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ON INTEGRAL MEANS FOR FRACTIONAL CALCULUS OF ANALYTIC FUNCTIONS

BY

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Abstract. Integral means inequalities are obtained for the fractional derivatives and the fractional integrals of analytic functions. Relevant connections with the integral means inequalities for fractional derivatives of univalent functions with negative coefficients are also pointed out.

1. Introduction. Let A denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are *analytic* in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let A_n denote the subclass of A consisting of all functions $f(z)$ of the

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form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (a_{n+1} \neq 0,\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

and let $g(z)$ be given by

$$g(z) = z + b_j z^j \quad (b_j \neq 0;\ j \geqq n+1). \tag{1.2}$$

We need the following subclasses of analytic functions with negative coefficients to apply our results for univalent functions with negative coefficients.

Let $A(n)$ denote the subclass of A consisting of all functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0,\ n \in \mathbb{N}). \tag{1.3}$$

$A(n)$ is said to be the subclass of analytic functions with negative coefficients. We denote by $T(n)$ the subclass of $A(n)$ of functions which are *univalent* in U, and by $T_\alpha(n)$ and $C_\alpha(n)$ the subclasses of $T(n)$ consisting of functions which are *starlike of order* $\alpha (0 \leqq \alpha < 1)$ and *convex of order* $\alpha (0 \leqq \alpha < 1)$ in U, respectively. The classes $A(n), T(n), T_\alpha(n)$ and $C_\alpha(n)$ were investigated by Chatterjea [1] (and Srivastava *et al.* [10]).

In particular, the subclasses:

$$T := T(1), \quad T^*(\alpha) := T_\alpha(1), \quad \text{and} \quad C(\alpha) := C_\alpha(1)$$

were considered earlier by Silverman [8]. Chatterjea [1] showed the following coefficient inequalities.

Lemma 1. *A function $f(z)\in A(n)$ is in the class $T_\alpha(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1).$$

Lemma 2. *A function $f(z)\in A(n)$ is in the class $C_\alpha(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1).$$

In this paper, we investigate the integral means inequalities for the fractional derivatives and for the fractional integrals of *normalized* analytic functions. Further, we show that our results hold true for the integral means inequalities for the fractional derivatives of univalent functions with negative coefficients (see, Kim and Choi [2]).

We shall make use of the following definitions of fractional calculus (that is, fractional integrals and fractional derivatives) (*cf.* Owa [4]; see also Srivastava and Owa [9]).

Definition 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad (\lambda > 0),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta \qquad (0 \leqq \lambda < 1),$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1, above.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \qquad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup 0)\,.$$

It readily follows from Definitions 1 and 2 that

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)} z^{k+\lambda} \qquad (\lambda > 0) \tag{1.4}$$

and

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \qquad (0 \leqq \lambda < 1), \tag{1.5}$$

respectively.

Further, we need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in U, the function $f(z)$ is said to be subordinate to $g(z)$ in U if there exists a function $w(z)$ analytic in U with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in U),$$

such that

$$f(z) = g(w(z)) \quad (z \in U).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Theorem 1 (Littlewood [3]). *If the functions $f(z)$ and $g(z)$ are analytic in U with*

$$g(z) \prec f(z),$$

then

$$\int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta \leqq \int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \quad (\mu > 0;\ 0 < r < 1).$$

2. Integral means inequalities. First we prove the following integral means inequalities for the fractional derivatives.

Theorem 2. *Let $f(z) \in A_n$ and $g(z)$ be given by (1.2), and supposed that*

$$\begin{aligned} &\sum_{k=n+1}^{\infty} (k-p)_{p+1}|a_k| \\ (2.1)\qquad \leqq\ &\frac{\Gamma(2-\nu-p)\Gamma(j+1)\Gamma(n+2-\lambda-p)}{\Gamma(2-\lambda-p)\Gamma(j+1-\nu-p)\Gamma(n+1-p)}|b_j| \quad (j \geqq n+1;\ n \in \mathbb{N}) \end{aligned}$$

for $p = 0$ or $1 (0 \leqq \lambda,\ \nu < 1)$ and $2 \leqq p \leqq n (0 < \lambda,\ \nu < 1)$, where $(k-p)_{p+1}$ denotes the Pochhammer symbol defined by $(k-p)_{p+1} = (k-p)(k-p+1)\cdots k$. Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{p+\lambda} f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|\frac{\Gamma(2-\nu-p)}{\Gamma(2-\lambda-p)} z^{\nu-\lambda} D_z^{p+\nu} g\,(z)\right|^\mu d\theta \quad (\mu>0). \tag{2.2}$$

Proof. By means of the fractional derivative formula (1.5) and Definition 3, we find from (1.1) that

$$\begin{aligned} D_z^{p+\lambda} f(z) &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} a_k z^{k-1}\right) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1 + \sum_{k=n+1}^{\infty} (k-p)_{p+1}\Gamma(2-\lambda-p)\Phi(k) a_k z^{k-1}\right) \end{aligned}$$

where

$$\Phi(k) := \frac{\Gamma(k-p)}{\Gamma(k+1-\lambda-p)} \left(\begin{cases} p = 0 \text{ or } 1 \ (0 \leqq \lambda < 1) \\ 2 \leqq p \leqq n (0 < \lambda < 1) \end{cases} ; \quad k \geqq n+1; \quad n \in \mathbb{N} \right).$$

Since $\Phi(k)$ is a decreasing function of k, we have

$$0 < \Phi(k) \leqq \Phi(n+1) = \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}$$

$$\left(\begin{cases} p = 0 \textit{ or } 1 \ (0 \leqq \lambda < 1) \\ 2 \leqq p \leqq n \ (0 < \lambda < 1) \end{cases} ; \quad k \geqq n+1; \quad n \in \mathbb{N} \right).$$

Similarly, by using (1.2), (1.5) and Definition 3, we obtain

$$D_z^{p+\nu} g(z) = \frac{z^{1-\nu-p}}{\Gamma(2-\nu-p)} \left(1 + \frac{\Gamma(2-\nu-p)\Gamma(j+1)}{\Gamma(j+1-\nu-p)} b_j z^{j-1} \right)$$

$$\left(\begin{cases} p = 0 \textit{ or } 1 \ (0 \leqq \nu < 1) \\ 2 \leqq p \leqq n \ (0 < \nu < 1) \end{cases} ; \quad k \geqq n+1; \quad n \in \mathbb{N} \right).$$

Thus we have

$$\frac{\Gamma(2-\nu-p)}{\Gamma(2-\lambda-p)} z^{\nu-\lambda} D_z^{p+\nu} g(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 + \frac{\Gamma(2-\nu-p)\Gamma(j+1)}{\Gamma(j+1-\nu-p)} b_j z^{j-1} \right).$$

For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left| 1 + \sum_{k=n+1}^{\infty} (k-p)_{p+1} \Gamma(2-\lambda-p) \Phi(k) a_k z^{k-1} \right|^{\mu} d\theta$$

$$\leqq \int_0^{2\pi} \left| 1 + \frac{\Gamma(2-\nu-p)\Gamma(j+1)}{\Gamma(j+1-\nu-p)} b_j z^{j-1} \right|^{\mu} d\theta \qquad (\mu > 0).$$

By applying Theorem 1 [3], it would suffice to show that

$$1 + \sum_{k=n+1}^{\infty} (k-p)_{p+1} \Gamma(2-\lambda-p) \Phi(k) a_k z^{k-1} \prec 1 + \frac{\Gamma(2-\nu-p)\Gamma(j+1)}{\Gamma(j+1-\nu-p)} b_j z^{j-1}. \tag{2.3}$$

By setting

$$1+\sum_{k=n+1}^{\infty}(k-p)_{p+1}\Gamma(2-\lambda-p)\Phi(k)a_k z^{k-1}=1+\frac{\Gamma(2-\nu-p)\Gamma(j+1)}{\Gamma(j+1-\nu-p)}b_j\{w(z)\}^{j-1},$$

we find that

$$\{w(z)\}^{j-1}=\frac{\Gamma(2-\lambda-p)\Gamma(j+1-\nu-p)}{\Gamma(2-\nu-p)\Gamma(j+1)b_j}\cdot\sum_{k=n+1}^{\infty}(k-p)_{p+1}\Phi(k)a_k z^{k-1},$$

which readily yields $w(0)=0$.

Therefore, we have

$$\begin{aligned}|w(z)|^{j-1} &\leqq \frac{\Gamma(2-\lambda-p)\Gamma(j+1-\nu-p)}{|b_j|\Gamma(2-\nu-p)\Gamma(j+1)}\sum_{k=n+1}^{\infty}(k-p)_{p+1}\Phi(k)|a_k||z|^{k-1}\\ &\leqq |z|^n\frac{\Gamma(2-\lambda-p)\Gamma(j+1-\nu-p)}{|b_j|\Gamma(2-\nu-p)\Gamma(j+1)}\cdot\Phi(n+1)\sum_{k=n+1}^{\infty}(k-p)_{p+1}|a_k|\\ &= |z|^n\frac{\Gamma(2-\lambda-p)\Gamma(j+1-\nu-p)}{|b_j|\Gamma(2-\nu-p)\Gamma(j+1)}\cdot\frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}\sum_{k=n+1}^{\infty}(k-p)_{p+1}|a_k|\\ &\leqq |z|^n<1,\end{aligned}$$

by means of the hypothesis (2.1) of Theorem 2.

In light of the last inequality above, we have the subordination (2.3), which evidently proves Theorem 2.

Remark 1. By applying the Hölder inequality to the right hand side of the inequality (2.2) of the theorem 2, we have the following:

$$\begin{aligned}\int_0^{2\pi}|F(z)|^{\mu}\,d\theta &\leqq \left(\int_0^{2\pi}|F(z)|^{\mu\cdot\frac{2}{\mu}}\,d\theta\right)^{\frac{\mu}{2}}\left(\int_0^{2\pi}1^{\frac{2}{2-\mu}}\,d\theta\right)^{\frac{2-\mu}{2}}\\ &= \left(\int_0^{2\pi}|F(z)|^2\,d\theta\right)^{\frac{\mu}{2}}\cdot(2\pi)^{\frac{2-\mu}{2}},\end{aligned}$$

where $0 < \mu < 2$ and

$$F(z) = \frac{\Gamma(2-\nu-p)}{\Gamma(2-\lambda-p)} z^{\nu-\lambda} D_z^{p+\nu} g(z).$$

This remark is applicable to the following results.

Next we have the following integral means inequalities for the fractional integrals.

Theorem 3. *Let $f(z) \in A_n$ and $g(z)$ be given by (1.2), and supposed that*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{\Gamma(2+\nu)\Gamma(j+1)\Gamma(n+2+\lambda)}{\Gamma(2+\lambda)\Gamma(j+1+\nu)\Gamma(n+1)} |b_j| \quad (j \geqq n+1;\ n \in \mathbb{N})$$

for λ and $\nu > 0$. Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^{-\lambda} f(z) \right|^{\mu} d\theta \leqq \int_0^{2\pi} \left| \frac{\Gamma(2+\nu)}{\Gamma(2+\lambda)} z^{\lambda-\nu} D_z^{-\nu} g(z) \right|^{\mu} d\theta \quad (\mu > 0).$$

Proof. By virtue of the fractional integral formula (1.4) and the fractional derivative formula (1.5), replacing λ with $-\lambda(\lambda > 0)$ and ν with $-\nu(\nu > 0)$, and putting $p = 0$ in Theorem 2, we complete the proof.

When $\lambda = \nu$, Theorem 2 readily yield

Corollary 1. *Let $f(z) \in A_n$ and $g(z)$ be given by (1.2), and supposed that*

$$\text{(2.4)} \quad \sum_{k=n+1}^{\infty} (k-p)_{p+1} |a_k| \leqq \frac{\Gamma(j+1)\Gamma(n+2-\lambda-p)}{\Gamma(j+1-\lambda-p)\Gamma(n+1-p)} |b_j| \quad (j \geqq n+1;\ n \in \mathbb{N})$$

for $0 \leqq p \leqq n$ and $0 \leqq \lambda < 1$, where $(k-p)_{p+1}$ denotes the Pochhammer symbol defined by $(k-p)_{p+1} = (k-p)(k-p+1)\cdots k$. Then, for $z = re^{i\theta}$

and $0<r<1$,

$$\int_0^{2\pi}\left|D_z^{p+\lambda}f(z)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|D_z^{p+\lambda}g(z)\right|^{\mu}d\theta \quad (\mu>0).$$

Also, from Theorem 3, when $\lambda=\nu$, we have the following:

Corollary 2. *Let* $f(z)\in A_n$ *and* $g(z)$ *be given by* (1.2), *and supposed that*

$$\sum_{k=n+1}^{\infty}k|a_k| \leqq \frac{\Gamma(j+1)\Gamma(n+2+\lambda)}{\Gamma(j+1+\lambda)\Gamma(n+1)}|b_j| \quad (j\geqq n+1;\ n\in\mathbb{N})$$

for $\lambda>0$. *Then, for* $z=re^{i\theta}$ *and* $0<r<1$,

$$\int_0^{2\pi}\left|D_z^{-\lambda}f(z)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|D_z^{-\lambda}g(z)\right|^{\mu}d\theta \quad (\mu>0).$$

As the special case $p=0$, Corollary 1 readily yields

Corollary 3. *Let* $f(z)\in A_n$ *and* $g(z)$ *be given by* (1.2) *and suppose that*

$$\sum_{k=n+1}^{\infty}k|a_k|\leqq\frac{\Gamma(j+1)\Gamma(n+2-\lambda)}{\Gamma(j+1-\lambda)\Gamma(n+1)}|b_j| \quad (0\leqq\lambda<1;\ j\geqq n+1;\ \in\mathbb{N}). \tag{2.5}$$

Then, for $z=re^{i\theta}$ *and* $0<r<1$,

$$\int_0^{2\pi}\left|D_z^{\lambda}f(z)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|D_z^{\lambda}g(z)\right|^{\mu}d\theta \qquad (\mu>0)\,.$$

Also, from Corollary 1, when $p=1$, we have the following:

Corollary 4. *Let* $f(z)\in A_n$ *and* $g(z)$ *be given by* (1.2) *and suppose that*

$$\sum_{k=n+1}^{\infty}k(n-1)|a_k|\leqq\frac{\Gamma(j+1)\Gamma(n+1-\lambda)}{\Gamma(j-\lambda)\Gamma(n)}|b_j| \quad (0\leqq\lambda<1;\ j\geqq n+1;\ n\in\mathbb{N}). \tag{2.6}$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^{1+\lambda} f(z) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| D_z^{1+\lambda} g(z) \right|^\mu d\theta \qquad (\mu > 0).$$

3. Applications. In Corollary 3, if $f(z) \in T_\alpha(n)$ and $g(z)$ is given by

$$g(z) = z - \frac{1}{j} z^j, \text{ that is, } b_j = -\frac{1}{j},$$

since

$$\frac{\Gamma(j+1)\Gamma(n+2-\lambda)}{\Gamma(j+1-\lambda)\Gamma(n+1)} \leqq j \quad (n \in \mathbb{N};\ 0 \leqq \lambda < 1;\ j \geqq n+1),$$

we have the coefficient inequality

$$\sum_{k=n+1}^{\infty} k a_k \leqq 1 \tag{3.1}$$

as the hypothesis (2.5) of Corollary 3. In fact, the coefficient inequality (3.1) holds true for the functions $f(z) \in T_\alpha(n)$, because by Lemma 1 [1],

$$\sum_{k=n+1}^{\infty} k a_k \leqq \sum_{k=n+1}^{\infty} \frac{k-\alpha}{1-\alpha} a_k \leqq 1.$$

Thus, we have the following:

Corollary 5. *Let* $f(z) \in T_\alpha(n)$ *and* $g(z)$ *be given by*

$$g(z) = z - \frac{1}{j} z^j \quad (j \geqq n+1;\ n \in \mathbb{N}).$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^{\lambda} f(z) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| D_z^{\lambda} g(z) \right|^\mu d\theta \qquad (0 \leqq \lambda < 1;\ \mu > 0). \tag{3.2}$$

Also, in Corollary 4, if $f(z) \in C_\alpha(n)$ and $g(z)$ is given by

$$g(z) = z - \frac{1}{j(j-1)} z^j, \text{ that is, } b_j = -\frac{1}{j(j-1)},$$

since

$$\frac{\Gamma(j+1)\Gamma(n+1-\lambda)}{\Gamma(j-\lambda)\Gamma(n)} \leqq j(j-1) \quad (0 \leqq \lambda < 1;\ j \geqq n+1;\ n \in \mathbb{N})$$

we have the coefficient inequality

$$\sum_{k=n+1}^{\infty} k(k-1)a_k \leqq 1 \tag{3.3}$$

as the hypothesis (2.6) of Corollary 4. By means of Lemma 2 [1], we have

$$\sum_{k=n+1}^{\infty} k(k-1)a_k \leqq \sum_{k=n+1}^{\infty} \frac{k(k-\alpha)}{1-\alpha} a_k \leqq 1.$$

Thus, the coefficient inequality (3.2) holds true for the function $f(z)$ of $C_\alpha(n)$. Hence, we have the following:

Corollary 6. *Let $f(z) \in C_\alpha(n)$ and $g(z)$ be given by*

$$g(z) = z - \frac{1}{j(j-1)} z^j \quad (j \geqq n+1;\ n \in \mathbb{N}).$$

Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^{1+\lambda} f(z) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| D_z^{1+\lambda} g(z) \right|^\mu d\theta \qquad (0 \leqq \lambda < 1; \quad \mu > 0)\,.$$

For $n = 1$, $j = 2$ and $\alpha = 0$, Corollary 5 and Corollary 6 are reduced to the integral means inequalities of Kim and Choi [2, Theorem 1, (i)] and [2, Theorem 2, (i)], respectively.

Remark 2. In Corollary 5, when $\lambda = 0$, by virtue of Remark 1, we have the following estimation for the right hand side of (3.2).

$$\int_0^{2\pi} \left| z - \frac{1}{j} z^j \right|^\mu d\theta \leqq \left(\frac{j^2+1}{j^2} \right)^{\frac{\mu}{2}} \cdot 2\pi < \frac{5}{2}\pi \quad (r \longrightarrow 1;\ 0 < \mu < 2).$$

In particular, when $j = 2$ and $\mu = 1$ above, we have

$$\int_0^{2\pi} \left| z - \frac{z^2}{2} \right| d\theta \leqq \sqrt{5}\pi = 7.024814 \cdots .$$

On the other hand, using *Mathematica*, we have the following:

$$\int_0^{2\pi} \left| z - \frac{z^2}{2} \right| d\theta \leqq \int_0^{2\pi} \sqrt{\frac{5}{4} - \cos\theta}\ d\theta = 6.068245 \quad (r \longrightarrow 1).$$

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