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ERROR BOUNDS FOR SOME NEW APPROXIMATION FORMS OF REGULAR FUNCTIONS

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Abstract. Approximation forms for regular functions $f(x)$ by use of a limit expression with Law of Large Numbers and expectation in probability theory were obtained, where appropriate error bounds for all the proposed approximation forms are provided herein. Such error bounds facilitate the applicability of the approximation forms.

1. Introduction. As pointed out in a previous work, Bernstein polynomials has been a broadly accepted tool for doing computer aided geometric design (CAGD) in industries and in simulated motion pictures. To allow more flexibility on application of Bernstein polynomials as approximation tool, more extensive approximation forms have been indicated by Kao (2002). It is usually possible to apply an approximation form for regular functions to obtain density estimates. However, error bounds are needed in order to have idea about how good the estimates will be and how fast the approximation forms converge to the actual value of the approximated function. Recently, Ghosal (2001) investigated the convergence rates when Bernstein polynomials were applied to do density estimation. It appears natural that the error bounds shall depend on assumptions given to the approximated function. Impens and Vernaeve (2001) provided asymptotics

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regarding Bernstein approximation form when the approximated function satisfies certain Lipschitz conditions. In addition, Petrone (1999) earlier used Bernstein polynomials to propose a Bayesian nonparametric procedure for density estimation. Regarding convergence of Bernstein approximation form, Farouki (1999) also investigated its convergent inversion approximations. In line with the desire to ensure the applicability of the approximation forms given by Feller (1966) and this author as mentioned in above, we are to obtain error bounds for such approximation forms in what follows.

2. The approximation forms. We consider f to be a continuous function, and for purpose of simplicity in practical applications f is assumed to be bounded. Then let there be a sequence of identically and independently distributed (iid) random variables $X_1, X_2, \ldots, X_n$ with mean μ and finite variance. According to strong law of large numbers,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{S_n}{n} \overset{n\to\infty}{\longrightarrow} \mu \text{ almost surely.}$$

Since f is continuous, $f(\bar{X})$ converges almost surely to $f(\mu)$ as $n \to \infty$. Then due to the assumption that f is bounded, the dominated convergence theorem implies

$$f(\mu) = \lim_{n\to\infty} Ef(\bar{X}) = \lim_{n\to\infty} Ef\left(\frac{S_n}{n}\right) \tag{2.1}$$

According to the results of approximation forms for $f(t)$ indicated by Kao (2002) from use of (2.1), we have the following:

Approximation Form I

By setting X_i's to be iid extended Bernoulli trials, i.e. $P\{X_i = x\} = p$ and $P\{X_i = 0\} = 1 - p, p \in (0,1)$, we have

$$f(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{t}{x}\right)^k \left(1 - \frac{t}{x}\right)^{n-k} f\left(\frac{kx}{n}\right) \quad \text{for any} \quad t \in R, \tag{2.2}$$

where x is arbitrary such that $0 \le t < x$ if $t \ge 0$, and $0 > t > x$ if $t < 0$.

This form can be regarded as Bernstein polynomial approximation form.

Approximation Form II

By setting X_i/x for any given non-zero real number x, $1 \le i \le n$, to be iid negative binomial random variables with distribution $NB(K,p)$, where K is a positive integer and $p \in (0,1)$, we have

$$f(t) = \lim_{n\to\infty} \sum_{k=nK}^{\infty} \binom{k-1}{nK-1} \left(\frac{t}{x}\right)^{nK} \left(1-\frac{t}{x}\right)^{k-nK} f\left(\frac{kx}{nK}\right) \tag{2.3}$$

for any $t \in R$, where x is arbitrary such that $0 \le t < x$ if $t \ge 0$, and $0 > t > x$ if $t < 0$.

This form can be regraded as negative binomial approximation form.

Approximation Form III

By setting X_i, $1 \le i \le n$, to be iid Poisson random variables with parameter $t > 0$, we have

$$f(t) = \lim_{n\to\infty} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nt)^k}{k!} e^{-nt} \tag{2.4}$$

Then by duality, we also have

$$f(t) = \lim_{n\to\infty} \sum_{k=0}^{\infty} f\left(-\frac{k}{n}\right) \frac{(-nt)^k}{k!} e^{nt} \quad \text{for} \quad t < 0. \tag{2.5}$$

Approximation Form IV

By setting X_i, $1 \le i \le n$, to be iid normally distributed with mean t and variance $\sigma^2 > 0$, we have

$$f(t) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sigma} \int_{-\infty}^{\infty} f(x) \phi\left(\frac{\sqrt{n}}{\sigma}(x-t)\right) dx, \quad -\infty < t < \infty \tag{2.6}$$

where ϕ is the probability density function of standard normal distribution.

Approximation Form V

By setting X_i, $1 \le i \le n$, to be iid exponential random variable with parameter $\frac{1}{t} > 0$, we have

$$f(t) = \lim_{n\to\infty} \frac{t^{-n}}{(n-1)!} \int_0^\infty f\left(\frac{x}{n}\right) x^{n-1} e^{-\frac{x}{t}}\, dx \quad \text{for} \quad t > 0 \tag{2.7}$$

Then by duality, we also have

$$f(t) = \lim_{n\to\infty} \frac{(-t)^{-n}}{(n-1)!} \int_0^\infty f\left(\frac{-x}{n}\right) x^{n-1} e^{-\frac{x}{t}}\, dx \quad \text{for} \quad t < 0 \tag{2.8}$$

The approximation forms presented in above may be applicable at user's discretion. Each of the approximation forms may be favorable to certain type of the discrete data values of f that are available for obtaining the approximation of $f(t)$.

3. Error bounds of the approximation forms. For simplicity in what follows and the reality in practical applications we assume that the considered function f has continuous and bounded second derivative in the defined domain. According to the stated fact that $f(\frac{S_n}{n})$ converges to $f(t)$ almost surely, where $t = E(X_i)$ and $S_n = X_1 + X_2 + \cdots + X_n$, we may have

$$f\left(\frac{S_n}{n}\right) = f(t) + f'(t)\left(\frac{S_n}{n} - t\right) + \frac{1}{2!} f''(\theta)\left(\frac{S_n}{n} - t\right)^2$$

with θ staying in between $\frac{S_n}{n}$ and t and θ approaching t as $n \to \infty$.

Therefore, by taking expectation, we obtain

$$\begin{aligned} Ef\left(\frac{S_n}{n}\right) &= f(t) + f'(t)\left[E\left(\frac{S_n}{n}\right) - t\right] + \frac{1}{2}E\left[f''(\theta)\left(\frac{S_n}{n} - t\right)^2\right] \\ &= f(t) + \frac{1}{2}E\left[f''(\theta)\left(\frac{S_n}{n} - t\right)^2\right] \end{aligned} \tag{3.1}$$

Assuming that the second derivative of f is uniformly bounded by C, i.e. $|f''(x)| \le C$ for all x, we then have

$$\left|Ef\left(\frac{S_n}{n}\right) - f(t)\right| \le \frac{C}{2}E\left[\left(\frac{S_n}{n} - t\right)^2\right] = \frac{C}{2}\mathrm{Var}\left(\frac{S_n}{n}\right) \tag{3.2}$$

By applying (3.2) to each of the approximation forms provided in the previous section, we immediately have error bound for each of the approximation forms.

For Approximation Form I:

$$\begin{aligned}\mathrm{Var}\,(X_i) &= x^2p(1-p) = t(x-t)\\ \mathrm{Var}\left(\frac{S_n}{n}\right) &= \frac{1}{n}x^2p(1-p) = \frac{t(x-t)}{n}\end{aligned}$$

Therefore, from (2.2) we have

$$\left|f(t) - \sum_{k=0}^{n}\binom{n}{k}\left(\frac{t}{x}\right)^k\left(1-\frac{t}{x}\right)^{n-k} f\left(\frac{kx}{n}\right)\right| \le \frac{Ct(x-t)}{2n} \tag{3.3}$$

for any t, where x is arbitrary such that $0 \le t < x$ if $t \ge 0$, and $x < t < 0$ if $t < 0$.

For Approximation Form II:

$$\begin{aligned}\mathrm{Var}\,(X_i) &= \frac{x^2K(1-p)}{p^2} = \frac{x^3K(x-t)}{t^2}\\ \text{and}\qquad \mathrm{Var}\left(\frac{S_n}{n}\right) &= \frac{x^2K(1-p)}{np^2} = \frac{x^3K(x-t)}{nt^2}\end{aligned}$$

Therefore, from (2.3) we have

$$\left|f(t) - \sum_{k=nK}^{\infty}\binom{k-1}{nK-1}\left(\frac{t}{x}\right)^{nK}\left(1-\frac{t}{x}\right)^{k-nK} f\left(\frac{kx}{nK}\right)\right| \le \frac{Cx^3K(x-t)}{2nt^2} \tag{3.4}$$

for any t, where x is arbitrary such that $0 \le t < x$ if $t \ge 0$, and $x < t < 0$ if

$t < 0$.

For Approximation Form III:

$$\mathrm{Var}\,(X_i) = t \quad \text{and} \quad \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{t}{n} \quad (t > 0)$$

Therefore, from (2.4) we have

$$\left| f(t) - \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nt)^k}{k!} e^{-nt} \right| \le \frac{Ct}{2n} \quad \text{for} \quad t > 0 \tag{3.5}$$

and from (2.5) we have

$$\left| f(t) - \sum_{k=0}^{\infty} f\left(\frac{-k}{n}\right) \frac{(-nt)^k}{k!} e^{nt} \right| \le \frac{C|t|}{2n} \quad \text{for} \quad t < 0 \tag{3.6}$$

For Approximation Form IV:

$$\mathrm{Var}\,(X_i) = \sigma^2 \quad \text{and} \quad \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$$

Therefore, from (2.6) we have

$$\left| f(t) - \frac{\sqrt{n}}{\sigma} \int_{-\infty}^{\infty} f(x) \phi\left(\frac{\sqrt{n}}{\sigma}(x-t)\right) dx \right| \le \frac{C\sigma^2}{2n}, \quad -\infty < t < \infty \tag{3.7}$$

where ϕ is the probability density function of $N(0,1)$.

For Approximation Form V:

$$\mathrm{Var}\,(X_i) = \frac{1}{t^2} \quad \text{and} \quad \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{nt^2}$$

Therefore, from (2.7) we have

$$\left| f(t) - \frac{t^{-n}}{(n-1)!} \int_0^{\infty} f\left(\frac{x}{n}\right) x^{n-1} e^{-\frac{x}{t}} dx \right| \le \frac{C}{2nt^2} \quad \text{for} \quad t > 0 \tag{3.8}$$

and from (2.8) we have

$$\left| f(t) - \frac{(-t)^{-n}}{(n-1)!} \int_0^\infty f\left(\frac{-x}{n}\right) x^{n-1} e^{\frac{x}{t}} dx \right| \le \frac{C}{2nt^2} \quad \text{for} \quad t < 0 \tag{3.9}$$

It should be further noted that when n is sufficiently large, C can be regarded to be approximately $|f''(t)|$ with the bounding inequality becoming an approximate equality. More precisely, when n is sufficiently large, we have

$$f(t) \approx A_n(t) - \frac{1}{2} f''(t) \mathrm{Var}\left(\frac{S_n}{n}\right) \tag{3.10}$$

where $A_n(t)$ is the approximation form applied and Var $(\frac{S_n}{n})$ corresponds to the applied approximation form. For example,

$$f(t) \approx \frac{\sqrt{n}}{\sigma} \int_{-\infty}^{\infty} f(x) \phi\left(\frac{\sqrt{n}}{\sigma}(x-t)\right) dx - \frac{\sigma^2}{2n} f''(t)$$

when n sufficiently large and if Approximation Form IV is considered.

In addition to the more precise fact of (3.10), from its derivation we may have the following theorems.

Theorem 1. *If $f(t)$ is a linear form of t, then for any approximation form $A_n(t)$ given herein we have*

$$f(t) = A_n(t), \quad t \in (-\infty, \infty).$$

Proof. Since it is assumed that $t = E(X_i)$ and $S_n = \sum_{i=1}^n X_i$, we have $E(\frac{S_n}{n}) = t$. When f is linear, it is immediate that $Ef(\frac{S_n}{n}) = f(E(\frac{S_n}{n}))$. Therefore,

$$A_n(t) = Ef\left(\frac{S_n}{n}\right) = f\left(E\left(\frac{S_n}{n}\right)\right) = f(t)$$

which establishes the proof.

Theorem 2. *If f is a convex function then when n is sufficiently large*

$$A_n(t) - \frac{C}{2}\mathrm{Var}\left(\frac{S_n}{n}\right) \le f(t) \le A_n(t)$$

where $A_n(t) = Ef(\frac{S_n}{n})$ *and* C *is a uniform bound of* $f''(x)$ *as given for Formula* (3.2).

Proof. According to Formula (3.2), it suffices to prove $f(t) \le A_n(t)$. When n is sufficiently large, $\frac{S_n}{n}$ approaches t and has its distributional support in the neighborhood of t where f is convex. By using the well-known Jensen's inequality in probability theory, we have

$$A_n(t) = Ef\left(\frac{S_n}{n}\right) \ge f\left(E\left(\frac{S_n}{n}\right)\right) = f(t)$$

This then proves the theorem.

In practical applications of the provided approximation forms, there are normally some known values of $f(a_k)$ with a_k beings monotone increasing in k, $1 \le k \le N$, where the $f(a_k)$'s can be used to obtain an approximation for any $f(t)$. It is advisable that we apply the integral approximation forms, including Approximation Forms IV and V, for the approximation purpose in view of the ease they offer for use. In light of Formula (2.6) we consider

$$\bar{U}_{n,N}(t) = \frac{\sqrt{n}}{\sigma}\sum_{k=1}^{N} f(a_k)\int_{l_k}^{u_k} \phi\left(\frac{\sqrt{n}(x-t)}{\sigma}\right)dx$$

where $l_k = \frac{a_{k-1}+a_k}{2}$, $u_k = \frac{a_k+a_{k+1}}{2}$, with $a_0 = a_1 - \frac{a_2-a_1}{2}$ and $a_{N+1} = a_N + \frac{a_N-a_{N-1}}{2}$ and ϕ is the probability density function of standard normal distribution, while corresponding to Formula (2.7) we consider

$$\bar{V}_{n,N}(t) = \frac{n^n t^{-n}}{(n-1)!}\sum_{k=1}^{N} f(a_k)\int_{l_k}^{u_k} y^{n-1}e^{-\frac{ny}{t}}dy$$

with l_k and u_k similarly defined as in above and a_0 taken to be 0 if $a_1 -$

$\frac{a_2-a_1}{2} < 0$. Then $\bar{U}_{n,N}(t)$ and $\bar{V}_{n,N}(t)$ are the proposed approximates for $f(t)$ based on the known values of $f(a_k)$, $1 \le k \le N$ and $a_k \uparrow$ in k.

Theorem 3. *Regarding the approximates $\bar{U}_{n,N}(t)$ and $\bar{V}_{n,N}(t)$, we have*

$$\lim_{n\to\infty} \bar{U}_{n,N}(a_j) = f(a_j) = \lim_{n\to\infty} \bar{V}_{n,N}(a_j)$$

for any $a_j, 1 \le j \le N$.

Proof. It appears sufficient to prove for $\bar{U}_{n,N}$, since proof for $\bar{V}_{n,N}$ can be established in similar way. According to the above, we set

$$\begin{aligned} \bar{U}_{n,N}(a_j) &= \sum_{k=1}^{N} f(a_k) \cdot \frac{\sqrt{n}}{\sigma} \int_{l_k}^{u_k} \phi\left(\frac{\sqrt{n}(x-a_j)}{\sigma}\right) dx \\ &= \sum_{k=1}^{N} f(a_k) \int_{s_{k,j}}^{t_{k,j}} \phi(y) dy \\ &= \sum_{n=1}^{N} f(a_k)[\Phi(t_{k,j}) - \Phi(s_{k,j})] \end{aligned}$$

where $t_{k,j} = \sqrt{n}(u_k - a_j)/\sigma$, $s_{k,j} = \sqrt{n}(l_k - a_j)/\sigma$ and Φ is the distribution function of standard normal distribution. For the case $k > j$, $0 \le \Phi(t_{k,j}) - \Phi(s_{k,j}) < 1 - \Phi(s_{k,j})$ and so as $n \to \infty$ we have

$$0 \le \Phi(t_{k,j}) - \Phi(s_{k,j}) \le 1 - \Phi\left(\frac{\sqrt{n}(a_{j+1}-a_j)}{2\sigma}\right) \to 1 - \Phi(\infty) = 0$$

For the case $k < j$, $0 \le \Phi(t_{k,j}) - \Phi(s_{k,j}) \le \Phi(t_{k,j})$ and as $n \to \infty$ we have

$$0 \le \Phi(t_{k,j}) - \Phi(s_{k,j}) \le \Phi(t_{k,j}) \le \Phi\left(\frac{\sqrt{n}(a_{j+1}-a_j)}{2\sigma}\right) \to \Phi(-\infty) = 0$$

for the case $k = j$, we have

$$t_{k,j} = \frac{\sqrt{n}}{\sigma}\left(\frac{a_k + a_{k+1}}{2} - a_j\right) = \frac{\sqrt{n}}{2\sigma}(a_{j+1} - a_j)$$

$$\text{and} \qquad s_{k,j} = \frac{\sqrt{n}}{\sigma}\left(\frac{a_{k-1}+a_k}{2} - a_j\right) = \frac{\sqrt{n}}{2\sigma}(a_{j-1} - a_j)$$

Therefore when $k = j$ and as $n \to \infty$ it holds that

$$\Phi(t_{k,j}) - \Phi(s_{k,j}) \to \Phi(\infty) - \Phi(-\infty) = 1$$

Summarizing the above, we then see that

$$\begin{aligned} \lim_{n\to\infty} \bar{U}_{n,N}(a_j) &= \sum_{k=1}^{N} \lim_{n\to\infty} f(a_k)[\Phi(t_{k,j}) - \Phi(s_{k,j})] \\ &= f(a_j)\cdot 1 = f(a_j) \quad \text{for any} \quad a_j,\ 1 \le j \le N, \end{aligned}$$

which furnishes the proof for $\bar{U}_{n,N}$.

The proof for $\bar{V}_{n,N}$ can be similarly established as for $\bar{U}_{n,N}$ except that Φ, which is the distribution function of standard normal attached to $\bar{U}_{n,N}$, is replaced by the distribution function of the gamma attached to $\bar{V}_{n,N}$.

It is apparent that Theorem 3 offers the assurance of consistency that the obtained approximates of $f(t)$ at $t = a_k$ $(1 \le k \le N)$ shall equal the originally given $f(a_k)$ when we let $n \to \infty$ in the approximation forms.

4. Asymptotics of differentiated Bernstein polynomials. As a result of being able to represent the considered function $f(x)$ in the various asymptotic forms shown in Section 2, we may equate one of any such asymptotic form to any of the other forms. Under assumption that ensure the differentiability of $f(x)$, we may also equate the derivatives of one forms to the corresponding derivatives of the other forms. In the work of Impens & Vernaeve (2001) a Theorem 1 is given as the following:

Take an integer $m \ge 1$, and let $m - \frac{1}{2} < r \le m$. If $f \in Lip(r)$, then

$$(4.1)\quad D_x^m B_n(f,x) - \frac{h_{n,x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t)(-D_t)^m (e^{-h_{n,x}^2 t^2})dt \overset{V}{\rightrightarrows} 0 \quad \text{as} \quad n \to \infty$$

where $x \in U = [0,1]$, $V = [\lambda, 1-\mu]$ with $0 < \lambda < 1-\mu < 1$, $h_{n,x} = \sqrt{\frac{n}{2x(1-x)}}$ $(0 < x < 1)$. In addition, $f_n \overset{A}{\rightrightarrows} f$ means that $\lim_{n\to+\infty} f_n(x) = f(x)$ uniformly for $x \in A$, D_x and D_t are the differentiation operators with respect to x and t and $Lip(r)$ $(r > 0)$ indicates the class of functions satisfying the r-Lipschitz condition therein defined.

Rather than the procedure of Impens & Vernaeve which proved the above theorem, it is possible to prove the theorem in a straightforward manner. According to Approximation Form IV in Section 2, for any $\sigma > 0$ we have

$$f(x) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(t) e^{-\frac{n}{2\sigma^2}(t-x)^2} dt \quad \text{for} \quad -\infty < x < \infty$$

if $f \in C^\circ(R)$. By setting $\xi = \sqrt{\frac{n}{2}} \cdot \frac{1}{\sigma}$, the above becomes

$$\begin{aligned} f(x) &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) e^{-\xi^2(t-x)^2} dt \\ &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t) e^{-\xi^2 t^2} dt \end{aligned}$$

In the event that f is originally defined only for $x \in U = [0,1]$, we may extend f by putting $f(x) = f(0)$ for $x < 0$ and $f(x) = f(1)$ for $x > 1$, then

$$\text{(4.2)} \qquad \begin{aligned} f(x) &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t) e^{-\xi^2 t^2} dt \\ \text{or} \qquad &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) e^{-\xi^2(t-x)^2} dt \quad \text{uniformly for} \quad x \in U \end{aligned}$$

This is the Gauss-Weierstrass integral.

On the other hand, the Bernstein polynomial approximation form

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

is known to satisfy that

$$f(x) = \lim_{n\to\infty} B_n(f,x) \quad \text{uniformly for} \quad x \in V \tag{4.3}$$

Therefore, the two representation forms for $f(x)$ uniformly on V and the assumed r-Lipschitz condition upon f assures that

$$\begin{aligned} D_x^m f(x) &= \lim_{n\to\infty} D_n^m B_n(f,x) \\ \text{and} \qquad D_x^m f(x) &= \lim_{\xi\to\infty} D_x^m \left[\frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) e^{-\xi^2(t-x)^2} dt\right] \end{aligned} \tag{4.4}$$

uniformly for $x \in V$.

Let $H(t-x) = e^{-\xi^2(t-x)^2}$, then from the above we have

$$D_x^m f(x) = \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t)[D_x^m H(t-x)]dt$$

However, it is obvious that

$$\begin{aligned} D_x^m H(t-x) &= \left[\frac{\partial(t-x)}{\partial x} \cdot \frac{\partial}{\partial(t-x)}\right]^m H(t-x) \\ &= \left[-\frac{\partial}{\partial(t-x)}\right]^m H(t-x) \\ &= (-D_{t-x})^m H(t-x) \end{aligned}$$

and

$$\begin{aligned} D_t^m H(t-x) &= \left[\frac{\partial(t-x)}{\partial t} \cdot \frac{\partial}{\partial(t-x)}\right]^m H(t-x) \\ &= \left[\frac{\partial}{\partial(t-x)}\right]^m H(t-x) \\ &= (D_{t-x})^m H(t-x) \end{aligned}$$

Therefore, we see that

$$D_x^m H(t-x) = (-D_t)^m H(t-x)$$

Accordingly it follows that

$$\begin{aligned} D_x^m f(x) &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t)(-D_t)^m H(t-x)dt \\ &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t)(-D_{t-x})^m H(t-x)dt \end{aligned}$$

By variable transformation in the integral using $s = t - x$, we then obtain

$$\begin{aligned} D_x^m f(x) &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+s)(-D_s)^m H(s)ds \\ &= \lim_{\xi\to\infty} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t)(-D_t)^m (e^{-\xi^2 t^2})dt \end{aligned} \tag{4.5}$$

Then take $\xi = h_{n,x} = \sqrt{\frac{n}{2x(1-x)}}$, which goes to ∞ as $n \to \infty$ in (4.5), to have

$$D_x^m f(x) = \lim_{n\to\infty} \frac{h_{n,x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t)(-D_t)^m (e^{-h_{n,x}^2 t^2})dt \tag{4.6}$$

The proof for (4.1) is hereby immediate when the equalities (4.4) and (4.6) are combined.

By following the arguments in above it appears feasible to obtain similar asymptotic equivalence shown by (4.1) between any other two representation forms for $f(x)$ provided in Section 2.

5. Examples for the error bounds. To demonstrate the performance of each of the proposed approximation forms, we selectively show heretofore some examples with the error bounds.

(1) Example 1: $f(t) = t^2$

We take Approximation Form I. Then it gives

$$\begin{aligned} A_n(t) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{t}{x}\right)^k \left(1 - \frac{t}{x}\right)^{n-k} \left(\frac{kx}{n}\right)^2 \\ &= \frac{x^2}{n^2} \left[\frac{nt}{x} \left(1 - \frac{t}{x}\right) + \frac{n^2 t^2}{x^2} \right] \end{aligned}$$

$$= \quad t^2 + \frac{1}{n} t(x-t),$$

which shows that (3.10) is exact.

(2) Example 2: $f(t) = e^{-t}$ for $t > 0$

We take Approximation Form III. Then by the fact that f is a convex function we obtain

$$\left[\sum_{k=0}^{\infty} e^{-\frac{k}{n}} \frac{(nk)^k}{k!} e^{-nt} \right] - \frac{t}{2n} \le e^{-t} \le \sum_{k=0}^{\infty} e^{-\frac{k}{n}} \frac{(nt)^k}{k!} e^{-nt} \quad \text{for} \quad t > 0.$$

(3) Example 3: $f(t) = \frac{1}{1+t}$ for $t > 0$

It is obvious that f is convex on the positive real and $|f''| \le 2$ on same domain. We take Approximation Form IV. Then it gives

$$\left[\frac{\sqrt{n}}{\sigma} \int_0^{\infty} \frac{1}{1+x} \phi\left(\frac{\sqrt{n}}{\sigma}(x-t) \right) dx \right] - \frac{\sigma^2}{n} \le \frac{1}{1+t} \le \frac{\sqrt{n}}{\sigma} \int_0^{\infty} \frac{1}{1+x} \phi\left(\frac{\sqrt{n}}{\sigma}(x-t) \right) dx$$

for any $t > 0$, where ϕ is the probability density function of standard normal distribution.

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