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BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH ALGEBRA

BY

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To Ky Fan, on the occasion of his sixtieth birthday

Abstract. Let B be a subalgebra of a semisimple commutative Banach algebra A and an ideal of A. In this paper we investigate the maximal ideal spaces between A and B, and study the multiplier algebras between A and B.

1. **Introduction.** In this note we denote by A a semisimple commutative Banach algebra and by B an ideal in A such that B forms a Banach subalgebra with another norm. We may assume naturally that the embedding of the subalgebra B into A is continuous.

It is proved in [5, Theorem 4] that the space of maximal ideals of $A_p(G)$ (for the notation, we refer to [5]) is homeomorphic to the space of maximal ideals of $L^1(G)$. However, we can get this kind of conclusion in the more general situation, and prove that if B is a dense ideal in A, then the maximal ideal space $\mathfrak{M}(B)$ of B is homeomorphic to the maximal ideal space $\mathfrak{M}(A)$ of A, provided A is a commutative semisimple Banach algebra.

It is known that if A has an approximate identity, then A is a strictly dense ideal of its multiplier algebra $M(A)$. In this case the maximal ideal space of A and the maximal ideal space of $M(A)$ need not be homeomorphic (cf. Wang [8], p. 1137).

The multipliers of the Banach algebra A and its subalgebra B are discussed in the final section. We show that if B is dense in A and B has a bounded approximate identity, then so does A. By Lai [4, Theorem 4.1], the multiplier algebra $M(B)$ of B is contained

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in the multiplier algebra $M(A)$ of A. Actually, in this case the dense ideal B would be identically equal to A, and so $M(A)=M(B)$.

2. The maximal ideal space of an ideal as a Banach algebra.

THEOREM 1. *Let A be a commutative Banach algebra and B be a dense ideal of A. Suppose that B forms a Banach algebra with another norm. Then the maximal ideal space $\mathfrak{M}(B)$ of B is homeomorphic to the maximal ideal space $\mathfrak{M}(A)$ of A.*

Proof. Let $\chi^{\sim}$ be a nonzero character of the algebra A. It is obvious that the restriction $\chi^{\sim}|_B$ of $\chi^{\sim}$ on B is also a nonzero character of B, since B is dense in A. We will show that any character χ of B can be uniquely extended to a character $\chi^{\sim}$ of the algebra A. In the following extension we need not use the spectral norm but use the ideal property. Take a character χ of B. Since B is an ideal of A, $a\in A$ and $b\in B$ implies $ab\in B$. It is required to show $\chi(ab)=\chi^{\sim}(a)\,\chi^{\sim}(b)$ with $\chi^{\sim}$ as the extension of χ. Hence we define, for $a\in A$,

$$\chi^{\sim}(a)=\frac{\chi(ab)}{\chi(b)} \tag{1}$$

for some $b\in B$ with $\chi(b)\neq 0$. We must show that

(i) $\chi^{\sim}$ is well-defined;

(ii) $\chi^{\sim}$ is the extension of χ;

(iii) $\chi^{\sim}$ is a character on A.

For (i), if $b, c\in B$ such that $\chi(b)\neq 0\neq\chi(c)$, then for any $a\in A$

$$\chi(abc)=\chi(bac)\Longleftrightarrow\chi(ab)\,\chi(c)=\chi(b)\,\chi(ac)$$

or

$$\frac{\chi(ab)}{\chi(b)}=\frac{\chi(ac)}{\chi(c)}.$$

Hence $\chi^{\sim}(a)$ does not depend on the choice of b in the definition of $\chi^{\sim}$. That is, $\chi^{\sim}$ is well-defined.

For (ii), if $a\in B$, then

$$\chi^{\sim}(a)=\frac{\chi(ab)}{\chi(b)}=\frac{\chi(a)\,\chi(b)}{\chi(b)}=\chi(a)$$

shows that $\chi^{\sim}$ is an extension of χ.

For (iii), take a, b in A and c, d in B with $\chi(cd) \neq 0$. Then

$$\chi^{\sim}(ab) = \frac{\chi(abcd)}{\chi(cd)} = \frac{\chi(ac)}{\chi(c)} \cdot \frac{\chi(bd)}{\chi(d)} = \chi^{\sim}(a)\,\chi^{\sim}(b)\,.$$

This extension is unique. Indeed, if $\bar{\chi}$ is another extension of χ, then for $a \in A$, $b \in B$,

$$\chi(ab) = \bar{\chi}(ab) = \bar{\chi}(a)\,\bar{\chi}(b) = \bar{\chi}(a)\,\chi(b)\,,$$

and since $\chi(b) \neq 0$,

$$\bar{\chi}(a) = \frac{\chi(ab)}{\chi(b)} = \chi^{\sim}(a)\,.$$

Hence $\bar{\chi} = \chi^{\sim}$.

The continuity of the restriction mapping $\chi^{\sim} \to \chi$ of the character in A to the character in B is trivial since it is equivalent to saying that the characters are continuous in the weak-$*$ topology, while the continuity of the extension mapping $\chi \to \chi^{\sim}$ follows immediately from the definition (1) of $\chi^{\sim}$. Indeed, for $a \in A$, $b \in B$ with $\chi(b) = 1$ for any character χ of B, as $\chi_\alpha \to \chi$ we have

$$|\chi_\alpha^{\sim}(a) - \chi^{\sim}(a)| = |\chi_\alpha(ab) - \chi(ab)| \to 0\,.$$

Therefore $\mathfrak{M}(B)$ is homeomorphic to $\mathfrak{M}(A)$. Q.E.D.

It is to be noted that even if B is not dense in A, the character of B is still uniquely extendable to a character of A in the same way as above. Thus we can prove more generally the following.

THEOREM 2. *Let B be an ideal of a commutative Banach algebra A. Suppose that B forms a Banach algebra with some norm. Then the maximal ideal space $\mathfrak{M}(B)$ of B is homeomorphic to an open subset of the space $\mathfrak{M}(A)$ of the maximal ideals of A.*

Proof. As in the proof of Theorem 1, we see that each nonzero character χ of B can be extended to a character $\chi^{\sim}$ of A by defining $\chi^{\sim}(a) = \chi(ab)/\chi(b)$ as in (1) for some b such that $\chi(b) \neq 0$ for all $a \in A$. As in the proof of Theorem 1 this extension is unique.

Letting $\phi(\chi) = \chi^{\sim}$, we see that ϕ is one-to-one and is still continuous for the same reason as in the proof of Theorem 1. On the other hand ϕ^{-1} with domain in $\phi(\mathfrak{M}(B))$ is clearly continuous, so that ϕ is a homeomorphism. Suppose $\chi^{\sim} \notin \phi(\mathfrak{M}(B))$; then $\chi^{\sim}|_B = 0$, since if $\chi^{\sim}|_B \neq 0$, $\chi^{\sim}$ becomes an extension of the character $\chi^{\sim}|_B$.

Since the topology in $\mathfrak{M}(A)$ is nothing but the weak-* topology as functionals on A, and element b in B as a functional in A is continuous over the character space $\mathfrak{M}(A)$ of A, the null set of b in B is closed and the intersection $\{\chi \in \mathfrak{M}(A) \mid \chi|_B = 0\}$ of these null sets is therefore closed in $\mathfrak{M}(A)$. Hence $\phi(\mathfrak{M}(B))$ is open. Q.E.D.

This theorem is more general than Birtel [2, Theorem 1] applying to multiplier algebras.

On the other hand, as B is a proper closed ideal of A (where B need not be a Banach algebra), then the maximal ideal space $\mathfrak{M}(A/B)$ of A/B is homeomorphic to the hull $h(B)$ of B with respect to the hull-kernel topologies (cf. Loomis [6, 20G]). Let $K=\mathfrak{M}(A)/h(B)$, the quotient of the maximal ideal space $\mathfrak{M}(A)$ and the hull $h(B)$ of B. Then the characters of A corresponding to the maximal ideals in K do not take all of B into 0, and the intersection of B and the elements of K are regular maximal ideals of B, i.e. $B \cap M_K \in \mathfrak{M}(B)$ with $M_K \in K$. It is immediately clear that no two distinct elements of K (that is, two characters χ_1, χ_2 of A such that χ_1, χ_2 are not identically equal to zero on B) can have the same intersection with B. It follows from Theorem 2 that every regular maximal ideal of B can be extended to a unique element in K. Thus $M_K \leftrightarrow M_K \cap B$ is one-to-one between K and $\mathfrak{M}(B)$.

COROLLARY 3. *Let B be a proper closed ideal of a commutative Banach algebra A. Then the maximal ideal space $\mathfrak{M}(A)$ of A can be written in the form*

$$\mathfrak{M}(A) = h(B) \cup K,$$

where $h(B)$ is the hull of B, is a closed subset in $\mathfrak{M}(A)$ and is homeomorphic to the maximal ideal space $\mathfrak{M}(A/B)$ of A/B, and K is an open subset in $\mathfrak{M}(A)$ in which there exists a natural homeomorphism from K onto $\mathfrak{M}(B)$.

3. **Maximal ideals in Segal algebras.** For the definition of a Segal algebra, we refer to Reiter [7, p. 126, Chapter 6, §2]. We denote by $S(G)$ the Segal algebra where G is a locally compact abelian group. It is a dense ideal of $L^1(G)$. Then the following theorem is an immediate consequence of Theorem 1.

THEOREM 4. *Any Segal algebra $S(G)$ is a semisimple commutative Banach algebra. The space of maximal ideals can be identified with the dual group $\hat{G}$.*

4. Multiplier algebras between the Banach algebra and its ideal as a subalgebra. Recall that a multiplier T of a commutative Banach algebra B is a bounded linear operator on B such that

$$T(xy) = Tx \cdot y = x \cdot Ty \qquad \text{for } x, y \in B.$$

The set of all multipliers of B is denoted by $M(B)$, which is a commutative Banach algebra with respect to composition and the uniform topology. For properties of $M(B)$, we refer to Wang [8], Birtel [2], Lai [4], etc.

If the Banach algebra B has a bounded approximate identity, then B embedded in $M(B)$ is a dense ideal in $M(B)$ with respect to the strict topology. This $M(B)$ is completely different from the A in Theorem 1. Indeed B is not dense in $M(B)$ with the uniform topology of $M(B)$ except when B has an identity. Thus in general the maximal ideal space $\mathfrak{M}(B)$ of B is not homeomorphic to the maximal ideal space $\mathfrak{M}(M(B))$ of $M(B)$. The relation of $\mathfrak{M}(B)$ and $\mathfrak{M}(M(B))$ can be found in Wang [8, Theorem 3.2], which is the special case of Corollary 3 in §2, but in the case of the multiplier algebra $M(B)$, we know that it has an identity, and the maximal ideal space $\mathfrak{M}(M(B))$ is compact and hence the closed subset $h(B)$, the hull of B, is compact. Furthermore $\mathfrak{M}(B)$ is dense in $\mathfrak{M}(M(B))$ with respect to the hull-kernel topology whenever B is semisimple (cf. Wang [8, Theorem 3.3]).

In this section the Banach algebra B has a bounded approximate identity, which is essential for the discussion of multiplier algebras, and we note that if the algebra is semisimple or with approximate identity, then it is without order (cf. Wang [8]). In the proof of Barnes [1, Proposition 3.3], some further condition seems necessary for Cohen's Theorem to be applicable; for instance, that B is an essential A-module or that $B = AB$. If this is so, the bounded approximate identity of A need not be a bounded approximate identity of B; for example, $B = A^p(G)$, $A = L^1(G)$ in Lai [3, Theorem 1 and the remark in pp. 573–574]. We replace part of the assumption in the propositon by the assumption that if $\{e_\alpha\}$ is a left [right] bounded approximate identity for B, then $\{e_\alpha\}$ is also a bounded approximate identity for A. We only need the approximate identity in the case of commutative algebra, a fact which we state as follows.

PROPOSITION 5. *Let B be a dense ideal of A. Suppose that B has a bounded approximate identity $\{e_\alpha\}$. Then $\{e_\alpha\}$ is also a bounded approximate identity of A.*

Proof. The proof is almost trivial. Since B is dense in A, for any $a\in A$ and any $\varepsilon>0$, there exists $b\in B$ such that

$$\|a-b\|_A<\varepsilon .$$

Since $\{e_\alpha\}$ is a bounded approximate identity for B, there is an $M>0$ such that $\|e_\alpha\|_B\leq M$ for all α, and for $b\in B$ there is an α_0 such that

$$\|e_\alpha b-b\|_B<\varepsilon, \qquad \text{for } \alpha>\alpha_0 .$$

Now by the continuity of the embedding of B into A, there is a constant $C>0$ with $\|b\|_A\leq C\|b\|_B$ for $b\in B$, and

$$\begin{aligned}\|ae_\alpha-a\|_A &\leq \|ae_\alpha-be_{\alpha_0}\|_A+\|be_{\alpha_0}-b\|_A+\|b-a\|_A\\ &\leq \|a-b\|_A\|e_\alpha\|_A+C\|be_\alpha-be_{\alpha_0}\|_B\\ &\quad +C\|be_{\alpha_0}-b\|_B+\|b-a\|_A\\ &<(M+3C+1)\varepsilon, \qquad \text{for } \alpha>\alpha_0 .\end{aligned}$$

Since ε is arbitrary, $\|ae_\alpha-a\|_A\to 0$ when the limit is taken over α. Therefore $\{e_\alpha\}$ is a bounded approximate identity in A, since the embedding of B into A is continuous. Q.E.D.

From this proposition, the multiplier algebra $M(B)$ of B can be extended to be the multipliers of A, i.e.

$$M(B)\subset M(A).$$

See Lai [4, Theorem 4.1]. Since B and A have a bounded approximate identity, B and A are dense ideals in $M(B)$ and $M(A)$ respectively, with respect to the strict topology.

The author expresses his gratitude to the referee, who points out that under the assumptions of Proposition 5, we would have

$$B=A.$$

Indeed, if A is regarded as a module over B, then

$$A = \{a \in A \mid e_\alpha a \to a\}$$

by Proposition 5, and by the factorization theorem of Cohen, any $a \in A$ can be written as $bc = a$ for $b \in B$, $c \in A$. Consequently,

$$A \subseteq B \Longrightarrow A = B .$$

Therefore

$$M(A) = M(B) .$$

Note that in this section, the conclusions hold for the non-commutative case by using left or right identities in some situations.

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