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IMMERSIONS OF LOCALLY COMPACT CONNECTED GROUPS

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Abstract. In this paper, we study locally compact connected groups. We prove several results regarding maximal compact connected abelian subgroups in a locally compact connected group which are analogous to known results regarding maximal tori in an analytic group. Also, the local decomposition of locally compact connected groups will be investigated. Finally, we extend theorems of Goto by proving three immersion theorems for locally compact connected groups.

0. Introduction. Let G and H be topological groups. A homomorphism $f : G \to H$ is said to be an immersion if f is one-to-one and continuous. If the image $f(G)$ is dense in H, then f is called a dense immersion. Dense immersions have been studied extensively by M. Goto (see [2, 3]). In this paper we extend three of Goto's immersion theorems. The first three sections of this paper will lay the groundwork for extending these theorems.

Many results are known regarding maximal tori in an analytic group. In section 1, we will prove analogous statements regarding maximal compact connected abelian subgroups (maximal pro-tori) in a locally compact connected group. For instance, it will be shown that all maximal pro-tori are conjugate to each other. In section 2, we study Iwasawa's local decomposition of locally compact connected groups (see [6, Theorem 11]).

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In [3], Goto defined a Z-decomposition of an analytic group H: $H = VN$, where V is a closed vector subgroup, N is a closed connected normal subgroup, and $V \cap N = 1$ [3, (6.2)]. Also, Goto [3] defined a generalized maximal torus (gm-torus) of an analytic group H: if $\mathcal{L}(H)$ denotes the Lie algebra of H and if $\alpha : H \to \text{Aut } \mathcal{L}(H)$ is the adjoint representation of H, then for a maximal torus T in $\overline{\alpha(H)}$, the group $B = \alpha^{-1}(T \cap \alpha(H))$ is called a generalized maximal torus of H. Then B is a closed connected abelian subgroup of H containing the center of H and a maximal torus of H [3, (7.2)].

In section 3, we define a Z-decomposition of a locally compact connected group G and a generalized maximal pro-torus (gm-pro-torus) of G. We also prove some basic properties of a Z-decomposition of G and of generalized maximal pro-tori of G. These properties are similar to the properties of a Z-decomposition of an analytic group and of gm-tori of an analytic group. For instance, it will be shown that a gm-pro-torus of G is a closed connected abelian subgroup of G containing the center of G and a maximal pro-torus of G.

In section 4, we extend the subsequent theorems of Goto [3] to a locally compact connected group G. The preparation work for extending these theorems will be discussed in sections 1 through 3.

Theorem A. *Let G be an analytic group, let H be a gm-torus of G, and let $v(H)$ be a vector part of H. Let $G = VN$ be a Z-decomposition of G. If M is a topological group and $f : G \to M$ is a dense immersion, then $M = \overline{f(v(H))}f(N)$.*

Theorem B. *Let G be an analytic group, let M be a topological group, and let $G = G^1 \supseteq G^2 \supseteq G^3 \cdots$ and $M = M^1 \supseteq M^2 \supseteq M^3 \cdots$ be the descending central series of G and M, respectively. If $f : G \to M$ is a dense immersion, then $f(G)^j = M^j$ for $j = 2, 3, \ldots$.*

Theorem C. *Let G be an analytic group with a gm-torus H. Let M be a topological group and $f : G \to M$ an immersion. If $\overline{f(v(H))}$ is locally compact, then $\overline{f(G)}$ is locally compact.*

Notation. We restrict our attention mostly to locally compact connected groups unless otherwise stated. Throughout this paper, all topological groups will be assumed to be Hausdorff. If G is a topological group, we use G^0 to denote its identity component, $Z(G)$ to denote its center, and 1 to denote the identity element of G. If A and B are subgroups of G, let $[A, B]$ denote the subgroup of G generated by $\{aba^{-1}b^{-1} : a \in A \text{ and } b \in B\}$. If K is a subset of G, its closure will be denoted by $\overline{K}$. If G is a locally compact connected group, Aut G will denote the group of all topological group automorphisms of G endowed with the Birkhoff topology. If $x \in G$, we denote the inner automorphism of G that is determined by x by I_x. If H is a subgroup of G, we denote the subgroup $\{I_x : x \in H\}$ of Aut G by Int H.

1. Maximal pro-tori.

Many results are known regarding maximal tori in a (compact) analytic group; see for example, Hochschild [4, Chapter 13] and Goto [3]. In this section, we will prove analogous results regarding maximal compact connected abelian subgroups in a locally compact connected group. We begin by recalling that every locally compact connected group can be approximated by Lie groups.

Definition 1.1. Let Γ be a set directed by a partial ordering $\leq$. For each $\alpha \in \Gamma$, let G_α be a topological group. Suppose that for all $\alpha, \beta \in \Gamma$, with $\alpha < \beta (\alpha \leq \beta$ and $\alpha \neq \beta)$, there is an open continuous homomorphism $\phi_{\beta\alpha}$ of G_β into G_α. Also, suppose that if $\alpha < \beta < \gamma$, then $\phi_{\gamma\alpha} = \phi_{\beta\alpha} \circ \phi_{\gamma\beta}$. The triple $(\Gamma, \{G_\alpha\}, \{\phi_{\beta\alpha}\})$ is called an *inverse mapping system*. Let $H =$

$\prod_{\alpha\in\Gamma} G_\alpha$ and let $G = \lim_{\leftarrow} G_\alpha = \{(x_\alpha) \in H : x_\alpha = \phi_{\beta\alpha}(x_\beta), \alpha < \beta\}$. Then G is called the *projective limit* of the inverse mapping system.

Theorem 1.2. [1, III, 7.3, Proposition 2] *Let $G_\alpha, \alpha \in \Gamma$, be a collection of closed normal subgroups of the topological group G. Suppose the following conditions hold*:

(a) *Every neighborhood of $1 \in G$ contains a G_α,*

(b) *Given α, $\beta \in \Gamma$, there is a $\gamma \in \Gamma$ such that $G_\gamma \subseteq G_\alpha \cap G_\beta$, and*

(c) *One of the G_α is compact.*

Then Γ is a directed set with partial order $\alpha \le \beta$ if and only if $G_\alpha \supseteq G_\beta$, and $G = \lim_{\leftarrow} G/G_\alpha$, the projective limit of the G/G_α.

Theorem 1.3. [7, Section 4.6] *Let G be a locally compact connected group. Let U be an arbitrary neighborhood of 1. Then there is a compact normal subgroup $K \subseteq G$ such that G/K is a Lie group and $K \subseteq U$.*

Let G be a locally compact connected group. Using Theorems 1.2 and 1.3, we can write $G = \lim_{\leftarrow} G/K_\alpha$, $\alpha \in \Gamma$, where $\{K_\alpha\}$ is a collection of compact normal subgroups of G such that G/K_α is a Lie group. We call any such collection of compact normal subgroups $\{K_\alpha\}$ a *canonical system of normal subgroups* in G.

We first show the existence of maximal compact connected abelian subgroups in locally compact connected groups.

Proposition 1.4. *Every locally compact connected group G has maximal compact connected abelian subgroups.*

Proof. Let Λ be the set of all compact connected abelian subgroups of G. Then $\Lambda \neq \phi$, since $\{1\} \in \Lambda$. Partially order Λ by set theoretic inclusion. Let $\mathcal{C} = \{C_i : i \in I\}$ be a chain in Λ. We want to show that $\mathcal{C}$ has an upper bound in Λ. Consider $C = \bigcup_{i\in I} C_i$: then C is a connected abelian group.

Therefore $\overline{C}$ is a locally compact connected abelian group. Thus we can write $\overline{C} = V \times M$, where V is a vector group and M is a compact connected abelian group. For each $i \in I$, C_i is contained in M; thus V is trivial and $\overline{C} = M$ is a compact connected abelian group. Therefore $\mathcal{C}$ has an upper bound in Λ. By Zorn's Lemma, Λ contains a maximal element.

Now we show how to construct maximal compact connected abelian subgroups in a compact connected group.

Let K be a compact connected group. Then $K = Z(K)^\circ S$, where S is the (almost-direct) product of a family of compact simple analytic subgroups [8, Section 25]. Write $S = \prod_{\lambda \in \Delta} S_\lambda$, where S_λ is a compact simple analytic group, $S_\lambda \cap S_\mu$ is a discrete (finite) central subgroup, and $[S_\lambda, S_\mu] = 1$. For each $\lambda \in \Delta$, let T_λ be a maximal torus in S_λ. Let $A = Z(K)^\circ \prod T_\lambda$. We claim that A is a maximal compact connected abelian subgroup of K. Actually, we will show that A is a maximal abelian subgroup of K. Let $A \subseteq B$, where B is a maximal abelian subgroup of K. Let Z_λ denote the center of S_λ. Since every maximal torus T_λ contains $Z_\lambda, A \supseteq Z(K)$. We now consider $K/Z(K) \supseteq B/Z(K) \supseteq A/Z(K)$. Then $K/Z(K) = \prod S_\lambda/Z_\lambda$ (direct product) $\supseteq B/Z(K) \supseteq A/Z(K) = \prod T_\lambda/Z_\lambda$. Since each T_λ/Z_λ is a maximal abelian subgroup of S_λ/Z_λ, we have $B/Z(K) = A/Z(K)$, which implies that $B = A$.

The following assertion is evident.

Proposition 1.5. *Let K be a compact connected group.*

(i) *All maximal compact connected abelian subgroups of K are conjugate to each other.*

(ii) *Every maximal compact connected abelian subgroup of K contains the center of K.*

(iii) *Any element in K is contained in some maximal compact connected abelian subgroup of K.*

Let A be a maximal compact connected abelian subgroup of K.

(iv) *A is a maximal abelian subgroup of K.*

(v) *A is its own centralizer in K.*

Corollary 1.6. *Let G be a locally compact connected group.*

(i) *All maximal compact connected abelian subgroups of G are conjugate to each other.*

(ii) *Every maximal compact connected abelian subgroup of G is a maximal compact abelian subgroup of G.*

Definition 1.7. Let G be a locally compact connected group. We shall call a compact connected abelian subgroup of G a *pro-torus* of G.

Proposition 1.8. *Let G be a locally compact connected group. Let A be a maximal pro-torus of G. Then the space G/A is simply connected.*

Proof. Let K be a maximal compact connected subgroup of G containing A. Then the space of G is homeomorphic with $E \times K$, where E is homeomorphic to a finite-dimensional Euclidean space. Now $K = Z(K)^0 S$, where $S = \prod_{\lambda \in \Delta} S_\lambda$ is the (almost-direct) product of a family of compact simple analytic subgroups. Then $A = Z(K)^0 \prod T_\lambda$, where T_λ is a maximal torus in S_λ. Then $G/A = E \times K/A$, where $K/A = \prod S_\lambda / T_\lambda$, the direct product of simply connected spaces. Hence G/A is simply connected.

Lemma 1.9. *Let A be a maximal pro-torus of a locally compact connected group G. Let K be a compact normal subgroup of G such that G/K is a Lie group. Then the image of A is a maximal torus of G/K.*

Proof. Let π denote the natural homomorphism $G \to G/K$. Then $\pi(A)$ is a torus. Let T be a maximal torus of G/K which contains $\pi(A)$. Let $F = \pi^{-1}(T)$. Then $A \subseteq F^\circ \subseteq F$, and $F^\circ K = F$. Let $D = F^\circ \cap K$. Then $F^\circ/D = F^\circ/(F^\circ \cap K) \cong F/K \cong T$. Now $F^0 = Z(F)^0 S$, where S is the (almost-direct) product of a family of compact simple analytic subgroups.

Since F^0/D is abelian, AD/D is a normal subgroup of F^0/D. Thus AD is a compact normal subgroup of F^0. Now $F^0 \supseteq AD \supseteq Z(F)^0$, so F^0/AD is semisimple or trivial. Hence $AD = F^0$, which implies that $\pi(A) = T$.

Proposition 1.10. *Let G and G' be locally compact connected groups and let $\beta : G \to G'$ be a continuous onto homomorphism. Let A' be a maximal pro-torus of G', and let H' be a closed connected subgroup of G' containing A'. Then $\beta^{-1}(H')$ is a closed connected subgroup of G and $\beta^{-1}(H')$ contains a maximal pro-torus of G.*

Proof. Let K be a compact normal subgroup of G such that G/K is a Lie group. Let $K' = \beta(K)$. Then K' is a compact normal subgroup of G', and since G/K is a Lie group, G'/K' is also a Lie group. Let $\pi : G \to G/K$ and $\pi' : G' \to G'/K'$ denote the natural homomorphisms. Also, let $\tilde{\beta}$ denote the induced map $G/K \to G'/K'$. Then $\pi'(H')$ is a closed connected subgroup of G'/K'. By Lemma 1.9, $\pi'(H')$ contains a maximal torus of G'/K', namely $\pi'(A')$. By [3, (7.1)], $\tilde{\beta}^{-1}(\pi'(H'))$ is a closed connected subgroup of G/K. Since $\tilde{\beta}^{-1}(\pi'(H')) = \pi(\beta^{-1}(H'))$, $\beta^{-1}(H')K/K$ is connected.

Let $\{K_\lambda\}$ be a canonical system of normal subgroups in G. Then $\beta^{-1}(H') = \lim_{\leftarrow} \beta^{-1}(H')K_\lambda/K_\lambda$ is connected. Hence $\beta^{-1}(H')$ is a closed connected subgroup of G.

Let A be a maximal pro-torus of $\beta^{-1}(H')$. Suppose $A \subseteq \tilde{A}$, where $\tilde{A}$ is a maximal pro-torus of G. Then $\beta(\tilde{A})$ is a pro-torus of G'. By Corollary 1.6, there exists $g' \in G'$, i.e., $g \in G$, such that $\beta(g) = g'$ and $\beta(g)B(\tilde{A})\beta(g)^{-1} = \beta(g\tilde{A}g^{-1}) \subseteq A' \subseteq H'$. Thus $g\tilde{A}g^{-1} \subseteq \beta^{-1}(H')$. By Corollary 1.6, there exists $\tilde{g} \in G$ such that $\tilde{g}\tilde{A}\tilde{g}^{-1} \subseteq A \subseteq \tilde{A}$. Therefore $A = \tilde{A}$ is a maximal pro-torus of G.

Proposition 1.11. *Let G be a locally compact connected group, $Z = Z(G)$ the center of G, and π the canonical homomorphism $G \to G/Z$. Let A'*

be a maximal pro-torus of G/Z. Then $\pi^{-1}(A')$ is a closed connected abelian subgroup of G.

Proof. Let $A = \pi^{-1}(A')$. Then, by Proposition 1.10, A is a closed connected subgroup of G, and A/Z is a compact connected abelian group. Let K be a compact normal subgroup of A such that A/K is a Lie group. Then $A/K \supseteq ZK/K$, and ZK/K is a central subgroup. Now $(A/K)/(ZK/K) \cong A/ZK$ is a compact connected abelian Lie group. Thus A/K is abelian.

Let $\{K_\lambda\}$ denote a canonical system of normal subgroups in A. Since $A = \lim_{\leftarrow} A/K_\lambda$, A is abelian. Hence $\pi^{-1}(A')$ is a closed connected abelian subgroup of G.

Corollary 1.12. *Let G be a locally compact connected group. Then $Z(G) \subseteq A$, where A is a closed connected abelian subgroup of G and A contains a maximal pro-torus of G.*

Proposition 1.13. *Let G be a locally compact connected group, N a closed connected normal subgroup of G, and A a maximal pro-torus of G. Then $A \cap N$ is a maximal pro-torus of N.*

Proof. The subgroup $(A \cap N)^0$ is a pro-torus of N. Let B be a maximal pro-torus of N which contains $(A \cap N)^0$. Then, by Corollary 1.6, there exists $g \in G$ such that $gBg^{-1} \subseteq A$. Since N is normal in G, $gBg^{-1} \subseteq N$, and thus $gBg^{-1} \subseteq A \cap N$. Hence $gBg^{-1} \subseteq (A \cap N)^0 \subseteq B$, which implies that $B = (A \cap N)^0$. Now $(A \cap N) \supseteq B$, where $A \cap N$ is a compact abelian subgroup of N. Since B is a maximal pro-torus of N, by Corollary 1.6, B is a maximal compact abelian subgroup of N. Thus $A \cap N = B$ is a maximal pro-torus of N.

Definition 1.14. Let E be a topological group. A subgroup F of E is called a *regular subgroup* of E if there exists a locally compact connected group G and an immersion $\phi : G \to E$ such that $\phi(G) = F$ [2, p. 155].

Proposition 1.15. *Let G be a locally compact connected group and F a regular subgroup of G. If F contains a maximal pro-torus of G, then F is closed.*

Proof. Without loss of generality, we may assume that $\overline{F} = G$. Then $G = F \cdot T$, where T is a pro-torus of G and F is normal in G. Now T is contained in a maximal pro-torus A of G. Therefore $G = F \cdot A$. Now F contains a maximal pro-torus A' of G by assumption. By Corollary 1.6, there exists $g \in G$ such that $A = gA'g^{-1}$. Now $A' \subseteq F$ implies that $gA'g^{-1} \subseteq gFg^{-1} = F$, since F is normal in G. Thus $A \subseteq F$, which implies that $G = F$.

2. The structure of locally compact connected groups. More than 50 years ago Iwasawa studied the structure of L-groups [6]. A few years later, Yamabe showed that every locally compact connected group is an L-group [9, 10]. In particular, Iwasawa's theorem [6, Theorem 11] on the local decomposition of L-groups can be applied to locally compact connected groups. This theorem can be stated as follows:

Theorem 2.1. *Let G be a locally compact connected group. Then we can find an arbitrary small neighborhood U of 1, a local Lie group L_l^*, and a compact normal subgroup K, such that U is the direct product of L_l^* and K: $U = L_l^* K$, $[L_l^*, K] = 1$, and $L_l^* \cap K = 1$.*

If L^* denotes the subgroup of G generated by L_l^*, then there exists a unique topology τ on L^* such that $L = (L^*, \tau)$ is a connected Lie group which maps continuously and isomorphically onto L^* by the identification map $\theta : L \to L^*$. Also, note that $G = L^* K$ and L^* is a normal subgroup of G. Let $D^* = L^* \cap K$, and let D be the inverse image of D^* in L. Then D is a discrete normal, hence central subgroup of L. We shall call such a local decomposition of G a *canonical decomposition*, including the notions of L,

L^*, D, D^*, and θ. In this section, we study this canonical decomposition of locally compact connected groups.

By carefully examining the proof of the above theorem of Iwasawa [6, Theorem 11] we have:

Theorem 2.2. *Let G be a locally compact connected group and let M be a compact normal subgroup of G with the property that G/M is a Lie group. Then there exists a compact normal subgroup $M' \subseteq M$ such that M/M' is abelian and G is locally the direct product of a local Lie group L and M'. In particular, if M is contained in a neighborhood U of 1, then so is M'. Thus the set of all compact normal subgroups of G which give rise to canonical decompositions forms a canonical system of normal subgroups in G.*

Definition 2.3. Let G be a locally compact connected group and K a maximal compact subgroup of G. Then the space of G is homeomorphic with $E \times K$, where E is a finite-dimensional Euclidean space. The *characteristic index* of G is defined to be the dimension of E [6, p. 549].

By means of Theorem 2.2, we obtain the following result:

Proposition 2.4. *Let G be a locally compact connected group. Let $G = L^*K$ be any canonical decomposition of G. Then the characteristic index of G/K is a constant.*

Proof. Let $G = L_1^*K_1 = L_2^*K_2$ be any two canonical decompositions of G. To show that the characteristic index of G/K_1 is equal to the characteristic index of G/K_2, we may assume that $K_2 \subseteq K_1$. Then $(G/K_2)/(K_1/K_2) \cong G/K_1$, and since K_1/K_2 is compact, the characteristic index of G/K_1 is equal to the characteristic index of G/K_2.

Lemma 2.5. *Let G be a locally compact connected group and let $G = L^*K$ be a canonical decomposition of G. Then $L^*/(L^* \cap K)$ is a Lie group.*

Proof. Consider the sequence $L \to L^* \to L^*K/K = G/K$. Then we have the sequence $L/D \to L^*/(L^* \cap K) \to G/K$, so $L/D \cong L^*/(L^* \cap K) \cong G/K$.

Corollary 2.6. *Suppose K is a compact normal subgroup of G and $G = L_1^*K = L_2^*K$ are two canonical decompositions of G. Then $L_1^*/(L_1^* \cap K) \cong L_2^*/(L_2^* \cap K) \cong G/K$, and L_1 and L_2 have isomorphic Lie algebras.*

Throughout the remainder of this section, we fix a locally compact connected group G and a canonical decomposition $G = L^*K$.

Lemma 2.7. *The closure of L^* in G is $L^*\overline{D^*}$.*

Proof. Since $D^* = L^* \cap K$ is central, $\overline{D^*}$ is a compact central subgroup of G. From the sequence $L/D \to L^*\overline{D^*}/\overline{D^*} \to G/K$, we see that $L^*\overline{D^*}/\overline{D^*}$ is locally compact. Thus $L^*\overline{D^*}$ is locally compact, and therefore $L^*\overline{D^*}$ is closed.

The following corollary is due to Goto [2, Proposition 7].

Corollary 2.8. *L^* is closed if and only if D^* is finite.*

Proof. If L^* is closed, then D^* is compact. Since D is discrete in L, D is countable, and therefore, so is D^*. Hence D^* must be finite.

The converse follows directly from Lemma 2.7.

Recall that L maps continuously and isomorphically onto L^* by the identification map $\theta : L \to L^*$. If B is a subgroup of L, then we denote its image under θ by B^*.

Proposition 2.9. *If B is a closed subgroup of L which contains D, then $\overline{B^*} = B^*\overline{D^*}$.*

Proof. Let ϕ denote the isomorphism $L/D \to L^*\overline{D^*}/\overline{D^*}$. Since B contains D, B/D is closed. Hence $\phi(B/D)$ is closed in $L^*\overline{D^*}/\overline{D^*}$, which implies that $B^*\overline{D^*}$ is closed.

Proposition 2.10. *$G = L^*\overline{D^*}K^0$ and $K = \overline{D^*}K^0$. In particular, D^*K^0 is dense in K.*

Proof. Since K^0 is compact, $\overline{L^*}K^0$ is a closed normal subgroup of G. Consider the isomorphism $G/\overline{L^*}K^0 \cong K/(\overline{L^*}K^0 \cap K)$. Since $\overline{L^*}K^0 \cap K$ contains K^0, $K/(\overline{L^*}K^0 \cap K)$ is totally disconnected. Since G is connected, $G/\overline{L^*}K^0$ is also connected. Hence $G = \overline{L^*}K^0$. Since $K \supseteq \overline{D^*}K^0$, we can write $K = K \cap L^*\overline{D^*}K^0 = (K \cap L^*)\overline{D^*}K^0 = D^*\overline{D^*}K^0 = \overline{D^*}K^0$. Hence $K = \overline{D^*}K^0$.

Corollary 2.11. $Z(K) = \overline{D^*}Z(K^0)$.

Proof. Since $K = \overline{D^*}K^0$, where $\overline{D^*}$ is central in K, every central element z of K^0 is also central in K. Conversely, if $z \in Z(K)$, then there exists $d \in \overline{D^*}$ such that $dz \in K^0$ (in fact $dz \in Z(K^0)$). Hence $Z(K) = \overline{D^*}Z(K^0)$.

Proposition 2.12. $Z(G) = Z(L^*)Z(K) = Z(L^*)\overline{D^*}Z(K^0)$.

Proof. Since $[L^*, K] = 1$, we have $Z(L^*) \subseteq Z(G)$ and $Z(K) \subseteq Z(G)$. Let $g = lk \in Z(G)$, with $l \in L^*$ and $k \in K$. Let $l^* \in L^*$. Then $gl^*g^{-1}l^{*-1} = 1 = (lk)l^*(k^{-1}l^{-1})l^{*-1} = lkl^*k^{-1}l^{*-1}l^*l^{-1}l^{*-1} = ll^*l^{-1}l^{*-1}$. Thus $l \in Z(L^*)$. Similarly, we have $k \in Z(K)$. Hence $Z(G) = Z(L^*)Z(K)$.

3. *Z*-decompositions and generalized maximal pro-tori. Let H be an analytic group. In [3], Goto defined a Z-decomposition of H and a generalized maximal torus of H. In this section, we will extend these definitions to locally compact connected groups, and we fix a locally compact connected group G and a canonical decomposition $G = L^*K$.

Let $L = VN$ be a Z-decomposition of L. Recall that V is a closed vector subgroup, N is a closed connected normal subgroup, $V \cap N = 1$, and $Z(L) \subseteq N$. Also $\overline{\text{Int}L} = \overline{(\text{Int}V)}(\text{Int}N)$, where $\overline{\text{Int}V}$ is a torus, and Int N is closed [3, (6.2)]. Then $L^* = V^*N^*$, where V^* and N^* denote the images of V and N, respectively, under the identification map θ. Hence $G = V^*N^*K = V^*P$, where $P = N^*K$.

We call the above decomposition $G = V^*P$ a Z-decomposition of G (relative to the canonical decomposition $G = L^*K$ and the Z-decomposition $L = VN$).

Proposition 3.1. *Let $G = V^*P$ be a Z-decomposition of G as above. Then V^* is a closed vector subgroup, $V^* \cap P = 1$, and P is a closed connected normal subgroup of G which contains $Z(G)$. Also $\overline{\text{Int}G} = \overline{(\text{Int}V^*)}(\text{Int}P)$, where $\overline{\text{Int}V^*}$ is a torus and* Int$P$ *is closed.*

Proof. First we will show that V^* is closed. Now $\overline{L^*} = L^*\overline{D^*} = V^*N^*\overline{D^*} = V^*\overline{N^*}$, so $L/N \to \overline{L^*}/\overline{N^*}$ is an isomorphism. From the sequence $V \to V^* \to \overline{L^*}/\overline{N^*}$, it follows that V^* is locally compact. Hence V^* is a closed vector subgroup.

Next we show that $V^* \cap P = V^* \cap N^*K$ is trivial. Suppose $v^* = n^*k \in V \cap N^*K$. Then $n^{*-1}v^* \in L^* \cap K = D^* \subseteq N^*$. Thus $v^* \in N^*$. Since $V \cap N = 1$, $V^* \cap N^* = 1$, so v^* must be trivial. Since $Z(L) \subseteq N$, $Z(L^*) \subseteq N^*$, which implies that $Z(G) = Z(L^*)Z(K) \subseteq P$. Since N is normal in L, N^* is normal in L^*. From this, it follows that P is normal in G. To show that P is closed, note that $P = N^*K = N^*\overline{D^*}K = \overline{N^*}K$, which is closed, since K is compact. Since $G/P \cong V^*$ and V^* is simply connected and G is connected, it follows that P is also connected.

Definition 3.2. Let G be a locally compact connected group, and let $\alpha : G \to \text{Aut } G$ be defined by $\alpha(x) = I_x$ for all $x \in G$. For a maximal pro-torus A' in $\overline{\text{Int}G}$, the group $\alpha^{-1}(A' \cap \text{Int}G)$ is called a *generalized maximal pro-torus* (*gm-pro-torus*) of G.

Since $\text{Int}G = (\text{Int}L^*)(\text{Int}K)$, it follows that $\overline{\text{Int}G} = \overline{(\text{Int}L^*)}(\text{Int}K)$. Let A' be a maximal pro-torus in $\overline{\text{Int}G}$. Then $A' = B'C'$, where B' is a maximal torus in $\overline{\text{Int}L^*}$($\overline{\text{Int}L^*} = \overline{\text{Int}L}$ is a Lie group) and C' is a maximal pro-torus in $\text{Int}K$. Let β denote the restriction of α to L^* and γ the restriction of α to K^0. Then $\alpha^{-1}(B') = \beta^{-1}(B')Z(G) = \beta^{-1}(B')Z(L^*)\overline{D^*}Z(K^0) = \beta^{-1}(B')\overline{D^*}Z(K^0)$, and $\alpha^{-1}(C') = \gamma^{-1}(C')Z(G) = \gamma^{-1}(C')Z(L^*)\overline{D^*}Z(K^0) = \gamma^{-1}(C')Z(L^*)\overline{D^*}$. Thus $\alpha^{-1}(A') = \beta^{-1}(B')\overline{D^*}\gamma^{-1}(C')$.

Now $\gamma^{-1}(C')$ is a maximal pro-torus C in K^0. Also $\beta^{-1}(B') = \theta(H)$, where H is a generalized maximal torus of L. Hence $\alpha^{-1}(A') = \theta(H)\overline{D^*}C = \overline{\theta(H)}C$.

Proposition 3.3. *$\theta(H)\overline{D^*}C = \overline{\theta(H)}C$ is a generalized maximal pro-torus of G, where H is a generalized maximal torus of L, and C is a maximal pro-torus of K^0. In fact, any generalized maximal pro-torus of G can be obtained in this way.*

Remark. $\theta(H)\overline{D^*}C = \overline{\theta(H)}C$ is a closed connected abelian subgroup of G containing $Z(G)$.

Definition 3.4. Let $G = L^*K$ be a canonical decomposition, and let B be a generalized maximal pro-torus of G. Then B can be written as $V \times C$, where V is a vector group and C is the unique maximal compact subgroup of B. Then V is called a *vector part* of the gm-pro-torus B (relative to the canonical decomposition $G = L^*K$) and is denoted by $v(B)$.

Proposition 3.5. *All gm-pro-tori in G are conjugate in the sense that given any two of them A_1 and A_2, there exists $\phi \in \overline{\text{Int}G}$ such that $\phi(A_1) = A_2$.*

Proof. Let A' and B' be maximal pro-tori in $\overline{\mathrm{Int}G}$. Then, by Corollary 1.6, there exists $\phi \in \overline{\mathrm{Int}G}$ such that $B' = \phi \circ A' \circ \phi^{-1}$. We claim that $\alpha^{-1}(B') = \alpha^{-1}(\phi \circ A' \circ \phi^{-1}) = \phi(\alpha^{-1}(A'))$, where $\alpha : G \to \mathrm{Aut}\, G$ is defined as above. Let $x \in G$. Then $x \in \alpha^{-1}(\phi \circ A' \circ \phi^{-1}) \longleftrightarrow \alpha(x) = I_x \in \phi \circ A' \circ \phi^{-1} \longleftrightarrow \phi^{-1} \circ I_x \circ \phi \in A' \longleftrightarrow I_{\phi^{-1}(x)} \in A' \longleftrightarrow x \in \phi(\alpha^{-1}(A'))$. Hence $\alpha^{-1}(B') = \phi(\alpha^{-1}(A'))$.

We fix a Z-decomposition $G = V^*P$ (relative to $G = L^*K$ and $L = VN$).

Let A' be a maximal pro-torus of $\overline{\alpha(G)}$ containing $\alpha(V^*)$. By Proposition 1.13, $A' \cap \alpha(P) = A'_{\alpha(P)}$ is a maximal pro-torus of $\alpha(P)$. Since $\overline{\alpha(G)} = \overline{\alpha(V^*)}\alpha(P)$ and $A' \supseteq \overline{\alpha(V^*)}$, we have $A' = \overline{\alpha(V^*)}A'_{\alpha(P)}$.

Let $A = A' \cap \alpha(G) = \alpha(V^*)A'_{\alpha(P)}$. Since $\alpha(G)/\alpha(P)$ is isomorphic with a vector group, $A'_{\alpha(P)}$ is a maximal pro-torus of $\alpha(G)$. Then $\alpha^{-1}(A) = B$ contains a maximal pro-torus of G. Since $\alpha(V^*) \subseteq A$, we have that $V^* \subseteq B$.

Thus we have the following proposition.

Proposition 3.6. *Let $G = L^*K$ be a canonical decomposition.*

(a) *A gm-pro-torus B of G is a closed connected abelian subgroup of G containing $Z(G)$ and a maximal pro-torus of G.*
(b) *Let B be a gm-pro-torus of G. Then for a suitable Z-decomposition $G = V^*P$, we have $B = V^*B_P$, $B_P = B \cap P$, $V^* \cap B_P = 1$, and $\alpha(B_P)$ is a maximal pro-torus of $\alpha(G)$.*

Remark. The properties of generalized maximal pro-tori of G in Propositions 3.5 and 3.6 are analogous to the properties of generalized maximal tori of an analytic group (see [3, (7.2)]).

4. Immersion theorems. With all of the preparation work done in sections 1 through 3, we are now able to extend Theorems A, B, and C to a locally compact connected group G.

Theorem 4.1. *Let G be a locally compact connected group and let $G = L^*K$ be a canonical decomposition of G. Let H' be a gm-pro-torus of G and let $v(H')$ be a vector part of H' (relative to the canonical decomposition $G = L^*K$). Let $G = V^*P$ be a Z-decomposition of G (relative to the canonical decomposition $G = L^*K$ and the Z-decomposition $L = VN$). If M is a topological group and $f : G \to M$ is a dense immersion, then $M = \overline{f(v(H'))}f(P)$.*

Proof. We can write $H' = \theta(H)\overline{D^*}C$, where H is a gm-torus of L, and C is a maximal pro-torus of K^0. Let $v(H)$ be a vector part of H. Then, by Theorem A, $\overline{f(L^*)} = \overline{f(\theta(L))} = \overline{f(\theta(v(H)))}f(N^*)$. Now $\theta(v(H)) \subseteq \overline{\theta(v(H))} \subseteq H' = v(H') \times K'$, where K' is the unique maximal compact subgroup of H'. Thus $\overline{f(\theta(v(H)))} \subseteq \overline{f(v(H'))f(K')} = \overline{f(v(H'))}f(K')$. Therefore $\overline{f(G)} = \overline{f(L^*)}f(K) = \overline{f(\theta(v(H)))}f(N^*)f(K) = \overline{f(v(H'))}f(K')f(P)$. It will follow that $\overline{f(G)} = \overline{f(v(H'))}f(P)$ once we show that $K' \subseteq P$.

Note that $\alpha(K')$ is a pro-torus of $\alpha(G)$, where $\alpha : G \to \text{Aut}\, G$ is defined by $\alpha(x) = I_x$. Since $\alpha(H' \cap P)$ is a maximal pro-torus of $\alpha(G)$ (Proposition 3.6.), by Corollary 1.6, there exists $y \in G$ such that $\alpha(y)\alpha(K')\alpha(y)^{-1} \subseteq \alpha(H' \cap P)$. Thus $yK'y^{-1} \subseteq H' \cap P$, and by the uniqueness of K', $K' = yK'y^{-1} \subseteq P$.

Let G be a locally compact connected group. We call G a (CA)-*group* if Int G is a closed subgroup of Aut G. From Theorem 4.1, we can conclude the following (see [3, page 729]):

Corollary 4.2. *Let G be a locally compact connected $(CA)-$group and let M be a topological group. If $f : G \to M$ is a dense immersion, then $M = \overline{f(Z(G))}f(G)$.*

Let A be a group, and let $[A, A] = A^2$, $[A^2, A] = A^3, \ldots, [A^j, A] = A^{j+1}$. The sequence $A = A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots$ of normal subgroups is called the *descending central series* of A. If $A^{j-1} \neq A^j = 1$, then A is called *nilpotent of degree* j.

Theorem 4.3. *Let G be a locally compact connected group and M a topological group. If $f : G \to M$ is a dense immersion, then $f(G)^j = M^j$ for $j = 2, 3, \ldots$. In particular, if G is nilpotent of degree j, then so is M.*

Proof. Let $G = L^*K$ be a canonical decomposition of G. Then $[L^*, K] = 1$, $\overline{f(G)} = \overline{f(L^*)}f(K)$, and $[\overline{f(L^*)}, f(K)] = 1$. Thus $M^2 = [M, M] = [\overline{f(G)}, \overline{f(G)}] = [\overline{f(L^*)}f(K), \overline{f(L^*)}f(K)] = [\overline{f(L^*)}, \overline{f(L^*)}][f(K),\ f(K)]$. Now, by Theorem B, we have that $[\overline{f(L^*)}, \overline{f(L^*)}] = [\overline{f \circ \theta(L)},\ \overline{f \circ \theta(L)}] = [f \circ \theta(L),\ f \circ \theta(L)] = [f(L^*), f(L^*)]$. Hence $M^2 = [f(L^*), f(L^*)][f(K), f(K)] = [f(L^*)f(K),\ f(L^*)f(K)] = [f(G), f(G)] = f(G)^2$.

Now we show that $M^3 = f(G)^3$. First $M^3 = [M^2, M] = [f(G)^2, \overline{f(G)}] = [f(L^*)^2 f(K)^2, \overline{f(L^*)}f(K)] = [f(L^*)^2, \overline{f(L^*)}][f(K)^2, f(K)] = [f(L^*)^2, \overline{f(L^*)}] \times f(K)^3$. Once again, by Theorem B, we have that $[f(L^*)^2,\ \overline{f(L^*)}] = [\overline{f(L^*)}^2, \overline{f(L^*)}] = [\overline{f \circ \theta(L)}^2, \overline{f \circ \theta(L)}] = [(f \circ \theta(L))^2, f \circ \theta(L)] = [f(L^*)^2, f(L^*)] = f(L^*)^3$. Therefore $M^3 = f(L^*)^3 f(K^3) = f(G^3)$.

By induction, it follows that $M^j = f(G)^j$ for all $j \geq 2$.

For a group A, we define the *derived series* of A by $A^{(0)} = A, A^{(1)} = A^2$, $A^{(2)} = [A^{(1)}, A^{(1)}], \cdots$. Then $A^{(0)} \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$, and if $A^{(j-1)} \neq A^{(j)} = 1$, then A is called *solvable of degree* j.

Corollary 4.4. *Let G be a locally compact connected group and M a topological group. If $f : G \to M$ is a dense immersion, then $f(G)^{(j)} = M^{(j)}$ for $j = 1, 2, \ldots$. In particular, if G is solvable of degree j, then so is M.*

Theorem 4.5. *Let G be a locally compact connected group and let H' be a gm-pro-torus of G. Let M be a topological group and $f : G \to M$ an immersion. If $\overline{f(v(H'))}$ is locally compact, then $\overline{f(G)}$ is locally compact.*

Proof. Let $G = L^*K$ be a canonical decomposition of G. Then $H' = \theta(H)\overline{D^*}C$, where H is a gm-torus of L and C is a maximal pro-torus of K^0. Let $v(H)$ be a vector part of H. Then $\theta(v(H)) \subseteq \overline{\theta(v(H))} \subseteq H' = v(H') \times K'$, where K' is the unique maximal compact subgroup of H'. Thus $\overline{f(\theta(v(H)))} \subseteq \overline{f(v(H'))f(K')} = \overline{f(v(H'))}f(K')$. Since $\overline{f(v(H'))}$ is locally compact, $\overline{f(v(H'))}f(K')$ is locally compact, which implies that $\overline{f(\theta(v(H)))}$ is locally compact. Then, by Theorem C, we have that $\overline{f(\theta(L))} = \overline{f(L^*)}$ is locally compact. Hence $\overline{f(G)} = \overline{f(L^*)}f(K)$ is locally compact.

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