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RANK FUNCTION FOR POSET MATROIDS[1]

BY

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Abstract. An excellent introduction to the topic of poset matroids is due to M. Barnabei, G. Nicoletti and L. Pezzoli [1, 2]. In this paper, we investigate the rank function and obtain the local rank axioms for poset matroids, thereby we can characterize poset matroids in terms of these "local" rank axioms. Some corresponding properties of combinatorial schemes are also obtained.

1. Introduction. The theory of poset matroid is the extension of classical matroids theory, developed by replacing the underlying set of a matroid by a partially ordered set. Consequently, the notion of subsets of a set is replaced by that of filter, or, dually, by the order idea of a partially ordered set.

By a fundamental theorem of G. Birkhoff, every finite distributive lattice is isomorphic to the lattice of all filters of a finite partially ordered set. Conversely, every finite partially ordered set is isomorphic to the partially ordered set of the meet-irreducible elements of a distributive lattice. By virtue of these isomorphisms, we may use the language of posets and the language of distributive lattice interchangeably. The notions of a poset matroid and a combinatorial scheme correspond to each other in this double language.

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In [2], M. Barnabei et al. have investigated some properties of the rank associated with an antichain in a distributive lattice, however we do not know why they didn't give the corresponding description of the rank properties for poset matroids. In fact the version between the two languages is not completely the same, a significant example is in [2], therefore we will describe the properties of the rank function relative to those of rank associated with an antichain in a distributive lattice given by Barnabei et al., however this is not our fundamental purpose. Actually, our purpose is to set up the systems of rank axioms for poset matroids by giving some sufficient and necessary conditions to determine a non-negative integer valued function to be a rank function of a poset matroid and investigate some interesting properties of rank function for poset matroids.

2. Notations and preliminary results. For completeness, we shall take the following notations: $\mathbb{P} := (P, \leq)$ denotes a finite partially ordered set. $\mathbb{N}$ denotes the non-negative integers.

A *filter* of $\mathbb{P}$ is any subset A of $\mathbb{P}$ such that for every x, $y \in \mathbb{P}$, if $x \geq y$ and $y \in A$, then $x \in A$. $\mathrm{F}(\mathbb{P})$ denotes the family of all filters of $\mathbb{P}$. An *incomparable set* of $\mathbb{P}$ is any subset A of $\mathbb{P}$ such that for every $x, y \in A : x \not< y$.

We call a sequence of filters $X_0 \subset X_1 \subset \cdots \subset X_k \subset \cdots \subset X_n$, where $|X_{i+1}| = |X_i| + 1, i = 0, 1, \ldots, n-1$, a *complete sequence.*

For any given subset A of $\mathbb{P}$, we define $\mathrm{Max}(A) = \{x \in A, x \text{ is maximal in } A\}$, $\mathrm{Min}(A) = \{x \in A, x \text{ is minimal in } A\}$.

For every x and y in a poset $\mathbb{P}$, $x \leq y$, the interval $[x, y]$ is defined to be a set $[x, y] := \{z \in \mathbb{P}, x \leq z \leq y\}$. If $\mathrm{card}[x, y] = 2$, then x is *covered* by y, in symbols, $x \lessdot y$. The least element of a finite lattice will be denoted by 0. The other terms and notations see [3] or [4].

We begin by recalling the definition of two fundamental notions of this work, namely, poset matroids and combinatorial schemes.

1. A *poset matroid* on the partially ordered set $\mathbb{P}$ is a family $\mathfrak{B}$ of filters of $\mathbb{P}$, called *bases*, following axiom
   (b.1) $\mathfrak{B} \neq \emptyset$.
   (b.2) For every B_1, $B_2 \in \mathfrak{B}$: $B_1 \not\subset B_2$.
   (b.3) For every B_1, $B_2 \in \mathfrak{B}$ and for every pair of filters X, Y of $\mathbb{P}$, such that $X \subseteq B_1$, $B_2 \subseteq Y$, $X \subseteq Y$, there exists $B \in \mathfrak{B}$ such that $X \subseteq B \subseteq Y$ (middle axiom).
2. Let $\mathbb{L}$ be a finite distributive lattice. A *combinatorial scheme* in $\mathbb{L}$ is a non-empty antichain A of $\mathbb{L}$ that satisfies the following axiom: for every a_1, $a_2 \in A$ and for every x, $y \in \mathbb{L}$, with $x \leq a_1$, $a_2 \leq y$, $x \leq y$, there exists $a \in A$ such that $x \leq a \leq y$.

Let A be an antichain in a distributive lattice $\mathbb{L}$. The *rank* ρ_A associated with A is the non-negative integer valued function on $\mathbb{L}$ defined as follows: for every $x \in \mathbb{L}$,

$$\rho_A(x) := \max\{\text{height}(y),\ y \leq x \text{ and } y \leq a \text{ for some } a \in A\}.$$

Let $\mathfrak{B}$ be a poset matroid on a partially ordered set $\mathbb{P}$, then the rank function of $\mathfrak{B}$ is the non-negative integer valued function on $F(\mathbb{P})$, namely $\rho : \mathrm{F}(\mathbb{P}) \to \mathbb{N}$, such that for every filter $A \in F(\mathbb{P})$

$$\rho(A) = \mathrm{Max}\{|X|,\ X \subseteq A, X \in \mathrm{F}(\mathbb{P}), \text{there exists } B \in \mathfrak{B} \text{ such that } X \subseteq B\},$$

we also called ρ the *rank function associated with* the poset matroid $\mathfrak{B}$. It is easy to see that once a poset matroid on the partially ordered set $\mathbb{P}$ is given, then its rank function is completely determined. Conversely, we can also use the non-negative integer valued function which satisfies certain conditions to define a poset matroid on poset $\mathbb{P}$. The rank for a poset matroid is denoted by $\rho(\mathfrak{B})$.

An *independent set* of a poset matroid $\mathfrak{B}$ on a partially ordered set $\mathbb{P}$ is a filter I such that there exists a basis $B \in \mathfrak{B}$ with $I \subseteq B$. A maximal independent set of a poset matroid is just a basis of that matroid. For a filter F, the family of all the independent sets contained in F is denoted by $\mathfrak{F}$, if $Y \in \{X \in \mathfrak{F} : X$ is a maximal element of $\mathfrak{F}\}$, then we say F *has* a maximal independent subset Y.

Theorem 2.1. (see [2]) *The family $\mathfrak{I}$ of all independent sets of a poset matroid $\mathfrak{B}$ on a partially ordered set $\mathbb{P}$ satisfies the following properties*:

(i.1) $\mathfrak{I} \neq \emptyset$.

(i.2) *If X, Y are filters of $\mathbb{P}$ such that $Y \in \mathfrak{I}$ and $X \subseteq Y$, then $X \in \mathfrak{I}$.*

(i.3) *For every X, $Y \in \mathfrak{I}$ with $|X| < |Y|$, then there exists $y \in \mathrm{Max}(Y - X)$ such that $X \cup \{y\} \in \mathfrak{I}$ (augmentation property).*

If X, $Y \in \mathfrak{I}$ with $X \subseteq Y$, by (i.3), X can be augmented into Y step by step as follows: choose $y_1 \in \mathrm{Max}(Y - X)$, such that $X_1 := X \cup \{y_1\} \in \mathfrak{I}$. After that, we choose $y_2 \in \mathrm{Max}\,(Y - X - \{y_1\})$ such that $X_2 := X_1 \cup \{y_2\} \in \mathfrak{I}$, and so on.

Theorem 2.2. (see [2]) *Let $\mathfrak{I}$ be the family of filters of $\mathbb{P}$ satisfying properties* (i.1), (i.2), (i.3), *then the family*

$$\mathfrak{B} := \{I \in \mathfrak{I},\ I \ \ \textit{is a maximal element of}\ \ \mathfrak{I}\}$$

is a poset matroid.

Theorem 2.3. (see [2]) *A map $\rho : \mathbb{L} \to \mathbb{N}$ is the rank associated with a combinatorial scheme if and only if it satisfies the following conditions*:

(r.1′) $\rho(\mathbf{0}) = 0$.

(r.2′) *For every x, $y \in \mathbb{L}$ such that $x \lessdot y$, we have $\rho(x) \leq \rho(y) \leq \rho(x) + 1$.*

(r.3′) *For every x, $y \in \mathbb{L}$, with $x < y$, $\rho(x) < \rho(y)$ and* $\mathrm{height}(y)$–$\mathrm{height}(x) \leq 2$, *there exists $z \in \mathbb{L}$ such that $x \lessdot z \lessdot y$ and $\rho(z) = \rho(x) + 1$.*

3. Rank axioms for poset matroids. In this section, first, we give the sufficient and necessary conditions for a non-negative integer valued function to be a rank function of a poset matroid, then we study some important properties of the rank function of poset matroids.

From Theorem 2.3, we obtain the following result:

Theorem 3.1. (rank axioms for poset matroids) *A map* $\rho : F(\mathbb{P}) \to \mathbb{N}$ *is the rank function of the poset matroid* $\mathfrak{B}$ *on the partially ordered set* $\mathbb{P}$ *if and only if it satisfies the following conditions*:

(r.1) $\rho(\emptyset) = 0$.

(r.2) *For every* $X, Y \in F(\mathbb{P})$ *such that* $X \subset Y$ *and* $|Y| = |X| + 1$*, we have* $\rho(X) \le \rho(Y) \le \rho(X) + 1$.

(r.3) *For every filters* $X, Y \in F(\mathbb{P})$*, with* $X \subset Y$, $\rho(X) < \rho(Y)$ *and* $|Y| - |X| \le 2$*, there exists filter* $Z \in F(\mathbb{P})$ *such that* $X \subset Z \subset Y$ *is a complete sequence and* $\rho(Z) = \rho(X) + 1$.

The following result characterizes poset matroids by their rank functions in terms of these "local" rank axioms. It will be used in the next section, and we shall prove it with two methods, one is a constructive method, the other is deduced from Theorem 3.1.

Theorem 3.2. (local rank axioms for poset matroids) *A function* $\rho : F(\mathbb{P}) \to \mathbb{N}$ *is the rank function of the poset matroid* $\mathfrak{B}$ *on the partially ordered set* $\mathbb{P}$ *if and only if the following conditions hold*:

(r.4) $\rho(\emptyset) = 0$.

(r.5) $\rho(X) \le \rho(X \cup \{y\}) \le \rho(X) + 1$.

(r.6) *For every filter* $X \in F(\mathbb{P})$*, if* $\rho(X \cup \{y\}) = \rho(X \cup \{z\}) = \rho(X)$ *for* y *and* z *in* $\mathrm{Max}(\mathbb{P}\backslash X)$ *then* $\rho(X \cup \{y\} \cup \{z\}) = \rho(X)$.

(r.7) *For every filter* $X \in F(\mathbb{P})$*, if* $\rho(X \cup \{y\}) = \rho(X)$ *for* $y \in \mathrm{Max}(\mathbb{P}\backslash X)$ *then* $\rho(X \cup \{y\} \cup \{x\}) = \rho(X)$ *for* $x \lessdot y$.

Proof of method 1. Assume ρ is the rank function associated with the poset matroid $\mathfrak{B}$ on the partially ordered set $\mathbb{P}$, by its definition, (r.4) and (r.5) are obvious true. For any filter X of $\mathbb{P}$ and $y,\, z \in \text{Max}(\mathbb{P}-X)$, set $A := X\cup\{y,\, z\}$. If (r.6) doesn't hold, we must have $\rho(A) > \rho(X)$. Assume $\widetilde{X}$ is a maximal independent subset of X, by Theorem 2.1, $\widetilde{X}$ must be augmented into a maximal independent subset of A, namely $\widetilde{X}\cup\{y\} \in \mathfrak{I}$ or $\widetilde{X}\cup\{z\} \in \mathfrak{I}$, however these contradict the assumptions $\rho(X\cup\{y\}) = \rho(X\cup\{z\}) = \rho(X)$, namely $\rho(A) = \rho(X)$.

In the following, we show that (r.7) holds. For all filter X of $\mathbb{P}$, $x \lessdot y \in \text{Max}(\mathbb{P} - X)$, set $A := X \cup \{y,\, x\}$. If (r.7) doesn't hold, we must have $\rho(A) > \rho(X)$. Assume $\widetilde{X}$ is a maximal independent subset of X, by Theorem 2.1, $\widetilde{X}$ can be augmented into a maximal independent subset of A, namely $\widetilde{X} \cup \{y\} \in \mathfrak{I}$ or $\widetilde{X} \cup \{y,\, x\} \in \mathfrak{I}$, both of which contradict the assumptions $\rho(X \cup \{y\}) = \rho(X)$.

Conversely, assume ρ is a non-negative integer valued function satisfying (r.4)-(r.7), we show it is the rank function associated with a poset matroid on the partially ordered set $\mathbb{P}$. We define

$$\mathfrak{I}_\rho = \{X|X \in F(\mathbb{P}),\, \rho(X) = |X|\}. \tag{3.1}$$

If we can show that $\mathfrak{I}_\rho$ satisfies (i.1)-(i.3) of Theorem 2.1, then it is a family of independent set, thus by Theorem 2.2,

$$\mathfrak{B} := \{I \in \mathfrak{I}_\rho,\, I \text{ is a maximal element of } \mathfrak{I}_\rho\}$$

is a poset matroid on partially ordered set of $\mathbb{P}$. Therefore, it is easy to see the non-negative integer valued function ρ is just the rank function associated with the poset matroid $\mathfrak{B}$ that we constructed using ρ.

Obviously $\mathfrak{I}_\rho \neq \emptyset$, namely $\mathfrak{I}_\rho$ satisfies (i.1) of Theorem 2.1. Assume $Y \in \mathfrak{I}_\rho$ and X is a filter of $\mathbb{P}$ with $X \subseteq Y$, we shall show that $X \in \mathfrak{I}_\rho$. If

$X \notin \Im_\rho$, we must have $\rho(X) < |X|$ since $\rho(X) \leq |X|$ for any of our rank functions. Assume $C_1 = \{c_{1_1}, c_{1_2}, \ldots, c_{1_{k_1}}\} = \mathrm{Max}(Y - X)$, by (r.5) we get

$$\rho(X \cup \{c_{1_1}\}) \leq \rho(X) + 1 < |X| + 1.$$

Using (r.5) repeatedly, we get

$$\rho(X \cup C_1) = \rho(X \cup \{c_{1_1}\} \cup \{c_{1_2}\} \cup \cdots \cup \{c_{1_{k_1}}\}) < |X| + k_1.$$

Assume $C_2 = \{c_{2_1}, c_{2_2}, \ldots, c_{2_{k_2}}\} = \mathrm{Max}(Y - X - C_1)$, with the same method as above we get

$$\rho(X \cup C_1 \cup C_2) < |X| + k_1 + k_2.$$

This process will continue until $X \cup C_1 \cup C_2 \cup \cdots \cup C_{k_t} = Y$ and we have

$$\rho(Y) = \rho(X \cup C_1 \cup C_2 \cup \cdots \cup C_{k_t}) < |X| + k_1 + k_2 + \cdots + k_t = |Y|.$$

However, this contradicts the assumption of $Y \in \Im_\rho$, so $\Im_\rho$ satisfies (i.2) of Theorem 2.1.

In the following we shall check that $\Im_\rho$ satisfies (i.3) of Theorem 2.1. Assume $X, Y \in \Im_\rho$ with $|X| < |Y|$, we shall show there exists $z_i \in \{z_1, z_2, \ldots, z_t\} = \mathrm{Max}(Y - X)$ such that $X \cup z_i \in \Im_\rho$, therefore $\Im_\rho$ satisfies (i.3) of Theorem 2.1. We proceed by contradiction. Assume that for all $i \in \{1, 2, \ldots, t\}$, we have $X \cup \{z_i\} \notin \Im_\rho$, namely $\rho(X) = \rho(X \cup \{z_i\}) = |X|$, by (r.6) we get $\rho(X \cup \{z_i\} \cup \{z_j\}) = \rho(X \cup z_i) = \rho(X) = |X|$, for $i, j \in \{1, 2, \ldots, t\}$. We can use (r.6) repeatedly and obtain

$$\begin{aligned} \rho(X) &= \rho(X \cup \{z_{i_1}\}) \\ &= \rho(X \cup \{z_{i_1}, z_{i_2}\}) \\ &\quad \vdots \\ &= \rho(X \cup \{z_{i_1}, z_{i_2}, \ldots, z_{i_{t-2}}\}) \qquad (3.2) \end{aligned}$$

$$
\begin{aligned}
&= \rho(X \cup \{z_{i_1},\, z_{i_2}, \ldots, z_{i_{t-2}},\, z_{i_{t-1}}\}) \\
&= \rho(X \cup \{z_1,\, z_2, \ldots, z_{t-1},\, z_t\}),
\end{aligned}
$$

where $i_1,\, i_2,\, \ldots, i_{t-1} \in \{1, 2, \ldots, t\}$. Set $X' := X \cup \{z_1,\, z_2, \ldots, z_t\}$, if $X' = Y$ then we get

$$\rho(Y) = \rho(X \cup \{z_1,\, z_2, \ldots, z_t\}) = \rho(X) = |X| < |Y|,$$

which contradicts the assumption $Y \in \mathfrak{I}_\rho$. Set $A_i := \{x \in \mathbb{P},\, x \lessdot\, z_i\}$, $i = 1$, 2, …, t. By Eqs (3.2), we have

$$\rho(X \cup \{z_1,\, z_2,\, \ldots, z_t\}) = \rho(X \cup \{z_2,\, z_3, \ldots, z_t\}) = \rho(X),$$

by (r.7) we have

$$\rho(X \cup \{z_1,\, z_2,\, \ldots, z_t\} \cup \{x_1\}) = \rho(X \cup \{z_2,\, z_3, \ldots, z_t\}) = \rho(X),$$

where $x_1 \in A_1$. Using (r.6) repeatedly , we obtain

$$
\begin{aligned}
\rho(X \cup \{z_1,\, z_2,\, \ldots, z_t\} \cup A_1) &= \rho(X \cup \{z_2,\, z_3, \ldots, z_t\}) \\
&\overset{(3.2)}{=} \rho(X \cup \{z_1,\, z_3, \ldots, z_t\}) \\
&= \rho(X).
\end{aligned}
$$

Now by (r.5), we have the following inequalities

$$
\begin{aligned}
\rho(X) &\le \rho(X \cup \{z_1\}) \le \cdots \le \rho(X \cup \{z_1\} \cup A_1) \\
&\le \rho(X \cup \{z_1,\, z_3\} \cup A_1) \\
&\qquad \vdots \\
&\le \rho(X \cup \{z_1,\, z_3,\, \ldots, z_t\} \cup A_1) \\
&\le \rho(X \cup \{z_1,\, z_2,\, z_3,\, \ldots, z_t\} \cup A_1) \\
&= \rho(X).
\end{aligned}
$$

Therefore, we obtain

$$\rho(X \cup \{z_1, z_3, \ldots, z_t\} \cup A_1) = \rho(X \cup \{z_1, z_2, z_3, \ldots, z_t\} \cup A_1) = \rho(X).$$

By (r.7), we have

$$\rho(X \cup \{z_1, z_2, z_3, \ldots, z_t\} \cup A_1 \cup \{x_2\}) = \rho(X), \quad \text{for all } x_2 \in A_2.$$

We can repeat the same argument for A_1 and z_2 to obtain

$$\rho(X \cup \{z_1, z_2, z_3, \ldots, z_t\} \cup A_1 \cup A_2) = \rho(X).$$

By induction, we obtain

$$\rho(X \cup \{z_1, z_2, z_3, \ldots, z_t\} \cup A_1 \cup A_2 \cup \ldots \cup A_t) = \rho(X). \tag{3.3}$$

With the same idea as we prove for Eqs (3.2) and (3.3), repeated using (r.5), (r.6) and (r.7) and by induction, we can obtain

$$\rho(Y) \leq \rho(X \cup Y) = \rho(X) = |X| < |Y|.$$

Obviously, this contracts the assumption that $Y \in \Im_\rho$. Therefore, $\Im_\rho$ satisfies (i.3) of Theorem 2.1, this finishes the proof of our assertion.

Proof of Method 2. The necessity is already given in Method 1. We shall show the sufficiency as following: assume ρ is an non-negative integer valued function satisfying (r.4)-(r.7), we show it is a rank function associated with a poset matroid on the partially ordered set $\mathbb{P}$. If we can show that the function ρ satisfies (r.1)-(r.3) of Theorem 3.1, then our assertion is true. Obviously, it is only necessary for us to check that ρ satisfies (r.3). Let X and Y be filters with $|Y| - |X| = 2$ and $\rho(Y) > \rho(X)$. If for all $y \in \text{Max}(\mathbb{P} \backslash X)$ such that $y \in Y \backslash X$, $\rho(X \cup \{y\}) = \rho(X)$ then by (r.6) and (r.7) we get $\rho(Y) = \rho(X)$. So, there is a $y \in Y \backslash X$ such that $\rho(X \cup \{y\}) > \rho(X)$.

Choose $Z = X \cup \{y\}$, then we obtain a complete sequence $X \subset Z \subset Y$ and $\rho(Z) = \rho(X) + 1$. By Theorem 3.1, ρ is the rank function of a poset matroid on the partially ordered set $\mathbb{P}$.

Corollary 3.3. *Let ρ be a rank function of a poset matroid $\mathfrak{B}$ on the partially ordered set $\mathbb{P}$. For any pair of filters A_1, A_2 of $\mathbb{P}$, if the equation $\rho(A_1 \cup \{e\}) = \rho(A_1)$ holds for every element $e \in A' := \mathrm{Max}(A_2 - A_1)$, then*

$$\rho(A_1 \cup A') = \rho(A_1).$$

Proof. The proof is by the induction on the size of A'. If $|A'| \leq 2$, by (r.6) of Theorem 3.2, the assertion is true. Assume that when $|A'| \leq k$ our assertion is true. We shall prove the assertion is also true when $|A'| = k+1$. Let $A' := A'' \cup \{e_1, e_2\}$, here $e_1, e_2 \in \mathrm{Max}(A' \backslash A'')$. Since $|A'' \cup \{e_1\}| = |A'' \cup \{e_2\}| = k$ and $A'' < k$, induction hypothesis apply to A'', $A'' \cup \{e_1\}$, $A'' \cup \{e_2\}$ we have

$$\begin{aligned} \rho(A_1 \cup A'') &= \rho(A_1), \\ \rho(A_1 \cup A'' \cup \{e_1\}) &= \rho(A_1), \\ \rho(A_1 \cup A'' \cup \{e_2\}) &= \rho(A_1). \end{aligned}$$

By (r.6) of Theorem 3.2, we get

$$\rho(A_1 \cup A'' \cup \{e_1, e_2\}) = \rho(A_1 \cup A'') = \rho(A_1),$$

namely $\rho(A_1 \cup A') = \rho(A_1)$.

In fact we can obtain a more general result as follows:

Proposition 3.4. *Let ρ be a rank function of a poset matroid $\mathfrak{B}$ on the partially ordered set $\mathbb{P}$. Let A, A_1, $A_2, \ldots, A_t$ be filters of the partially ordered set $\mathbb{P}$. If for every A_i of $\{A_1, A_2, \ldots, A_t\}$, $i = 1, 2, \ldots, t$, the equation*

$\rho(A_i \cup A) = \rho(A)$ *holds, then*

$$\rho(A \cup A_1 \cup A_2 \cup \ldots \cup A_t) = \rho(A).$$

Proof. By assumption $\rho(A \cup A_1) = \rho(A)$ and $\rho(A \cup A_2) = \rho(A)$, we shall prove when $t = 2$ that

$$\rho(A \cup A_1 \cup A_2) = \rho(A).$$

In fact we set $\widetilde{A}$ as the maximal independent subset of A, therefore $\rho(A) = \rho(\widetilde{A}) = |\widetilde{A}|$. Since $\widetilde{A} \subseteq A \subseteq A \cup A_1$ and $\rho(A \cup A_1) = \rho(A) = \rho(\widetilde{A})$, we obtain $\widetilde{A}$ is also a maximal independent subset of $A \cup A_1$. Similarly, $\widetilde{A}$ is also a maximal independent subset of $A \cup A_2$, and therefore of $\widetilde{A} \cup A_2$. Thereby there is a common maximal independent subset between $\widetilde{A} \cup A_2$ and $A \cup A_1 \cup A_2$, then we have

$$\rho(A) \leq \rho(A \cup A_1 \cup A_2) = \rho(\widetilde{A} \cup A_2) \leq \rho(A \cup A_2) = \rho(A),$$

therefore, we have $\rho(A \cup A_1 \cup A_2) = \rho(A)$.

Assume that $t \leq k-1$, the assertion is true. We shall prove the assertion is also true when $t = k$. Since $\rho(A \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1}) = \rho(A) = \rho(\widetilde{A}) = |\widetilde{A}|$ and $\widetilde{A} \subseteq A \subseteq A_1 \cup A_2 \cup \cdots \cup A_{k-1}$, we get that $\widetilde{A}$ is also a maximal independent subset of $A \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1}$, by Theorem 2.1, $\widetilde{A} \cup A_k$ and $A \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k$ have a common maximal independent set, therefore we obtain

$$\rho(A) \leq \rho(A \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k) = \rho(\widetilde{A} \cup A_k) \leq \rho(A \cup A_k) = \rho(A),$$

namely

$$\rho(A \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k) = \rho(A).$$

By induction, we have $\rho(A \cup A_1 \cup A_2 \cup \cdots \cup A_t) = \rho(A)$.

Corollary 3.5. *Let ρ be the rank function of a poset matroid $\mathfrak{B}$ on the partially ordered set $\mathbb{P}$. For every A, $A' \in F(\mathbb{P})$, if $A \subseteq A'$, then*

$$0 \leq \rho(A') - \rho(A) \leq |A'| - |A|,$$

the first equality holds if and only if A' and A has a common maximal independent subset; the second equality holds if and only if the cardinality for the symmetric difference of the maximal independent subsets of A' and A is equal to $|A'| - |A|$.

Proof. Let $A_1 := \mathrm{Max}(A' - A) = \{e_{1_1}, e_{1_2}, \cdots, e_{1_n}\}$, by (r.5) of Theorem 3.2, we have

$$\begin{aligned} 0 &\leq \rho(A \cup \{e_{1_1}\}) - \rho(A) \leq 1, \\ 0 &\leq \rho(A \cup \{e_{1_1}, e_{1_2}\}) - \rho(A \cup \{e_{1_1}\}) \leq 1, \\ &\vdots \\ 0 &\leq \rho(A \cup A_1) - \rho(A \cup \{e_{1_1}, e_{1_2}, \ldots, e_{1_{n-1}}\}) \leq 1, \end{aligned}$$

then we let $A_2 := \mathrm{Max}(A' - A - A_1) = \{e_{2_1}, e_{2_2}, \ldots, e_{2_m}\}$, by (r.5) of Theorem 3.2 we have

$$\begin{aligned} 0 &\leq \rho(A \cup A_1 \cup \{e_{2_1}\}) - \rho(A \cup A_1) \leq 1, \\ 0 &\leq \rho(A \cup A_1 \cup \{e_{2_1}, e_{2_2}\}) - \rho(A \cup A_1 \cup \{e_{2_1}\}) \leq 1, \\ &\vdots \\ 0 &\leq \rho(A \cup A_1 \cup A_2) - \rho(A \cup A_1 \cup \{e_{2_1}, e_{2_2}, \ldots, e_{2_{m-1}}\}) \leq 1, \\ &\vdots \end{aligned}$$

At last, we get $A_t := \mathrm{Max}(A' - A - A_1 - A_2 - \cdots - A_{t-1}) = \{e_{t_1}, e_{t_2}, \ldots, e_{t\mu}\} \subseteq \mathrm{Min}(A')$, by (r.5) of Theorem 3.2 we have

$$0 \leq \rho(A \cup A_1 \cup \cdots \cup A_{t-1}\{e_{t_1}\}) - \rho(A \cup A_1 \cup \cdots \cup A_{t-1}) \leq 1,$$

$$0 \leq \rho(A \cup A_1 \cup \cup \cdots \cup A_{t-1}\{e_{t_1}, e_{t_2}\}) - \rho(A \cup A_1 \cup \cup \cdots \cup A_{t-1} \cup \{e_{t_1}\}) \leq 1,$$

$$\vdots$$

$$0 \leq \rho(A') - \rho(A \cup A_1 \cup \cdots \cup A_{t-1} \cup \{e_{t_1}, e_{t_2}, \ldots, e_{t\mu-1}\}) \leq 1.$$

Adding all the inequations, we get $0 \leq \rho(A') - \rho(A) \leq |A' - A|$, however $A \subseteq A'$ implies $|A' - A| = |A'| - |A|$, therefore we get $0 \leq \rho(A') - \rho(A) \leq |A'| - |A|$.

In fact $\rho(A') - \rho(A) = 0$ implies that the maximal independent subsets of A' and A have equal cardinality, together with $A \subseteq A'$, so we conclude that A' and A have a common maximal independent subset. The inverse is obviously true.

Let $\widetilde{A}$ be a maximal independent subset of A, by Theorem 2.1, $\widetilde{A}$ can be augmented into a maximal independent subset of A', denoted by $\widetilde{A'}$, therefore $\widetilde{A}\Delta\widetilde{A'} = \widetilde{A'} - \widetilde{A}$, together with $\widetilde{A} \subseteq \widetilde{A'}$, we get $|\widetilde{A}\Delta\widetilde{A'}| = |\widetilde{A'}| - |\widetilde{A}|$. Therefore,

$$\rho(A') - \rho(A) = |A'| - |A|,$$

namely,

$$|\widetilde{A'}| - |\widetilde{A}| = |A'| - |A|,$$

if and only if

$$|\widetilde{A}\Delta\widetilde{A'}| = |A'| - |A|.$$

So we finish the proof of our assertion.

4. Combinatorial schemes. We now give an equivalent definition of poset matroids that uses the language of distributive lattice, which was first put forward by Banabei et al. in [2]. Let Inc($\mathbb{P}$) be the distributive lattice of all filters of the partially ordered set $\mathbb{P}$, ordered by inclusion.

A poset matroid $\mathfrak{B}$ on partially ordered set $\mathbb{P}$ can be seen to be a nonempty antichain A of the distributive lattice $\mathrm{Inc}(\mathbb{P})$ satisfying the following properties:

(a.1) For every $a_1, a_2 \in A$ and for every $x, y \in \mathrm{Inc}(\mathbb{P})$, $x \le a_1$, $a_2 \le y$, $x \le y$, there exists $a \in A$ such that $x \le a \le y$.

Conversely, any nonempty antichain A of a finite distributive lattice $\mathbb{L}$ that satisfies (a.1) is the lattice counterpart of a poset matroid.

The previous considerations lead to the following definition: a nonempty antichain A of a distributive lattice $\mathbb{L}$ that satisfies the property:

(a.1) For every $a_1, a_2 \in A$ and for every $x, y \in \mathrm{Inc}(\mathbb{P})$, $x \le a_1$, $a_2 \le y$, $x \le y$, there exists $a \in A$ such that $x \le a \le y$ (*middle property*) will be called a *combinatorial scheme*. By abuse of language, the elements of a combinatorial scheme will also be called *bases*.

From Theorem 3.2, we obtain

Theorem 4.1. (local rank axioms for combinatorial schemes) *A function $\rho : \mathbb{L} \to \mathbb{N}$ is the rank function associated with a combinatorial scheme A in the distributive lattice $\mathbb{L}$ if and only if the following conditions hold*:

(r.4′) $\rho(\mathbf{0}) = 0$.

(r.5′) *For every $x, y \in \mathbb{L}$ with $x \lessdot y$, we have $\rho(x) \le \rho(y) \le \rho(x) + 1$.*

(r.6′) *For every $x, y, z \in \mathbb{L}$ with $x \lessdot y$, $x \lessdot z$, if $\rho(y) = \rho(z) = \rho(x)$, then $\rho(y \vee z) = \rho(x)$.*

(r.7′) *For every $x, y, z \in \mathbb{L}$ with $x \lessdot y \lessdot z$, if $\rho(y) = \rho(x)$, then $\rho(z) = \rho(x)$.*

From Corollary 3.3 and Proposition 3.4, we get

Corollary 4.2. *Let $\rho : \mathbb{L} \to \mathbb{N}$ be the rank function associated with a combinatorial scheme A in the distributive lattice $\mathbb{L}$, if for every $x, y_1, y_2,$*

$\ldots, y_t \in \mathbb{L}$ *with* $x \lessdot y_i$ *and* $\rho(x \vee y_i) = \rho(x)$, *for* $i = 1, \ldots, t$, *then*

$$\rho(x \vee y_1 \vee y_2 \vee \ldots \vee y_t) = \rho(x).$$

Proposition 4.3. *Let* $\rho : \mathbb{L} \to \mathbb{N}$ *be the rank function associated with a combinatorial scheme* A *in the distributive lattice* $\mathbb{L}$, $x, y_1, y_2, \ldots, y_t \in \mathbb{L}$, *if* $\rho(x \vee y_i) = \rho(x)$, *for* $i = 1, \ldots, t$, *then*

$$\rho(x \vee y_1 \vee y_2 \vee \ldots \vee y_t) = \rho(x).$$

From Corollary 3.5, we obtain

Corollary 4.4. *Let* $\rho : \mathbb{L} \to \mathbb{N}$ *be the rank function associated with a combinatorial scheme* A *in the distributive lattice* $\mathbb{L}$. *For every* $x, y \in \mathbb{L}$, *if* $x \leq x'$, *then*

$$0 \leq \rho(x') - \rho(x) \leq \text{height}(x') - \text{height}(x).$$

Up to this now, it is just as G.-C. Rota said (see [5]), "it may well be the case that some of the problems of today's matroid theory will reveal their secrets only when look at in this new and more proper setting".

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References

1. M. Barnabei, G. Nicoletti, L. Pezzoli, *Symmetric property for poset matroids,* Adv. Math., **102**(1993), 230-239.

2. M. Barnabei, G. Nicoletti and L. Pezzoli, *Matroids on partially ordered sets,* Adv. in Appl. Math., **21**(1998), 78-112.

3. D. J. A. Welsh, *Matroid Theory,* Academic Press, London, 1976.

4. G. D. Birkhoff, *Lattice Theory.* 3rd Ed., Colloquium Publication 25, American Mathematical Society, Providence, RI, 1967.

5. G.-C. Rota, *Report on the present state of combinatorics,* Discrete Math. **153**(1996), 289-303.

6. Li Shu-chao, *Global rank axioms for poset matroids,* accepted by Acta Mathematic Sinica.

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