+\frac{1}{2}x_1\right)-6f\left(\frac{2}{3}y_3+\frac{1}{3}x_0\right)=0.
\end{aligned}\tag{2.7}$$

Letting $y_4=\frac{2}{3}y_3+\frac{1}{3}x_0$ in the equation (2.7), we obtain

$$\begin{aligned}
&f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)\\
&-f(x_0+x_2+x_3)+f(x_0+x_1)+f(x_0+x_2)+f(x_0+x_3)-f(x_0)\\
&+6f\left(y_4+\frac{1}{3}x_3+\frac{1}{2}x_1-x_2\right)-6f\left(y_4+\frac{1}{3}x_3-x_2\right)\\
&-6f\left(y_4+\frac{1}{3}x_3+\frac{1}{2}x_1\right)+6f\left(y_4+\frac{1}{3}x_3\right)\\
&-6f\left(y_4+\frac{1}{2}x_1-x_2\right)+6f\left(y_4-x_2\right)
\end{aligned}\tag{2.8}$$

$$+6f\left(y_4+\frac{1}{2}x_1\right)-6f\left(y_4\right)=0.$$

Now we replace $x_0$ by $x_0+x_4$ in the equation (2.8) to get

$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_4)\\
&-f(x_0+x_1+x_3+x_4)-f(x_0+x_2+x_3+x_4)\\
&+f(x_0+x_1+x_4)+f(x_0+x_2+x_4)+f(x_0+x_3+x_4)-f(x_0+x_4)\\
&+6f\left(y_4+\frac{1}{3}x_3+\frac{1}{2}x_1-x_2\right)-6f\left(y_4+\frac{1}{3}x_3-x_2\right)\\
&-6f\left(y_4+\frac{1}{3}x_3+\frac{1}{2}x_1\right)+6f\left(y_4+\frac{1}{3}x_3\right)\\
&-6f\left(y_4+\frac{1}{2}x_1-x_2\right)+6f\left(y_4-x_2\right)\\
&+6f\left(y_4+\frac{1}{2}x_1\right)-6f\left(y_4\right)=0.
\end{aligned}\tag{2.9}$$

Subtracting (2.8) from (2.9), we obtain

$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_3)\\
&-f(x_0+x_1+x_2+x_4)-f(x_0+x_1+x_3+x_4)\\
&-f(x_0+x_2+x_3+x_4)+f(x_0+x_1+x_2)+f(x_0+x_1+x_3)\\
&+f(x_0+x_1+x_4)+f(x_0+x_2+x_3)+f(x_0+x_2+x_4)+f(x_0+x_3+x_4)\\
&-f(x_0+x_1)-f(x_0+x_2)-f(x_0+x_3)-f(x_0+x_4)+f(x_0)=0
\end{aligned}$$

which is

$$\Delta_{x_1,\ldots,x_4}f(x_0)=0\tag{2.10}$$

for all $x_0,x_1,x_2,x_3,x_4\in\mathbb{R}$. Hence from Lemma 1.1 we have

$$f(x)=A^3(x)+A^2(x)+A^1(x)+A^0(x),\qquad\forall\,x\in\mathbb{R},\tag{2.11}$$

where $A^n(x)$ is the diagonal of the $n$-additive map $A_n:\mathbb{R}^n\to\mathbb{R}$ for $n=1,2,3$, and $A^0(x)=A^0$ is an arbitrary constant. Letting (2.11) into (1.4) and noting that

$$A^3(x+y)+A^3(x-y)=2\,A^3(x)+6\,A^{1,2}(x,y),$$

$$A^2(x+y)+A^2(x-y)=2\,A^2(x)+2\,A^2(y),$$

and $A^{1,2}(x,2y) = 4A^{1,2}(x,y)$, we conclude that $f$ in (2.11) satisfies (1.4). The proof of the theorem is now complete.

The following corollary follows from the above theorem.

**Corollary 2.2.** *The continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.4) *for all $x,y\in\mathbb{R}$ if and only if $f$ is of the form*

$$f(x)=a_3x^3+a_2x^2+a_1x+a_0, \qquad \forall\, x\in\mathbb{R},$$

*where $a_3,a_2,a_1,a_0$ are arbitrary real constants.*

**3. Solution of the equation (1.4) on commutative groups.** In this section, we solve the functional equation (1.4) on commutative groups with some additional requirements.

A group $\mathbb{G}$ is said to be *divisible* if for every element $b\in\mathbb{G}$ and every $n\in\mathbb{N}$, there exists an element $a\in\mathbb{G}$ such that $na=b$. If this element $a$ is unique, then $\mathbb{G}$ is said to be *uniquely divisible*. In a uniquely divisible group, this unique element $a$ is denoted by $\frac{b}{n}$. The equation $na=b$ has a solution is equivalent to say that the multiplication by $n$ is surjective. Similarly, the equation $na=b$ has a unique solution is equivalent to say that the multiplication by $n$ is bijective. Thus the notions of $n$-divisibility and $n$-unique divisibility refer, respectively, to surjectivity and bijectivity of the multiplication by $n$.

The proof of Theorem 2.1 can be generalized to abstract structures by using a more general result of Hosszu [2] instead of Lemma 1.1. Since the proof of the following theorem is identical to the proof of Theorem 2.1 we omit its proof.

**Theorem 3.1.** *Let $\mathbb{G}$ and $\mathbb{S}$ be uniquely divisible abelian groups. The function $f : \mathbb{G} \to \mathbb{S}$ satisfies the functional equation* (1.4) *for all $x, y \in \mathbb{G}$ if and only if $f$ is of the form*

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \qquad \forall\, x \in \mathbb{G},$$

*where $A^n(x)$ is the diagonal of the $n$-additive map $A_n : \mathbb{G}^n \to \mathbb{S}$ for $n = 1, 2, 3,$ and $A^0(x) = A^0$ is an arbitrary element in $\mathbb{S}$.*

The unique divisibility requirement of the groups in Theorem 3.1 can be weaken using two important results due to Székelyhidi [4]. With the use of the two important results, the proof becomes even shorter but not so elementary any more. The results needed for this improvements are the followings (see [4], pp.70-72):

**Theorem 3.2.** *Let $\mathbb{G}$ be a commutative semigroup with identity, $\mathbb{S}$ a commutative group and $n$ a nonnegative integer. Let the multiplication by $n!$ be bijective in $\mathbb{S}$. The function $f : \mathbb{G} \to \mathbb{S}$ is a solution of Fréchet functional equation*

$$\Delta_{x_1,\dots,x_{n+1}} f(x_0) = 0 \qquad \forall\ x_0, x_1, \dots, x_{n+1} \in \mathbb{G} \tag{3.1}$$

*if and only if $f$ is a polynomial of degree at most $n$.*

**Theorem 3.3.** *Let $\mathbb{G}$ and $\mathbb{S}$ be commutative groups, $n$ a nonnegative integer, $\phi_i$, $\psi_i$ additive functions from $\mathbb{G}$ into $\mathbb{G}$ and $\phi_i(\mathbb{G}) \subseteq \psi_i(\mathbb{G})$ $(i = 1, 2, \dots, n+1)$. If the functions $f, f_i : \mathbb{G} \to \mathbb{S}$ $(i = 1, 2, \dots, n+1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i\left(\phi_i(x) + \psi_i(y)\right) = 0$$

*then $f$ satisfies Fréchet functional equation* (3.1).

The following corollary follows from the above two theorems.

**Corollary 3.4.** *Let $\mathbb{G}$ and $\mathbb{S}$ be commutative groups, $n$ a nonnegative integer, $k_i$ a nonzero integer, $i \in \{1, 2, \ldots, n+1\}$. Let the multiplication by $k_i$ be surjective in $\mathbb{G}$, $i \in \{1, 2, \ldots, n+1\}$, and let the multiplication by $n!$ be bijective in $\mathbb{S}$. If the functions $f, f_i : \mathbb{G} \to \mathbb{S}$, $i \in \{1, 2, \ldots, n+1\}$ satify*

$$f(x) + \sum_{i=1}^{n+1} f_i\,(x + k_i y) = 0 \tag{3.2}$$

*for all $x, y \in \mathbb{G}$ then $f$ is a polynomial of degree at most $n$, that is $f$ is of the form*

$$f(x) = A^0(x) + A^1(x) + A^2(x) + \cdots + A^n(x), \tag{3.3}$$

*where $A^0(x) = A^0$ is an arbitrary constant in $\mathbb{S}$, $A_1 \in Hom(\mathbb{G}, \mathbb{S})$, and $A^n(x)$ is the diagonal of a $n$-additive symmetric function $A_n : \mathbb{G}^n \to \mathbb{S}$, $n \in \{2, 3, \ldots, n\}$.*

Using Corollary 3.4, an improved version of Theorem 3.1 can be established in the general setting of Theorem 3.2. and Theorem 3.3.

**Theorem 3.5.** *Let $\mathbb{G}$ and $\mathbb{S}$ be divisible abelian groups. Let the multiplication by 2 be surjective in $\mathbb{G}$ and let the multiplication by 6 be bijective in $\mathbb{S}$. The function $f : \mathbb{G} \to \mathbb{S}$ satisfies the functional equation* (1.4) *for all $x, y \in \mathbb{G}$ if and only if $f$ is of the form*

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \qquad \forall\, x \in \mathbb{G},$$

*where $A^n(x)$ is the diagonal of the $n$-additive symmetric map $A_n : \mathbb{G}^n \to \mathbb{S}$ for $n = 1, 2, 3$, and $A^0(x) = A^0$ is an arbitrary element in $\mathbb{S}$.*

*Proof.* To prove the theorem it is enough to observe that the unique divisibility of $\mathbb{S}$ by 6 allows one to write (1.4) in the form of (3.2) where $f_1 = f_2 = \frac{1}{6}f$, $f_3 = f_4 = -\frac{2}{3}f$, $k_1 = 2$, $k_2 = -2$, $k_3 = 1$, $k_4 = -1$. By Corollary 3.4 we get that $f$ is of the form (3.3). The same argument as used in the last five lines of the proof of Theorem 2.1 shows that any function of the form (3.3) actually satisfies (1.4).

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