BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 32, Number 1, March 2004

A STRONG LAW FOR COMPACTLY UNIFORMLY INTEGRABLE SEQUENCES OF INDEPENDENT RANDOM ELEMENTS IN BANACH SPACES

BY

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Abstract. For a compactly uniformly integrable sequence of independent random elements $\{V_n, n \geq 1\}$ in a real separable Banach space, conditions are provided for the strong law of large numbers $\sum_{i=1}^{n}(V_i - EV_i)/b_n \to 0$ almost certainly to hold where $\{b_n, n \geq 1\}$ is a sequence of positive constants. The main result is general enough to include as special cases a strong law of Adler, Rosalsky, and Taylor [3] for compactly uniformly integrable sequences and a strong law of Taylor and Wei [11] for uniformly tight sequences. Illustrative examples are provided which show that the main result is sharp or which show how it improves upon or is different from other results in the literature.

1. Introduction. Throughout this paper, let $\mathcal{X}$ be a real separable Banach space with norm $||\cdot||$ and let $\{V_n, n \geq 1\}$ be a sequence of independent random elements in $\mathcal{X}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. The sequence $\{V_n, n \geq 1\}$ is said to obey the *strong law of large numbers* (SLLN) with (nonrandom) centering elements $\{c_n, n \geq 1\} \subseteq \mathcal{X}$ and norming

Received by the editors April 2, 2002.

AMS 2002 Subject Classification: Primary 60B12, 60F15; Secondary 60B11.

Key words and phrases: Real separable Banach space, compactly uniformly integrable sequence of independent random elements, strong law of large numbers, almost certain convergence.

constants $\{b_n > 0, n \geq 1\}$ if

$$\frac{\sum_{i=1}^{n}(V_i - c_i)}{b_n} \to 0 \quad \text{almost certainly (a.c.).}$$

In this paper, the main result (Theorem 3.1) establishes a SLLN for the sequence $\{V_n, n \geq 1\}$ under the assumption that it is compactly uniformly integrable. (Technical definitions such as this will be discussed in Section 2.) No conditions are imposed on the underlying Banach space $\mathcal{X}$. Theorem 3.1 is new even when specialized to the (real-valued) random variable case and it includes as corollaries

- a SLLN of Adler, Rosalsky, and Taylor [3] for a compactly uniformly integrable sequence of independent random elements (Corollary 3.2)
- a SLLN of Taylor and Wei [11] for a uniformly tight sequence of independent random elements (Corollary 3.3).

The plan of the paper is as follows. For convenience, technical definitions and preliminary results are consolidated into Section 2. Theorem 3.1 and its corollaries will be established in Section 3. In Section 4, an example is provided showing that Theorem 3.1 is sharp. Moreover, two examples are given showing that the hypotheses of Theorem 3.1 can hold but not those of Corollary 3.2, and for one of these examples, the hypotheses fail for a related result of Cantrell and Rosalsky [4] which provided a SLLN for sums of independent random elements in *Rademacher type* p $(1 \leq p \leq 2)$ Banach spaces.

2. Preliminaries.

Some definitions and preliminary results will be presented prior to establishing the main result.

We define the *expected value* or *mean* of a random element V, denoted EV, to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathcal{X}$. A sufficient condition for EV to exist is that $E||V|| < \infty$ (see, e.g., Taylor [10], p.40).

A sequence of random elements $\{V_n, n \geq 1\}$ is said to be *compactly uniformly integrable* if for every $\varepsilon > 0$, there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathcal{X}$ such that

$$\sup_{n\geq 1} E||V_n I(V_n \notin \mathcal{K}_\varepsilon)|| \leq \varepsilon.$$

A sequence of random elements being compactly uniformly integrable is the natural extension of a sequence of random variables being uniformly integrable in that the two definitions are equivalent when the Banach space is the real line.

A sequence of random elements $\{V_n, n \geq 1\}$ is said to be *uniformly tight* if for every $\varepsilon > 0$, there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathcal{X}$ such that

$$\sup_{n\geq 1} P\{V_n \notin \mathcal{K}_\varepsilon\} \leq \varepsilon.$$

It was shown by Daffer and Taylor [7] that compact uniform integrability implies uniform tightness. Cuesta and Matrán [6] observed that compact uniform integrability of $\{V_n, n \geq 1\}$ also implies that the sequence of random variables $\{||V_n||, n \geq 1\}$ is uniformly integrable. Conversely, Cuesta and Matrán [6] observed that if $\{V_n, n \geq 1\}$ is uniformly tight and $\{||V_n||, n \geq 1\}$ is uniformly integrable, then $\{V_n, n \geq 1\}$ is compactly uniformly integrable. See Wang and Bhaskara Rao [12] and Cuesta and Matrán [6] for further discussion concerning compact uniform integrability and uniform tightness.

A real separable Banach space is said to be of *Rademacher type p* $(1 \leq p \leq 2)$ if there exists a constant $0 < C < \infty$ such that

$$E\Big|\Big|\sum_{i=1}^{n} V_i\Big|\Big|^p \leq C\sum_{i=1}^{n} E||V_i||^p$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent mean 0 random elements. If a real separable Banach space is of Rademacher type p for some

$1<p\leq 2$, then it is of Rademacher type q for all $1\leq q<p$. Every real separable Banach space is of Rademacher type (at least) 1 while the $\mathcal{L}_p$-spaces and ℓ_p-spaces are of Rademacher type $2\wedge p$ for $p\geq 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line is of Rademacher type 2.

Two related results of Cantrell and Rosalsky [4] and a lemma of Adler and Rosalsky [1] will now be stated. Proposition 2.1 will be used in the proof of Theorem 3.1. Lemma 2.1 will be used in Example 4.2. Moreover, it will be shown in Remark 4.1(i) that apropos of Example 4.3 the assumptions of Theorem 3.1 can be satisfied but not those of Proposition 2.2.

Proposition 2.1. (Cantrell and Rosalsky [4]). *Let $\{V_n, n\geq 1\}$ be a sequence of independent random elements in a real separable Rademacher type p $(1\leq p\leq 2)$ Banach space, and let $\{a_n, n\geq 1\}$ and $\{b_n, n\geq 1\}$ be sequences of positive constants with $b_n\uparrow\infty$ such that either*

$$\sum_{n=1}^{\infty}\frac{a_n^p}{b_n^p}<\infty \quad \text{or} \quad \sum_{i=1}^{n}a_i=\mathcal{O}(b_n).$$

Suppose that for some $\lambda>0$ and for all $\varepsilon>0$

$$\sum_{n=1}^{\infty}P\{||V_n||>\lambda b_n\}<\infty$$

and

$$\sum_{n=1}^{\infty}\frac{1}{b_n^p}E||V_nI(\varepsilon a_n<||V_n||\leq\lambda b_n)-E(V_nI(\varepsilon a_n<||V_n||\leq\lambda b_n))||^p<\infty. \tag{2.1}$$

Then the SLLN

$$\frac{\sum_{i=1}^{n}\Big(V_i-E(V_iI(||V_i||\leq\lambda b_i))\Big)}{b_n}\to 0 \ a.c.$$

obtains.

Proposition 2.2. (Cantrell and Rosalsky [4]). *Let $\{V_n, n \geq 1\}$ be a sequence of independent random elements in a real separable Rademacher type p $(1 \leq p \leq 2)$ Banach space, and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive constants. Suppose that all of the hypotheses of Proposition* 2.1 *are satisfied, that the $\{V_n, n \geq 1\}$ all have expected values, and that*

$$\frac{\sum_{i=1}^{n} E(V_i I(||V_i|| > \lambda b_i))}{b_n} \to 0. \tag{2.2}$$

Then the SLLN

$$\frac{\sum_{i=1}^{n}(V_i - EV_i)}{b_n} \to 0 \quad a.c.$$

obtains.

Propositions 2.1 and 2.2 are general enough results to contain, respectively:

- a well-known SLLN due to Heyde [9] for the random variable case
- the Adler, Rosalsky, and Taylor [2] extension to a Banach space setting of Feller's [8] famous generalization of the Marcinkiewicz-Zygmund SLLN.

An example illustrating the sharpness of Proposition 2.1 as well as examples satisfying the hypotheses of Proposition 2.1 but not those of Heyde's [9] theorem may be found in Cantrell and Rosalsky [4].

The following lemma will be used in the proof of Corollary 3.3.

Lemma 2.1. *If $\{V_n, n \geq 1\}$ is a sequence of random elements with $\sup_{n\geq 1} E||V_n||^r < \infty$ for some $r > 0$, then $\sup_{n\geq 1} E||V_n||^p < \infty$ for all $0 < p < r$.*

Proof. Let $0 < p < r$ and note that for $n \ge 1$

$$\begin{aligned} E||V_n||^p &= E(||V_n||^p I(||V_n|| \le 1)) + E(||V_n||^p I(||V_n|| > 1)) \\ &\le 1 + E\left(\frac{||V_n||^r}{||V_n||^{r-p}} I(||V_n|| > 1)\right) \\ &\le 1 + E(||V_n||^r I(||V_n|| > 1)) \\ &\le 1 + E||V_n||^r. \end{aligned}$$

Thus

$$\sup_{n\ge 1} E||V_n||^p \le 1 + \sup_{n \ge 1} E||V_n||^r < \infty.$$

The following lemma of Adler and Rosalsky [1] will be used in Example 4.3.

Lemma 2.2. (Adler and Rosalsky [1]). *Let $\{Y_n, n \ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable Y in the sense that for some constant $0 < D < \infty$*

$$P\{|Y_n| > t\} \le DP\{|DY| > t\}, \quad t > 0, \quad n \ge 1.$$

Let $\{b_n, n \ge 1\}$ be a sequence of positive constants such that

$$\Big(\max_{1 \le i \le n} b_i^p\Big) \sum_{i=n}^{\infty} \frac{1}{b_i^p} = \mathcal{O}(n) \quad \textit{for some } p > 0$$

and

$$\sum_{n=1}^{\infty} P\{|Y| > Db_n\} < \infty.$$

Then for all $0 < M < \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(|Y_n|^p I(|Y_n| \le Mb_n) < \infty.$$

Finally, some remarks about notation are in order. It proves convenient to define $\log x = \log_e(e \vee x)$, $x > 0$ where $\log_e$ denotes the logarithm to the base e. Moreover, the symbol C denotes throughout a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

3. Mainstream. With the preliminaries accounted for, Theorem 3.1 may be presented. As will become apparent, the proof of Theorem 3.1 owes much to the work of Cuesta and Matrán [6].

Theorem 3.1. *Let $\{V_n, n \geq 1\}$ be a sequence of independent random elements in a real separable Banach space, let $1 \leq p \leq 2$, and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive constants with $b_n \uparrow$ and $n = \mathcal{O}(b_n)$ such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \tag{3.1}$$

or

$$\sum_{i=1}^{n} a_i = \mathcal{O}(b_n). \tag{3.2}$$

Suppose that for some $\lambda > 0$ and for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E\Big| ||V_n|| I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(||V_n|| I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))\Big|^p < \infty. \tag{3.4}$$

Then if

(3.5) $\{V_n, n \geq 1\}$ *is compactly uniformly integrable,*

the SLLN

$$\frac{\sum_{i=1}^{n}(V_i - EV_i)}{b_n} \to 0 \quad a.c.$$

obtains.

Proof. Let C be such that $n \le Cb_n, n \ge 1$. Clearly $b_n \uparrow \infty$. In view of the work of Cuesta and Matrán ([6], Section 4), it suffices to verify that

$$\frac{\sum_{i=1}^{n}(||V_i|| - E||V_i||)}{b_n} \to 0 \ \text{ a.c.} \tag{3.6}$$

and that

$$\frac{\sum_{i=1}^{n}\Big(g(V_i) - Eg(V_i)\Big)}{b_n} \to 0 \ \text{ a.c.} \tag{3.7}$$

for every bounded and continuous real function g on $\mathcal{X}$. To prove (3.6), note that since R is of Rademacher type p for every $p \in [1,2]$, we can apply Proposition 2.1 to the sequence of random variables $\{||V_n||, n \ge 1\}$ thereby yielding

$$\frac{\sum_{i=1}^{n}\Big(||V_i|| - E(||V_i||I(||V_i|| \le \lambda b_i))\Big)}{b_n} \to 0 \ \text{ a.c.} \tag{3.8}$$

It will now be shown that

$$E(||V_n||I(||V_n|| > \lambda b_n)) \to 0. \tag{3.9}$$

Let $\varepsilon > 0$ be arbitrary. By (3.5), there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathcal{X}$ such that

$$\sup_{n\ge 1} E||V_n I(V_n \notin \mathcal{K}_\varepsilon)|| \le \varepsilon.$$

Since $\mathcal{K}_\varepsilon$ is compact, it is bounded and so there exists a constant $M < \infty$ such that

$$\mathcal{K}_\varepsilon \subseteq \{v \in \mathcal{X} : ||v|| \le M\}.$$

Thus, whenever n is such that $\lambda b_n \ge M$,

$$[||V_n|| > \lambda b_n] \subseteq [V_n \notin \mathcal{K}_\varepsilon].$$

Then since $b_n \uparrow \infty$, for all large n

$$E(||V_n||I(||V_n|| > \lambda b_n)) \leq E(||V_n||I(V_n \notin \mathcal{K}_\varepsilon)) \leq \varepsilon$$

thereby establishing (3.9) since $\varepsilon > 0$ is arbitrary. But then

$$\frac{\sum_{i=1}^{n} E(||V_i||I(||V_i|| > \lambda b_i))}{b_n} \leq \frac{C\sum_{i=1}^{n} E(||V_i||I(||V_i|| > \lambda b_i))}{n} \to 0 \tag{3.10}$$

by (3.9) and the Cesàro mean summability theorem. Combining (3.8) and (3.10) yields (3.6).

Next, to verify (3.7), let g be a bounded and continuous real function defined on $\mathcal{X}$. Then, letting $B = \sup\{|g(v)| : v \in \mathcal{X}\}$,

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}\,(g(V_n))}{b_n^2} \leq \sum_{n=1}^{\infty} \frac{E(g(V_n))^2}{b_n^2} \leq C^2B^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and hence by the Khintchine-Kolmogorov convergence theorem

$$\sum_{n=1}^{\infty} \frac{g(V_n) - E(g(V_n))}{b_n} \quad \text{converges a.c.}$$

and (3.7) follows immediately by the Kronecker lemma.

Remarks 3.1. (i) Observe that (3.1) is weaker for larger p.

(ii) A sufficient condition for (3.4) is of course that

$$\sum_{n=1}^{\infty} \frac{E(||V_n||^p I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))}{b_n^p} < \infty. \tag{3.11}$$

Corollary 3.1. *Let $\{V_n, n \geq 1\}$ be a compactly uniformly integrable sequence of independent random elements in a real separable Banach space. If $\{b_n, n \geq 1\}$ is a sequence of positive constants with $b_n \uparrow$, $n = \mathcal{O}(b_n)$, and*

$$\sum_{n=1}^{\infty} E\left(\frac{||V_n||^2}{||V_n||^2 + b_n^2}\right) < \infty, \tag{3.12}$$

then the SLLN

$$\frac{\sum_{i=1}^{n}(V_i - EV_i)}{b_n} \to 0 \quad a.c. \tag{3.13}$$

obtains.

Proof. By the argument of Heyde [9], the condition (3.12) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} P\{||V_n|| > b_n\} < \infty \tag{3.14}$$

and

$$\sum_{n=1}^{\infty} \frac{E(||V_n||^2 I(||V_n|| \le b_n))}{b_n^2} < \infty. \tag{3.15}$$

Let $p = 2$ and $\lambda = 1$. Now (3.15) certainly ensures that (3.11) holds for all $\varepsilon > 0$ and for any choice of $\{a_n, n \ge 1\}$. Choose in particular $a_n = 1$, $n \ge 1$. Then (3.2) holds by the $n = \mathcal{O}(b_n)$ hypothesis. The conclusion (3.13) then follows from (3.14) and (3.11) by Theorem 3.1 and Remark 3.1(ii).

Corollary 3.2. (Adler, Rosalsky, and Taylor [3]). *Let $\{V_n, n \ge 1\}$ be a compactly uniformly integrable sequence of independent random elements in a real separable Banach space and let $\{b_n, n \ge 1\}$ be a sequence of positive constants with $b_n \uparrow$ and $n = \mathcal{O}(b_n)$. If*

$$\sum_{n=1}^{\infty} \frac{E||V_n||^p}{b_n^p} < \infty \quad \text{for some } 1 \le p \le 2, \tag{3.16}$$

then the SLLN

$$\frac{\sum_{i=1}^{n}(V_i - EV_i)}{b_n} \to 0 \quad a.c. \tag{3.17}$$

obtains.

Proof. The condition (3.16) of course entails

$$\sum_{n=1}^{\infty} E\left(\frac{||V_n||^p}{||V_n||^p + b_n^p}\right) < \infty$$

which, by an argument similar to that of Heyde [9], is equivalent to the pair of conditions (3.14) and

$$\sum_{n=1}^{\infty} \frac{E(||V_n||^p I(||V_n|| \le b_n))}{b_n^p} < \infty. \tag{3.18}$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{E(||V_n||^2 I(||V_n|| \le b_n))}{b_n^2} &= \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\left(||V_n||^p \left(\frac{||V_n||}{b_n}\right)^{2-p} I(||V_n|| \le b_n)\right) \\ &\le \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\Big(||V_n||^p I(||V_n|| \le b_n)\Big) \\ &< \infty \text{ (by (3.18))}.\end{aligned}$$

Using the equivalence cited above, (3.12) holds and the conclusion (3.17) then follows from Corollary 3.1.

Corollary 3.3. (Taylor and Wei [11]). *Let $\{V_n, n \ge 1\}$ be a uniformly tight sequence of independent random elements in a real separable Banach space such that*

$$\sup_{n \ge 1} E||V_n||^p < \infty \textit{ for some } p > 1. \tag{3.19}$$

Then the SLLN

$$\frac{\sum_{i=1}^{n}(V_i - EV_i)}{n} \to 0 \quad a.c.$$

obtains.

Proof. In view of Lemma 2.1, it may be assumed that $1 < p \le 2$. Let $b_n = n, n \ge 1$. It follows from (3.19) that the sequence of random variables $\{||V_n||, n \geq 1\}$ is uniformly integrable (see Chow and Teicher [5], p. 102). Then this and the uniform tightness hypothesis ensure that the sequence $\{V_n, n \geq 1\}$ is compactly uniformly integrable (recall the discussion in Section 2). Next, by (3.19) there exists a constant $C < \infty$ such that $E||V_n||^p \leq C, n \geq 1$ implying

$$\sum_{n=1}^{\infty} \frac{E||V_n||^p}{n^p} \leq \sum_{n=1}^{\infty} \frac{C}{n^p} < \infty$$

since $p > 1$. The conclusion follows from Corollary 3.2.

4. Some interesting examples. We conclude by presenting some examples to:

- illustrate the sharpness of Theorem 3.1
- show that the hypotheses of Corollary 3.1 can be satisfied when the hypotheses of Corollary 3.2 fail
- show that the hypotheses of Theorem 3.1 can be satisfied when the hypotheses of Proposition 2.2 fail.

The first example, which is due to Adler, Rosalsky, and Taylor [3], shows that Theorem 3.1 can fail if the compact uniform integrability condition is weakened to uniform tightness. Example 4.1 concerns the real separable Banach space ℓ_1 of absolutely summable real sequences $v = \{v_j, j \geq 1\}$ with norm $||v|| = \sum_{j=1}^{\infty} |v_j|$. Let $v^{(n)}$ denote the element of ℓ_1 having 1 in its n^{th} position and 0 elsewhere, $n \geq 1$.

Example 4.1. Consider the real separable Banach space ℓ_1. Define a sequence $\{V_n, n \geq 1\}$ of independent random elements in ℓ_1 by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$P\{V_n = n^{1/2} v^{(n)}\} \quad = \quad P\{V_n = -n^{1/2} v^{(n)}\}$$

$$= \frac{1}{2} - \frac{1}{2}P\{V_n = 0\} = \frac{1}{2n^{1/2}}, \quad n \geq 1.$$

Setting $a_n = 1$, $b_n = n$, $n \geq 1$, and $\lambda = 1$, (3.2) holds and

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} = \sum_{n=1}^{\infty} P\{||V_n|| > n\} = \sum_{n=1}^{\infty} 0 < \infty$$

establishing (3.3). Also note that for $1 \leq p \leq 2$ and $0 < \varepsilon < 1$, the expression in (3.11) reduces to

$$\sum_{n=1}^{\infty} \frac{E(||V_n||^p I(\varepsilon < ||V_n|| \leq n))}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p}\Big(\frac{n^{p/2}}{n^{1/2}}\Big) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p+1}{2}}} < \infty$$

since $p > 1$ thereby establishing (3.4) via Remark 3.1(ii). It was shown by Adler, Rosalsky, and Taylor [3] that

$$\frac{||\sum_{i=1}^{n} V_i||}{n} \to 1 \ \text{ a.c.}$$

and so the conclusion of Theorem 3.1 fails. It remains to show that $\{V_n, n \geq 1\}$ is uniformly tight but not compactly uniformly integrable. Let $\varepsilon > 0$, select N_ε such that $N_\varepsilon^{-1/2} \leq \varepsilon$, and let

$$\mathcal{K}_\varepsilon = \{0, v^{(1)}, -v^{(1)}, \sqrt{2}v^{(2)}, -\sqrt{2}v^{(2)}, \ldots, \sqrt{N_\varepsilon}v^{(N_\varepsilon)}, -\sqrt{N_\varepsilon}v^{(N_\varepsilon)}\}.$$

Then $\mathcal{K}_\varepsilon$ is compact and for $n \geq 1$

$$P\{V_n \notin \mathcal{K}_\varepsilon\} = \begin{cases} 0 < \varepsilon, & \text{if } n \leq N_\varepsilon \\ n^{-1/2} < \varepsilon, & \text{if } n > N_\varepsilon. \end{cases}$$

Thus $\{V_n, n \geq 1\}$ is uniformly tight. However, if $\mathcal{K}$ is any compact set, then $\mathcal{K}$ can contain at most finitely many elements of $\{\pm\sqrt{n}v^{(n)}, n \geq 1\}$ and so

$$\sup_{n\geq 1} E||V_n I(V_n \notin \mathcal{K})|| = 1$$

thereby showing that (3.5) fails. Therefore (3.5) cannot be replaced by uniform tightness in Theorem 3.1.

In the following two examples, the hypotheses of Corollary 3.1 (and Theorem 3.1) are satisfied but those of Corollary 3.2 are not.

Example 4.2. Let $\{V_n, n \geq 1\}$ be a sequence of independent random variables with

$$P\{V_n = n^2\} = P\{V_n = -n^2\} = \frac{1}{2n^2 \log n} = \frac{1}{2} - \frac{1}{2}P\{V_n = 0\}, \quad n \geq 1.$$

Note that $EV_n = 0$, $n \geq 1$. Since

$$n^2 P\{|V_n| = n^2\} = \frac{1}{\log n} = o(1),$$

it follows from the uniformly integrability criterion (see, e.g., Chow and Teicher [5], p. 94) that the sequence $\{V_n, n \geq 1\}$ is (compactly) uniformly integrable. Let $b_n = n$, $n \geq 1$. Note that for every $1 \leq p \leq 2$,

$$\sum_{n=1}^{\infty} \frac{E|V_n|^p}{b_n^p} = \sum_{n=1}^{\infty} \frac{n^{2p}}{n^p n^2 \log n} = \sum_{n=1}^{\infty} \frac{1}{n^{2-p} \log n} = \infty$$

since $2 - p \leq 1$. Hence Corollary 3.2 cannot be applied.

To show that the hypotheses of Corollary 3.1 are satisfied, it needs to be verified that (3.12) holds or, equivalently, that the conditions (3.14) and (3.15) hold. Now

$$\sum_{n=1}^{\infty} P\{|V_n| > b_n\} = \sum_{n=1}^{\infty} P\{|V_n| > n\} = \sum_{n=2}^{\infty} \frac{1}{n^2 \log n} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{E(V_n^2 I(|V_n| \leq b_n))}{b_n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2 \log n} < \infty$$

and so (3.14) and (3.15) hold. Thus by Corollary 3.1,

$$\frac{\sum_{i=1}^{n} V_i}{n} \to 0 \text{ a.c.} \tag{4.1}$$

Hence the conclusion (3.17) does indeed hold by Corollary 3.1 (hence by Theorem 3.1) but not by Corollary 3.2. It may be noted that (4.1) can also be obtained by Proposition 2.2 recalling that the real line is of Rademacher type 2.

The next example, Example 4.3, differs from Example 4.2 in that:

- Example 4.3 concerns a sequence of random elements rather than a sequence of random variables.
- The random elements in Example 4.3 are unbounded whereas the random variables in Example 4.2 are bounded.
- Proposition 2.2 is not necessarily applicable in Example 4.3 but, as was noted above, it is applicable in Example 4.2.

Example 4.3. Let $\{V_n, n \geq 1\}$ be a uniformly tight sequence of independent random elements where $||V_n||$ has distribution given by

$$P\{||V_n|| \geq x\} = \frac{e}{(\log n)x(\log((\log n)x))^2}, \quad x \geq \frac{e}{\log n}, \quad n \geq 1.$$

Then $\{||V_n||, n \geq 1\}$ is stochastically dominated by $||V_1||$ and

$$E||V_1|| = e + \int_e^\infty \frac{e}{x(\log x)^2} dx = 2e < \infty$$

which combined with the uniform tightness assumption ensure that (see Cuesta and Matrán [6]) the sequence $\{V_n, n \geq 1\}$ is compactly uniformly integrable. Let $b_n = n$, $n \geq 1$. Now for all $n \geq 1$,

$$E||V_n|| \quad \geq \quad \int_{\frac{e}{\log n}}^\infty \frac{e}{(\log n)x(\log((\log n)x))^2} dx$$

$$\begin{aligned} &= \int_e^\infty \frac{e}{t(\log t)^2(\log n)} dt \\ &= \frac{e}{\log n} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{E||V_n||}{n} = \infty. \tag{4.2}$$

Note that for all $1 < p \leq 2$ and $n \geq 1$

$$\begin{aligned} E||V_n||^p &\geq \int_{\frac{e}{\log n}}^\infty \frac{x^{p-1}e}{(\log n)x(\log((\log n)x))^2} dx \\ &= \int_e^\infty \frac{e}{(\log n)^p t^{2-p}(\log t)^2} dt \\ &= \infty \quad (\text{since } 2-p<1). \end{aligned}$$

Thus, recalling (4.2),

$$\sum_{n=1}^{\infty} \frac{E||V_n||^p}{b_n^p} = \sum_{n=1}^{\infty} \frac{E||V_n||^p}{n^p} = \infty \quad \text{for all } 1 \leq p \leq 2$$

and so Corollary 3.2 is not applicable.

To show that the hypotheses of Corollary 3.1 are satisfied, it needs to be verified that (3.12) holds or, equivalently, that the conditions (3.14) and (3.15) hold. Since $\{V_n, n \geq 1\}$ is stochastically dominated by $||V_1||$ (with $D = 1$) and since $E||V_1|| < \infty$, we have

$$\sum_{n=1}^{\infty} P\{||V_n|| > b_n\} \leq \sum_{n=1}^{\infty} P\{||V_1|| > n\} < \infty \tag{4.3}$$

whence (3.14) holds. Next, note that

$$\left(\max_{1\leq i\leq n} b_i^2\right) \sum_{i=n}^{\infty} \frac{1}{b_i^2} = n^2 \sum_{i=n}^{\infty} \frac{1}{i^2} = n^2 \mathcal{O}\left(\frac{1}{n}\right) = \mathcal{O}(n)$$

and recalling (4.3), we have by Lemma 2.2 with $Y_n = ||V_n||$, $n \geq 1$ and

$Y = ||V_1||$ that (3.15) holds. Thus by Corollary 3.1,

$$\frac{\sum_{i=1}^{n} V_i}{n} \to 0 \ \text{ a.c.}$$

Hence, as in the previous example, the conclusion (3.17) does indeed hold by Corollary 3.1 (hence by Theorem 3.1) but not by Corollary 3.2.

Remarks 4.1. (i) Note that if the underlying Banach space is not of Rademacher type p for any $1 < p < 2$ (thus the Banach space is only of Rademacher type 1), then apropos of Example 4.3 it will now be shown that condition (2.1) of Proposition 2.2 can fail (with $p = 1$) and so Proposition 2.2 would not be applicable. Assume that V_n is symmetric, $n \geq 1$ and that $a_n = 1$, $n \geq 1$. (Such a choice of $\{a_n, n \geq 1\}$ satisfies the hypotheses of Theorem 3.1 according to the *proof* of Corollary 3.1.) Now integration by parts yields for all $n \geq 1$ and $a > 0$

$$E(||V_n||I(||V_n|| \leq a)) = \int_0^a P\{||V_n|| > x\}dx - aP\{||V_n|| > a\}.$$

Let $\lambda > 0$ be arbitrary and let $\varepsilon = 1$. Then for all large n

$$\begin{aligned}
& E(||V_n||I(\varepsilon a_n < ||V_n|| \leq \lambda b_n)) \\
= \ & E(||V_n||I(||V_n|| \leq \lambda n)) - E(||V_n||I(||V_n|| \leq 1)) \\
= \ & \int_0^{\lambda n} P\{||V_n|| > x\}dx - \int_0^1 P\{||V_n|| > x\}dx \\
& -\lambda n P\{||V_n|| > \lambda n\} + P\{||V_n|| > 1\} \\
\geq \ & \int_1^{\lambda n} P\{||V_n|| > x\}dx - \lambda n P\{||V_n|| > \lambda n\} \\
= \ & \int_1^{\lambda n} \frac{e}{(\log n)x(\log((\log n)x)^2} dx - \lambda n P\{||V_n|| > \lambda n\} \\
= \ & \int_{\log n}^{\lambda n \log n} \frac{e}{t(\log t)^2 \log n} dt - \lambda n P\{||V_n|| > \lambda n\} \\
= \ & \frac{e}{(\log n) \log\log n} - \frac{e}{(\log n)(\log n + \log\log n + \log \lambda)} - \lambda n P\{||V_n|| > \lambda n\} \\
\geq \ & \frac{e}{2(\log n) \log\log n} - \lambda n P\{||V_n|| > \lambda n\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{E(||V_n||I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))}{b_n}\\
\geq\ & C+\sum_{n=1}^{\infty}\Big(\frac{e}{2n(\log n)\log\log n}-\lambda P\{||V_n||>\lambda n\}\Big)\\
=\ & C+\frac{e}{2}\sum_{n=1}^{\infty}\frac{1}{n(\log n)\log\log n} \quad \text{(by arguing as in (4.3))}\\
=\ & \infty
\end{aligned}$$

and so (2.1) fails with $p = 1$. Since the $\{V_n, n \geq 1\}$ are symmetric and $E||V_n|| < \infty, n \geq 1$, (2.2) holds. Thus, it is solely the failure of (2.1) which renders Proposition 2.2 inapplicable.

(ii) It is interesting to observe that the sequence of random elements in Example 4.3 does not satisfy the hypotheses to Theorem 7 of Cuesta and Matrán [6] solely because

$$E||V_1||^2 = \int_e^{\infty}\frac{e}{(\log x)^2}dx = \infty.$$

(iii) The conditions (2.1) of Proposition 2.2 and (3.4) of Theorem 3.1 are not comparable in general. However, since for any random element V with $E||V|| < \infty$ we have

$$E|||V|| - E||V|||^2 \leq E|||V|| - ||EV|||^2 \leq E||V-EV||^2,$$

the implication (2.1) $\Rightarrow$ (3.4) holds when $p = 2$.

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