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ON OSCILLATION OF A NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATION

BY

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Abstract. Sufficient conditions are established for the oscillation of solutions of neutral partial functional differential equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left(p(t)\frac{\partial}{\partial t}(u(x,t)+\lambda(t)u(x,t-\tau))\right)+q(x,t)u(x,t) \\ &+\sum_{j=1}^{m} q_j(x,t)f_j(u(x,t-\sigma_j)) \\ = \ &a(t)\Delta u(x,t)+\sum_{k=1}^{s} a_k(t)\Delta u(x,t-\rho_k) \end{aligned}$$

for $(x,t)\in\Omega\times[0,\infty)=G$, where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean $\mathbb{N}-$ space $\mathbb{R}^{\mathbb{N}}$, subject to the condition $\int_{t_0}^{\infty}\frac{1}{p(t)}dt<\infty, t_0\geq 0$.

1. Introduction. Recently there has been a lot of interest towards the study of qualitative properties of solutions of partial functional differential equations. This is due to a fact that partial functional differential equations provide a natural description to a number of real world problems arising in various branches of science and technology. See, for example the recent monograph of Wu [13] for theory and applications of partial functional differ-

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ential equations. However, only a few results have appeared in the literature dealing with oscillation and nonoscillation of partial functional differential equations, see [2, 3, 5, 9, 10] and the references cited therein.

In this paper, we study the oscillatory behavior of solutions of nonlinear neutral partial functional differential equations of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}(u(x,t)+\lambda(t)u(x,t-\tau))\right]+q(x,t)u(x,t)\\
\text{(E)}\qquad &+\sum_{j=1}^{m} q_j(x,t)f_j(u(x,t-\sigma_j))\\
= \quad &a(t)\Delta u(x,t)+\sum_{k=1}^{s} a_k(t)\Delta u(x,t-\rho_k), \qquad (x,t)\in\Omega\times[0,\infty)\equiv G,
\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$ with a piecewise smooth boundary $\partial\Omega$ and $\Delta u(x,t)=\sum_{r=1}^{N}\left(\partial^2 u(x,t)/\partial x_r^2\right)$. Equation (E) is supplemented by one of the following boundary conditions, namely

$$\text{(1)}\qquad \frac{\partial u(x,t)}{\partial \nu}+g(x,t)u(x,t)=0, \quad (x,t)\in\partial\Omega\times[0,\infty),$$

where ν is the unit exterior normal vector to $\partial\Omega$, and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, and

$$\text{(2)}\qquad u(x,t)=0, \qquad (x,t)\in\partial\Omega\times[0,\infty).$$

In what follows, we always assume without mentioning that

(A_1) $p\in C^1([0,\infty);[0,\infty))$;

(A_2) $\lambda\in C^2([0,\infty);[0,\infty))$, $0\le\lambda(t)<1$ and τ is a nonnegative constant;

(A_3) $q,q_j\in C(\overline{G};(0,\infty))$, $q(t)=\min_{x\in\bar\Omega} q(x,t)$, $q_j(t)=\min_{x\in\bar\Omega} q_j(x,t)$, $j\in \mathrm{I}_m=\{1,2,\ldots,m\}$;

(A_4) $a,a_k\in C([0,\infty);[0,\infty))$, ρ_k and σ_j are nonnegative constants, $j\in \mathrm{I}_m$; $k\in \mathrm{I}_s=\{1,2,\ldots,s\}$;

(A_5) $f_j\in C(\mathbb{R}:\mathbb{R})$ are convex in $[0,\infty)$, and $uf_j(u)>0$ for $u\neq 0$, $j\in \mathrm{I}_m$.

It is easy to see that the method of steps yields a weak solution to (E) in G. For further results concerning the existence and uniqueness of solutions of (E), one can refer to [1, 4, 6, 7].

A function $u \in C^2(G) \bigcap C'(\overline{G})$ is called a solution of the problem (E), (1)((E), (2)) if it satisfies (E) in the domain G and the boundary condition (1)((2)). The solution $u(x,t)$ of the equation (E), (1) (or (E), (2)) is said to be oscillatory in the domain G if for any positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that $u(x_0, t_0) = 0$ holds.

Very recently Li and Cui [9] considered the equation (E) and obtained several criteria for the oscillation of all solutions of (E) under the condition $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} ds = \infty$. This motivated the present research and the principal reason is that the condition $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} ds = \infty$ is not usually satisfied in practical problem, see Wong [12] [p.228]. Therefore it is important for applications to get rid of the aforementioned condition.

The purpose of this paper is to establish some new oscillation criteria for the problem (E),(1)((E),(2)) which complement and extend results in [2, 3, 5, 6] by using integral averaging technique. Examples illustrating the results are inserted in the text of the paper.

2. Oscillation of the problem (E), (1). Following Philos [11], we introduce a class of functions $\mathcal{P}$. Let $\mathcal{D}_0 = \{(t,s) : t > s \geq t_0\}$ and $\mathcal{D} = \{(t,s) : t \geq s \geq t_0\}$. The function $H \in C(\mathcal{D}; \mathbb{R})$ is said to belong to the class $\mathcal{P}$ if

(H_1) $H(t,t) = 0$ for $t \geq t_0, H(t,s) > 0$ on $\mathcal{D}_0$;

(H_2) H has a continuous and nonpositive partial derivative on $\mathcal{D}_0$ with respect to the second variable.

Corresponding to each solution $u(x,t)$ of the problem (E), (1), we consider the function

$$V(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dt,\quad t\geq T\geq 0 \tag{3}$$

where $|\Omega|=\int_\Omega dx$. The following lemma holds.

Lemma 2.1. *If $u(x,t)$ is a solution of the problem (E), (1) for which $u(x,t)>0$ in $G_T=\Omega\times[T,\infty)$, $T\geq 0$, then the function $V(t)$ defined by (3) satisfy the inequality*

$$\frac{d}{dt}\Big(p(t)\frac{d}{dt}(V(t)+\lambda(t)V(t-\tau))\Big)+q(t)V(t)+\sum_{j=1}^m q_j(t)f_j(V(t-\sigma_j))\leq 0 \tag{4}$$

with $V(t)>0$, $V(t-\tau)>0$ and $V(t-\sigma_j)>0$, $j\in \mathrm{I}_m$, for $t\geq T$.

Proof. Let $t\geq T$. Integrating (E) with respect to x over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\Big(p(t)\frac{d}{dt}\Big(\int_\Omega u(x,t)dx+\lambda(t)\int_\Omega u(x,t-\tau)dx\Big)\Big)\\ &+\int_\Omega q(x,t)u(x,t)dx+\sum_{j=1}^m\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))dx\\ =\ &a(t)\int_\Omega \Delta u(x,t)dx+\sum_{k=1}^s a_k(t)\int_\Omega \Delta u(x,t-\rho_k)dx.\end{aligned} \tag{5}$$

From Green's formula and boundary condition (1), it follows that

$$\int_\Omega \Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\nu}dS=-\int_{\partial\Omega}g(x,t)u(x,t)dS\leq 0,\quad t\geq T \tag{6}$$

and

$$\begin{aligned}\int_\Omega \Delta u(x,t-\rho_k)dx &= \int_{\partial\Omega}\frac{\partial u(x,t-\rho_k)}{\partial\nu}dS\\ &= -\int_{\partial\Omega}g(x,t-\rho_k)u(x,t-\rho_k)dS\leq 0,\quad t\geq T,\ k\in I_s,\end{aligned} \tag{7}$$

where dS is the surface element on $\partial\Omega$. Also, from (A_3), (A_5) and Jenson's inequality, we obtain

$$\int_\Omega q(x,t)u(x,t)dx \geq q(t)\int_\Omega u(x,t)dx, \quad t \geq T, \tag{8}$$

and

$$\begin{aligned}\int_\Omega q_j f_j(u(x,t-\sigma_j))dx &\geq q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))dx\\ &\geq q_j(t)\int_\Omega dx f_j\Big(\int_\Omega u(x,t-\sigma_j)dx\Big(\int_\Omega dx\Big)^{-1}\Big), \quad t\geq T.\end{aligned} \tag{9}$$

Inview of (3), (6–9), (5) yield

$$\frac{d}{dt}\Big[p(t)\frac{d}{dt}(V(t)+\lambda(t)V(t-\tau))\Big]+q(t)V(t)+\sum_{j=1}^{m}q_j(t)f_j(V(t-\sigma_j)) \leq 0, \quad t\geq T.$$

This proves the lemma.

Lemma 2.2. *Let $u(x,t)$ be a positive solution of the problem (E), (1) defined on G_T then the function $Z(t) = V(t) + \lambda(t)V(t-\tau)$ where $V(t)$ is defined by* (3)*, satisfies one of the following conditions:*

(I) $Z(t) > 0$, $Z'(t) > 0$, $(p(t)Z'(t))' \leq 0$,

(II) $Z(t) > 0$, $Z'(t) < 0$, $(p(t)Z'(t))' \leq 0$,

for all $t \geq T$.

Proof. From lemma 2.1, the function $V(t)$ satisfies the inequality (4) and $V(t) > 0, V(t-\tau) > 0$ and $V(t-\sigma_j) > 0, j \in I_m$ for $t \geq T$. From (4) and the hypothesis we have $Z(t) > 0$ and $(p(t)Z'(t))' \leq 0$ for $t \geq T$. Hence $p(t)Z'(t)$ is monotonic and eventually of one sign. This proves Lemma 2.2.

Lemma 2.3. *Let $u(x,t)$ be a positive solution of the problem (E), (1)* *defined on G_T and suppose Case (I) of Lemma 2.2 holds, then*

$$V(t) \geq (1-\lambda(t))Z(t) \tag{10}$$

for $t \geq T \geq t_0$.

Proof. From Case (I), $Z(t)$ is positive and increasing for $t \geq T$, and hence from the definition of $Z(t)$, we obtain $Z(t) \geq V(t)$ and $V(t) = Z(t) - \lambda(t)V(t-\tau) \geq Z(t) - \lambda(t)Z(t-\tau) \geq (1-\lambda(t))Z(t)$ for $t \geq T$.

Lemma 2.4. *Let $u(x,t)$ be a positive solution of the problem (E), (1) defined on G_T and suppose Case (II) of Lemma 2.2 holds. Then*

$$V(t-\tau) \geq \frac{Z(t)}{1+\lambda(t)} \geq (1-\lambda(t))Z(t), \quad t \geq T. \tag{11}$$

Proof. In this case the function $Z(t)$ is positive and nonincreasing for $t \geq T$ and therefore without loss of generality we may assume from the definition of $Z(t)$ that $V(t)$ is also nonincreasing (see [2]) for $t \geq T$. Hence

$$Z(t) = V(t) + \lambda(t)V(t-\tau) \leq V(t-\tau) + \lambda(t)V(t-\tau)$$

which implies (11).

Theorem 2.1. *Assume that $\frac{f_j(u)}{u} \geq \alpha_j$ for $u \neq 0$ where α_j are positive constants $j \in I_m$. Let $h, H : \mathcal{D} \to \mathbb{R}$ be continuous functions such that $H \in \mathcal{P}$ and*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in \mathcal{D}_0. \tag{12}$$

Assume also that there exits a continuously differentiable function $\rho : [t_0, \infty) \to \mathbb{R}_+$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)A(s)\rho(s) - \frac{p(s-\sigma)\rho(s)}{4} Q^2(t,s) \right] ds = \infty \tag{13}$$

where

$$A(t) = q(t)(1-\lambda(t)) + \sum_{j=1}^{m} \alpha_j q_j(t)(1-\lambda(t-\sigma_j)),$$

$$Q(t,s) = h(t,s) - \frac{\rho'(t)}{\rho(t)}\sqrt{H(t,s)},$$

$$\sigma = \max\{\sigma_1, \ldots, \sigma_m\}$$

and

$$\liminf_{t\to\infty} \int_{t+\tau-\sigma^*}^{t} \frac{1}{p(s)} \left(\int_{s-\sigma^*}^{s} B(r)dr \right) ds > \frac{1}{e} \tag{14}$$

where $B(t) = \sum_{j=1}^{m} \alpha_j q_j(t)/(1+\lambda(t+\tau-\sigma_j))$ *and* $\sigma^* = \min\{\sigma_1, \ldots, \sigma_m\} > \tau$. *Then every solution* $u(x,t)$ *of the problem* (E), (1) *is oscillatory in* G.

Proof. Assume to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (1) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0$, $u(x,t-\tau) > 0$, $u(x,t-\rho_k) > 0$ and $u(x,t-\sigma_j) > 0$ in $\Omega\times[T,\infty)$, $T \ge t_0$, $k \in I_s$, $j \in I_m$, where T is chosen so large that Lemmas 2.1 to 2.4 hold for $t \ge T$. From Lemma 2.1 the function $V(t)$ defined by (3) satisfy the inequality

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}(V(t)+\lambda(t)V(t-\tau))\right)+q(t)V(t)+\sum_{j=1}^{m} q_j(t)f_j(V(t-\sigma_j)) \le 0. \tag{15}$$

Let $Z(t) = V(t) + \lambda(t)V(t-\tau)$. Then $Z(t)$ satisfies either Case (I) or Case (II) of Lemma 2.2.

Case (I): For this case $Z(t) > 0$, $Z'(t) > 0$ and $(p(t)Z'(t))' \le 0$, $t \ge T$. Using Lemma 2.3 and (A_5), (15) yields

$$\begin{aligned} &(p(t)Z'(t))' + q(t)(1-\lambda(t))Z(t-\sigma) \\ &+ \left[\sum_{j=1}^{m} \alpha_j q_j(t)(1-\lambda(t-\sigma_j))\right] Z(t-\sigma) \le 0, \end{aligned} \tag{16}$$

for $t \geq T$. Define

$$W(t) = \rho(t)p(t)\frac{Z'(t)}{Z(t-\sigma)}, \tag{17}$$

then

$$W'(t) = \frac{\rho'(t)}{\rho(t)}W(t) - A(t)\rho(t) - \frac{Z'(t-\sigma)}{Z(t-\sigma)}W(t). \tag{18}$$

From $(p(t)Z'(t))' \leq 0$ for $t \geq T$ we have $p(t-\sigma)Z'(t-\sigma) \geq p(t)Z'(t)$, and consequently by (18), for $t \geq T$, we obtain that

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - A(t)\rho(t) - \frac{W^2(t)}{\rho(t)p(t-\sigma)}. \tag{19}$$

Hence from (19) for all $t \geq T \geq t_0$, we have

$$\begin{aligned}
&\int_T^t H(t,s)A(s)\rho(s)ds \\
&\leq \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}W(s)ds - \int_T^t H(t,s)W'(s)ds - \int_T^t H(t,s)\frac{W^2(s)}{\rho(s)p(s-\sigma)}ds \\
&= H(t,T)W(T) - \int_T^t \left[-\frac{\partial H}{\partial s}(t,s)W(s) - H(t,s)\frac{\rho'(s)}{\rho(s)}W(s) + \frac{H(t,s)}{\rho(s)p(s-\sigma)}W^2(s)\right]ds \\
&= H(t,T)W(T) - \int_T^t \left[\sqrt{\frac{H(t,s)}{\rho(s)p(s-\sigma)}}W(s) + \frac{1}{2}\sqrt{p(s-\sigma)\rho(s)}Q(t,s)\right]^2 ds \\
&\quad + \int_T^t \frac{p(s-\sigma)\rho(s)}{4}Q^2(t,s)ds.
\end{aligned}$$

Thus, for all $t \geq T \geq t_0$, we conclude that

$$\int_T^t \left[H(t,s)A(s)\rho(s) - \frac{p(s-\sigma)\rho(s)}{4}Q^2(t,s)\right]ds \leq H(t,T)W(T). \tag{20}$$

By virtue of (20) and (H_2), for every $t \geq t_0$, we obtain

$$\begin{aligned}
\int_T^t \left[H(t,s)A(s)\rho(s) - \frac{p(s-\sigma)\rho(s)}{4}Q^2(t,s)\right]ds &\leq H(t,T)|W(t_0)| \\
&\leq H(t,t_0)|W(t_0)|.
\end{aligned} \tag{21}$$

Then, by (13) and (H_2), we have

$$\begin{aligned}\int_{t_0}^{t}\left[H(t,s)A(s)\rho(s)-\frac{p(s-\sigma)\rho(s)}{4}Q^2(t,s)\right]ds \\ \leq\ H(t,t_0)\left[\int_{t_0}^{t_1}A(s)\rho(s)ds+|W(t_0)|\right].\end{aligned}\tag{22}$$

Inequality (22) yields

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)A(s)\rho(s)-\frac{p(s-\sigma)\rho(s)}{4}Q^2(t,s)\right]ds\\ \leq\ &\int_{t_0}^{t_1}A(s)\rho(s)ds+|W(t_0)|<\infty,\end{aligned}$$

which contradicts (13).

Case (II): Now assume $Z(t)$ satisfies (10). In view of the hypothesis and Lemma 2.3, we have from (15), for $t\geq T\geq t_0$

$$(p(t)Z'(t))'+\left[\sum_{j=1}^{m}\alpha_j\frac{q_j(t)}{1+\lambda(t+\tau-\sigma_j)}\right]Z(t+\tau-\sigma^*)\leq 0\tag{23}$$

Integrating (23) from $t-\sigma^*$ to t, we have

$$p(t)Z'(t)-p(t-\sigma^*)Z'(t-\sigma^*)+\int_{t-\sigma^*}^{t}B(s)Z(s+\tau-\sigma^*)ds\leq 0,$$

or

$$Z'(t)+\left(\frac{1}{p(t)}\int_{t-\sigma^*}^{t}B(s)ds\right)Z(t+\tau-\sigma^*)\leq 0.$$

From [4], condition (14) implies that the last inequality has no eventually positive solution, a contradiction. This completes the proof of the theorem.

Corollary 2.1. *Let conditions of Theorem* 2.1 *be hold. If the inequality* (15) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (E), (1) *is oscillatory in* G.

Corollary 2.2. *Assume that the conditions of Theorem* 2.1 *hold with* (13) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)A(s)ds \;=\; \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} p(s-\sigma)\rho(s)Q^2(t,s)ds \;<\; \infty,$$

then every solution $u(x,t)$ *of the problem* (E), (1) *is oscillatory in* G.

Remark 2.1. With the appropriate choice of functions H and h, it is possible to derive from Theorem 2.1, a number of oscillation criteria for the problem (E), (1). Defining for example, for some integer $n > 2$ the function $H(t,s)$ by

$$H(t,s) = (t-s)^{n-1}, \qquad (t,s) \in \mathcal{D}$$

we can easily check that $H \in \mathcal{P}$. Furthermore, the function

$$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \qquad (t,s) \in \mathcal{D}$$

is continuous and satisfies the required condition given in Theorem 2.1. Therefore, as a consequence of Theorem 2.1, we obtain the following oscillation criteria.

Corollary 2.3. *Assume that the conditions of Theorem* 2.1 *hold with* (13) *replaced by*

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \Big[(t-s)^{n-1}\rho(s)A(s) - \frac{p(s-\sigma)\rho(s)}{4}(t-s)^{n-3}\left((n-1) - \frac{\rho'(s)}{\rho(s)}(t-s)\right)^2\Big]ds = \infty$$

for some integer $n > 2$. *Then every solution* $u(x,t)$ *of the problem* (E), (1) *is oscillatory in* G.

Remark 2.2. With a different choice of function H and h, it is possible to derive many more corollaries. In fact, another possibility is to choose the functions H and h as follows

$$H(t,s) = \left(\log \frac{t}{s}\right)^n, \qquad t \ge s \ge t_0$$

$$h(t,s) = \frac{n}{s}\left(\log \frac{t}{s}\right)^{\frac{n}{2}-1}, \quad t \ge s \ge t_0.$$

One may also choose the more general forms for the functions H and h

$$H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \qquad t \ge s \ge t_0$$

$$h(t,s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{\frac{n}{2}-1}, \quad t \ge s \ge t_0.$$

where $n > 1$ is an integer, and $\theta : [t_0, \infty) \to \mathbb{R}_+$ is a continuous function satisfying condition

$$\lim_{t\to\infty} \int_{t_0}^t \frac{du}{\theta(u)} = \infty.$$

Next we establish conditions for the oscillation of all solutions of the problem (E), (1) subject to the following conditions:

(H_3) $\displaystyle\int_{t_0}^\infty \frac{dt}{p(t)} < \infty;$

(H_4) $\frac{f_j(u)}{u^\gamma} \ge M_j > 0$ for $u \ne 0$ and $\gamma \in (1,\infty)$ is a ratio of odd integers.

Theorem 2.2. *In addition to conditions* (H_3) *and* (H_4) *assume* $\sigma_j \ge \tau$ *for all* $j \in I_m$. *Then all solutions of the problem* (E), (1) *are oscillatory if*

$$\sum_{j=1}^m \int_{t_0}^\infty q_j(t)(1-\lambda(t-\sigma_j))^\gamma Q^\gamma(t-\sigma_j)dt = \infty, \tag{24}$$

and

$$\sum_{j=1}^{m}\int_{t_0}^{\infty} q_j(t)(1-\lambda(t+\tau-\sigma_j))^{\gamma}Q^{\gamma}(t)dt \;=\; \infty \tag{25}$$

where $Q(t)=\int_t^{\infty}\frac{1}{p(s)}ds.$

Proof. Suppose the contrary; there exists a nonoscillatory solution $u(x,t)$ of the problem (E), (1) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $u(x,t)>0$, $u(x,t-\tau)>0$, $u(x,t-\rho_k)>0$ and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\ge t_0$, $k\in I_s$, $j\in I_m$. Then the function $V(t)$ defined by (3) satisfies the inequality (15).

Let $Z(t)=V(t)+\lambda(t)V(t-\tau)$, then $Z(t)>0$ for $t\ge t_1$. From (15), we have

$$(p(t)Z'(t))' \;\le\; 0 \quad \text{for} \quad t\ge T\ge t_1, \tag{26}$$

and

$$p(t)Z'(t) \;\le\; p(T)Z'(T) \quad \text{for} \quad t\ge T$$

or

$$Z'(t) \;\le\; \frac{p(T)Z'(T)}{p(t)} \quad \text{for} \quad t\ge T.$$

Integrating the last inequality from T to t, we have

$$Z(t)-Z(T)\le (p(T)Z'(T))\int_T^t \frac{1}{p(s)}ds, \quad t\ge T. \tag{27}$$

So $Z(t)$ is bounded above. From (27) we obtain

$$Z(T)\ge -(p(T)Z'(T))\int_T^t \frac{1}{p(s)}ds, \quad t\ge T.$$

Letting $t\to\infty$ we obtain

$$Z(T)\ge -(p(T)Z'(T))Q(T), \tag{28}$$

where $Q(T)$ is defined by (25) and T is an arbitrary large number.

From Lemma 2.2 there are two possible cases for $Z(t)$. First we consider that $Z(t) > 0, Z'(t) > 0$ for $t \geq T$. Integrating (15) from T to t we have

$$\int_T^t (p(s)Z'(s))'ds + \sum_{j=1}^m \int_T^t q_j(s)f_j(V(s-\sigma_j))ds \leq 0. \tag{29}$$

In view of condition (H_4) and Lemma 2.3, we have from (29)

$$p(t)Z'(t) - p(T)Z'(T) + \sum_{j=1}^m M_j \int_T^t q_j(s)(1-\lambda(s-\sigma_j))^\gamma Z^\gamma(s-\sigma_j)ds \leq 0 \tag{30}$$

or

$$\sum_{j=1}^m M_j \int_T^t q_j(s)(1-\lambda(s-\sigma_j))^\gamma Z^\gamma(s-\sigma_j)ds \leq p(T)Z'(T).$$

Letting $t \to \infty$, we have

$$\sum_{j=1}^m M_j \int_T^\infty q_j(s)(1-\lambda(s-\sigma_j))^\gamma Z^\gamma(s-\sigma_j)ds \leq \infty. \tag{31}$$

For this case $Z(t)$ is increasing, so there exists a number c such that $Z(t) \geq c > 0$ for $t \geq T$. Thus there exists a $T_1 \geq T$ such that

$$Z(t-\sigma_j) \geq Q(t-\sigma_j), \quad t \geq T_1, \tag{32}$$

and $j \in I_M$, since $Q(t) \to 0$ as $t \to \infty$. Combining (31) and (32), we have

$$\sum_{j=1}^m M_j \int_{T_1}^\infty q_j(t)(1-\lambda(t-\sigma_j))^\gamma Q^\gamma(t-\sigma_j)dt < \infty \tag{33}$$

which contradicts (24).

Now we consider the other case that $Z(t) > 0$ and $Z'(t) < 0$ for $t \geq T$. From (28), we have

$$Z(t) \geq -(p(t)Z'(t))Q(t) \tag{34}$$

for $t \geq T$. Consider

$$
\begin{aligned}
&((p(t)Z'(t)^{-\gamma+1})' \\
= \; & (-\gamma+1)(p(t)Z'(t))^{-\gamma}(p(t)Z'(t))' \\
\leq \; & (-\gamma+1)(p(t)Z'(t))^{-\gamma}\Big(-\sum_{j=1}^{m} q_j(t)f_j(V(t-\sigma_j))\Big) \\
\leq \; & (-\gamma+1)(p(t)Z'(t))^{-\gamma}\Big(-\sum_{j=1}^{m} M_j q_j(t)V^{\gamma}(t-\sigma_j))\Big) \\
\leq \; & (-\gamma+1)(p(t)Z'(t))^{-\gamma}\Big(-\sum_{j=1}^{m} M_j q_j(t)(1-\lambda(t+\tau-\sigma_j))^{\gamma}Z^{\gamma}(t))\Big) \\
\leq \; & -(\gamma-1)\sum_{j=1}^{m} M_j q_j(t)(1-\lambda(t+\tau-\sigma_j))^{\gamma}Q^{\gamma}(t),
\end{aligned}
$$

where we have used condition (H_4), (34) and Lemma 2.4. Integrating the last inequality from T to t, we obtain

$$(\gamma-1)\sum_{j=1}^{m} M_j \int_T^t q_j(s)(1-\lambda(s+\tau-\sigma_j))^{\gamma}Q^{\gamma}(s)ds \leq (p(T)Z'(T))^{-\gamma+1}$$

and so letting $t \to \infty$, we obtain

$$\sum_{j=1}^{m} M_j \int_T^{\infty} q_j(t)(1-\lambda(t+\tau-\sigma_j))^{\gamma}Q^{\gamma}(t)dt < \infty$$

which contradicts (25). The proof is now complete.

Next we consider the problem (E), (1) subject to the following condition:

(H_5) $\frac{f_j(u)}{u^{\gamma}} \geq M_j > 0$ for $u \neq 0$ and $\gamma \in (0,1)$ is a ratio of odd positive integers.

Theorem 2.3. *In addition to conditions (H_3) and (H_5) assume that*

$$\int_{t_0}^{\infty} \sum_{j=1}^{m} q_j(t)(1-\lambda(t-\sigma_j))^{\gamma}Q(t-\sigma_j)dt = \infty \tag{35}$$

and

$$\int_{t_0}^{\infty}\sum_{j=1}^{m} q_j(t)(1-\lambda(t+\tau-\sigma_j))^{\gamma}Q(t)dt=\infty. \tag{36}$$

Then every solution of the problem (E), (1) *is oscillatory in* G.

Proof. Without loss of generality we may assume that $u(x,t)>0$, $u(x,t-\tau)>0$, $u(x,t-\rho_k)>0$ and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\ge t_0$, $k\in I_s$, $j\in I_m$, is a solution of the problem (E), (1). Therefore

$$(p(t)Z'(t))'\le 0 \quad \text{for} \quad t\ge T\ge t_1.$$

If $Z'(t)>0$ for $t\ge T$, we have (31) and (33). For large t we have $Q(t)\le 1$ and $Q^{\gamma}(t)\le Q(t)$. Therefore from (33), we obtain

$$\int_{T_1}^{\infty}\sum_{j=1}^{m} M_j q_j(t)(1-\lambda(t-\sigma_j))^{\gamma}Q(t-\sigma_j)dt<\infty$$

which contradicts (35).

For this case $Z'(t)<0$ for $t\ge T$, from (30)

$$p(t)Z'(t)-p(T)Z'(T)+\int_{T}^{t}\sum_{j=1}^{m} M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^{\gamma}Z^{\gamma}(s+\tau-\sigma_j)ds<0, \quad t\ge T.$$

or

$$-Z'(t)\ge\frac{1}{p(t)}\int_{T}^{t}\sum_{j=1}^{m} M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^{\gamma}Z^{\gamma}(s+\tau-\sigma_j)ds, \quad t\ge T.$$

We consider the differential $(Z^{2\varepsilon}(t))'$ where $\varepsilon>0$ such that $2\varepsilon<1-\gamma$.

$$\begin{aligned}-(Z^{2\varepsilon}(t))' &= -2\varepsilon(Z^{2\varepsilon-1}(t)Z'(t))\\ &\ge 2\varepsilon(Z^{2\varepsilon-1}(t)\frac{1}{p(t)}\int_{T}^{t}\sum_{j=1}^{m} M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^{\gamma}Z^{\gamma}(s+\tau-\sigma_j)ds\\ &\ge 2\varepsilon\frac{1}{p(t)}\int_{T}^{t}\sum_{j=1}^{m} M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^{\gamma}Z^{2\varepsilon+\gamma-1}(s+\tau-\sigma_j)ds\end{aligned}$$

$$\geq 2\varepsilon \frac{1}{p(t)} \int_T^t \sum_{j=1}^m M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^\gamma Z^{2\varepsilon+\gamma-1}(s)ds$$

according as $\tau \geq \sigma_j$ or $\tau \leq \sigma_j$ and $Z(t)$ is decreasing. Since $c_1 \geq Z(t) > 0$ for $t \geq T$ where $c_1 > 0$ is a constant, there exists positive number k such that

$$-(Z^{2\varepsilon}(t))' \geq \frac{k}{p(t)} \int_T^t \sum_{j=1}^m M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^\gamma ds.$$

Integrating and rearranging we obtain

$$Z^{2\varepsilon}(T) - Z^{2\varepsilon}(t) \geq \int_T^t \sum_{j=1}^m M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^\gamma \left(\int_s^t \frac{1}{p(r)}dr\right) ds$$

and so letting $t \to \infty$, we have

$$\int_T^\infty \sum_{j=1}^m M_j q_j(s)(1-\lambda(s+\tau-\sigma_j))^\gamma Q(s)ds < \infty$$

which contradicts (36). This completes the proof.

3. Oscillation of the problem (E), (2). In this section we establish sufficient conditions for the oscillation of all solutions of the problem (E), (2). For this we need the following:

The smallest eigen value β_0 of the Dirichelet problem

$$\Delta\omega(x) + \beta\omega(x) = 0 \quad \text{in} \quad \Omega$$

$$\omega(x) = 0 \quad \text{on} \quad \partial\Omega,$$

is positive and the corresponding eigen function $\phi(x)$ is positive in Ω.

Theorem 3.1. *Let all conditions of Theorem* 2.1 *be hold. Then every solution of the problem* (E), (2) *oscillates in* G.

Proof. Assume the contrary: then there exists a nonoscillatory solution $u(x,t)$ of the problem (E), (2) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $u(x,t)>0$, $u(x,t-\tau)>0$, $u(x,t-\rho_k)>0$ and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\geq t_0$, $k\in I_s$, $j\in I_m$.

Multiply both sides of equation (E) by $\phi(x)>0$ and then integrating with respect to x over the domain Ω, we obtain for $t\geq t_1$,

$$\begin{aligned}&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)\phi(x)dx+\lambda(t)\int_\Omega u(x,t-\tau)\phi(x)dx\right)\right]\\ &\qquad+\int_\Omega q(x,t)u(x,t)\phi(x)dx+\sum_{j=1}^m\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\phi(x)dx \qquad (37)\\ &=a(t)\int_\Omega \Delta u(x,t)\phi(x)dx+\sum_{k=1}^s a_k(t)\int_\Omega \Delta u(x,t-\rho_k)\phi(x)dx.\end{aligned}$$

From Green's formula and boundary condition (2), it follows that

$$\begin{aligned}\int_\Omega \Delta u(x,t)\phi(x)dx &= \int_\Omega u(x,t)\Delta\phi(x)dx\\ &= -\beta_0\int_\Omega u(x,t)\phi(x)dx\leq 0,\quad t\geq t_1,\end{aligned}\tag{38}$$

and for $k\in I_s$.

$$\begin{aligned}\int_\Omega \Delta u(x,t-\rho_k)\phi(x)dx &= \int_\Omega u(x,t-\rho_k)\Delta\phi(x)dx\\ &= -\beta_0\int_\Omega u(x,t-\rho_k)\phi(x)dx\leq 0,\quad t\geq t_1,\end{aligned}\tag{39}$$

From (A_3), (A_5) and Jenson's inequality, it follows that

$$\int_\Omega q(x,t)u(x,t)\phi(x)dx\geq q(t)\int_\Omega u(x,t)\phi(x)dx,\quad t\geq t_1,\tag{40}$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\phi(x)dx$$

$$\begin{aligned} (41) \qquad &\geq q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))\phi(x)dx \\ &\geq q_j(t)\int_\Omega \phi(x)dx f_j\left(\int_\Omega u(x,t-\sigma_j)\phi(x)dx\left(\int_\Omega \phi(x)dx\right)^{-1}\right) \end{aligned}$$

for $t \geq t_1$ and $j \in I_m$.

Set

$$(42) \qquad V(t) = \int_\Omega u(x,t)\phi(x)dx\left(\int_\Omega \phi(x)dx\right)^{-1}, \qquad t \geq t_1.$$

Inview of (37)-(42), we obtain

$$(43)\ \frac{d}{dt}\left[p(t)\frac{d}{dt}(V(t)+\lambda(t)V(t-\tau)\right)+q(t)V(t)+\sum_{j=1}^{m} q_j(t)f_j(V(t-\sigma_j)) \leq 0$$

for $t \geq t_1$. Rest of the proof is similar to that of Theorem 2.1 and hence the details are omitted.

Corollary 3.1. *If the inequality* (43) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (E), (2) *is oscillatory in* G.

The following theorems and corollaries can be proved analogously.

Corollary 3.2. *Let the conditions of Corollary* 2.2 *hold; then every solution* $u(x,t)$ *of the problem* (E), (2) *is oscillatory in* G.

Corollary 3.3. *Let the conditions of Corollary* 2.3 *hold; then every solution* $u(x,t)$ *of the problem* (E), (2) *is oscillatory in* G.

Theorem 3.2. *Let the conditions of Theorem* 2.2 *hold; then every solution* $u(x,t)$ *of the problem* (E), (2) *is oscillatory in* G.

Theorem 3.3. *Let the conditions of Theorem* 2.3 *hold; then every* *solution $u(x,t)$ of the problem (E), (2) is oscillatory in G.*

4. Examples. In this section we give some examples to illustrate our results established in Sections 2 and 3.

Example 1. Consider the differential equation

$$(E_1)\qquad \begin{aligned} &\frac{\partial}{\partial t}\left((t+\pi)^2\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{t+\pi}u(x,t-2\pi)\right)\right)+(t+\pi)^2u(x,t)\\ &+(t+\pi)u\left(x,t-4\pi\right)\\ =\ &2(t+\pi)\Delta u\left(x,t-\frac{\pi}{2}\right)\end{aligned}$$

for $(x,t)\in(0,\pi)\times[0,\infty)$, with boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\geq 0. \tag{44}$$

Here $N=1$, $s=1$, $m=1$, $p(t)=(t+\pi)^2$, $\lambda(t)=\frac{1}{t+\pi}$, $\tau=2\pi$, $a_1(t)=2(t+\pi)$, $\rho_1=\frac{\pi}{2}$, $q(x,t)=(t+\pi)^2$, $q_1(x,t)=t+\pi$, $\sigma_1=4\pi$ and $f_1(u)=u$. It is easy to see that $q(t)=(t+\pi)^2$, $q_1(t)=(t+\pi)$, $A(t)=(t+\pi)^2\left(1-\frac{1}{t+\pi}\right)+(t+\pi)\left(1-\frac{1}{t-3\pi}\right)$, $B(t)=\frac{(t-\pi)(t+\pi)}{t+1-\pi}$. Choose $n=3$ and $\rho(t)=1$, then it is easy to see that all conditions of Corollary 3.3 are satisfied. Hence every solution of the problem (E_1), (44) oscillates in $(0,\pi)\times[0,\infty)$. In fact $u(x,t)=\sin x\cos t$ is such a solution.

Example 2. Consider the differential equation

$$(E_2)\qquad \begin{aligned} &\frac{\partial}{\partial t}\left((t+\pi)^2\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{(t+\pi)}u(x,t-2\pi)\right)\right)+2(t+\pi)u\left(x,t-\frac{5\pi}{2}\right)\\ &=\left((t+\pi)^2+(t+\pi)\right)\Delta u(x,t)\end{aligned}$$

for $(x,t)\in(0,\pi)\times[0,\infty)$, with boundary condition

$$u_x(0,t)=u_x(\pi,t)=0,\quad t\geq 0. \tag{45}$$

It is easy to check that the conditions of Corollary 2.3 are satisfied. Therefore, every solution of the problem (E_2), (45) is oscillatory in $(0, \pi) \times [0, \infty)$. In fact $u(x, t) = \cos x \sin t$ is such a solution.

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