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ON THE CONVERGENCE OF A TRUNCATED CLASS OF OPERATORS

BY

OCTAVIAN AGRATINI

Abstract. In this paper we are dealing with a general class of positive approximation processes of discrete type expressed in series. We modify them into finite sums and investigate their approximation properties in weighted spaces of continuous functions. Some special cases are also revealed.

1. Introduction. The approximation of functions by linear positive operators is an important problem in many mathematical theories. The following examples are two of the best known and intensively studied approximation processes on unbounded intervals.

The former is the n-th Favard-Mirakjan-Szász operator defined by

$$(S_nf)(x) := \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad s_{n,k}(x) := \frac{(nx)^k}{k!}e^{-nx}, \tag{1}$$

for every f belonging to the Banach lattice E_2, $x \in [0,\infty)$ and $n \in \mathbb{N}$. Here $E_2 = E_2([0,\infty)) := \{f \in C([0,\infty)) |\ \lim_{x\to\infty}(1+x^2)^{-1}f(x)$ is finite$\}$ is endowed with the norm $\|\cdot\|_*$, $\|f\|_* := \sup_{x\geq 0}(1+x^2)^{-1}|f(x)|$.

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The later is the n-th Baskakov operator defined by

$$(2)\qquad (V_nf)(x) := \sum_{k=0}^{\infty} v_{n,k}(x)f\left(\frac{k}{n}\right),\ v_{n,k}(x) := \binom{n+k-1}{k}x^k(1-x)^{-n-k},$$

for every $f \in E_2$, $x \in [0,\infty)$ and $n \in \mathbb{N}$.

Note that the series which appear both in (1) and (2) are absolutely convergent for every $f \in E_2$. Furthermore, we point out that every S_n and V_n map $C_B([0,\infty))$ into itself, where $C_B(I)$ denotes the space of all real-valued continuous and bounded functions defined on the interval I endowed with the usual sup-norm $\|\cdot\|_\infty$, $\|f\|_\infty := \sup_{x\in I}|f(x)|$. Also, note that E_2 is isomorphic to $C([0,1])$, see e.g. [2; Proposition 4.2.5].

However, we notice that the construction of the above operators requires an estimation of infinite series which in a certain sense restricts the operators usefulness from the computational point of view. In this respect, in order to approximate a function f, it is interesting and useful to consider partial sums of S_nf or V_nf which only have finite terms depending upon n and x. This approach of the above operators has already been made in the late decades. J. Grof [4] examined the operator $(S_{n,N}f)(x) = \sum_{k=0}^{N} s_{n,k}(x)f(k/n)$ establishing that if $(N(n))_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty}(N(n)/n) = \infty$ then $\lim_{n\to\infty}(S_{n,N}f)(x) = f(x)$ for all $x \geq 0$ and $f \in C([0,\infty))$ satisfying a growth condition of the form $|f(t)| \leq Ae^{mt}$ ($A \in \mathbb{R}_+$, $m \in \mathbb{N}$). Following a little different course from the one of Grof, the next modified operators of Szász respectively Baskakov-type were investigated

$$(3)\qquad \begin{aligned} (S_{n,\delta}f)(x) &= \sum_{k=0}^{[n(x+\delta)]} s_{n,k}(x)f\left(\frac{k}{n}\right), \\ (V_{n,\delta}f)(x) &= \sum_{k=0}^{[n(x+\delta)]} v_{n,k}(x)f\left(\frac{k}{n}\right), \quad x \geq 0. \end{aligned}$$

Here $[\alpha]$ indicates the largest integer not exceeding α. The first was studied by Heinz-Gerd Lehnhoff [5] and the second has recently been approached by Jianli Wang and Songping Zhou [6], the authors giving a necessary and sufficient condition which guarantees the convergence of $(V_{n,\delta}f)_{n\geq 1}$ to f. In (3) f belongs to certain subspaces of $C([0,\infty))$.

At this point we find appropriate to mention the important research achievement in this field obtained by G.Z. Zhou and S.P. Zhou [7].

The present paper is dealing with a general class of linear positive operators expressed in series. The sequence is constructed in the next section. Further on, the operators are truncated fading away their "tails" and sufficient conditions are provided to ensure their convergence on certain spaces of functions. Particular cases are also punctuated.

2. Building up the operators. We set $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $o(\cdot)$, $\mathcal{O}(\cdot)$ the Landau symbols and $e_j, j\in\mathbb{N}_0$, stands for the j-th monomial, $e_j(t)=t^j$. Throughout the paper K represents a compact subinterval of $\mathbb{R}_+=[0,\infty)$.

For each $n\in\mathbb{N}$ we consider the following.

(i) A net on $\mathbb{R}_+$ named $\Delta_n=(x_{n,k})_{k\geq 0}$ is fixed with the property: for every $k\in\mathbb{N}_0$, γ_k exists such that $x_{n,k}=\mathcal{O}(n^{-\gamma_k})$ $(n\to\infty)$.

(ii) A sequence $(\phi_{n,k})_{k\geq 0}$ is given, where every $\phi_{n,k}$ belongs to $C^1(\mathbb{R}_+)$, the space of all real-valued functions continuously differentiable in $\mathbb{R}_+$. We assume that this sequence is a blending system with a certain connection with Δ_n, more precisely the following conditions hold:

$$\phi_{n,k}\geq 0,\quad k\in\mathbb{N}_0,\quad \sum_{k=0}^{\infty}\phi_{n,k}=e_0,\quad \sum_{k=0}^{\infty}x_{n,k}\phi_{n,k}=e_1. \tag{4}$$

(iii) A positive function $\psi\in\mathbb{R}^{\mathbb{N}\times\mathbb{R}_+}$, $\psi(n,\cdot)\in C(\mathbb{R}_+)$, exists with the property

$$\psi(n,x)\phi'_{n,k}(x)=(x_{n,k}-x)\phi_{n,k}(x), \quad k\in\mathbb{N}_0, \quad x\geq 0. \tag{5}$$

By using the above three requirements we define the operators

$$(L_nf)(x)=\sum_{k=0}^{\infty}\phi_{n,k}(x)f(x_{n,k}), \quad x\geq 0, \quad f\in\mathcal{F}, \tag{6}$$

where $\mathcal{F}$ stands for the domain of L_n containing the set of all continuous functions on $\mathbb{R}_+$ for which the series in (6) is convergent.

Remark 1. $L_n, n\in\mathbb{N}$, are positive linear operators and consequently, they become monotone. Obviously $C_B(\mathbb{R}_+)\subset\mathcal{F}$ and every operator L_n map continuously $C_B(\mathbb{R}_+)$ into itself. Indeed, taking into account (4) for $f\in C_B(\mathbb{R}_+)$ and $x\geq 0$ we have $|(L_nf)(x)|\leq\|f\|_\infty$. Moreover, $L_ne_0=e_0$ implies $\|L_n\|_{C_B}:=\sup_{\|f\|_\infty\leq 1}\|L_nf\|_\infty=1$.

Further on we are going to present a technical result.

Lemma 1. *Let $L_n, n\in\mathbb{N}$, be defined by* (6) *and $\Lambda_{n,r}$ be the r-th central moment of L_n. For every $x\in\mathbb{R}_+$ the following identities*

$$\Lambda_{n,0}(x)=1, \quad \Lambda_{n,1}(x)=0, \tag{7}$$

$$\Lambda_{n,r+1}(x)=\psi(n,x)(\Lambda'_{n,r}(x)+r\Lambda_{n,r-1}(x)), \quad r\in\mathbb{N}, \tag{8}$$

$$\Lambda_{n,2}(x)=\psi(n,x), \tag{9}$$

hold true.

Proof. Firstly we recall $\Lambda_{n,r}(x):=L_n((e_1-xe_0)^r,x)$, $r\in\mathbb{N}_0$. For $r=0$ and $r=1$ the values result easily from relations (4). For every $r\in\mathbb{N}$, with the help of (5) we can write

$$\begin{aligned}\frac{d}{dx}L_n((e_1-xe_0)^r,x) &= \sum_{k=0}^{\infty}\{\phi'_{n,k}(x)(x_{n,k}-x)^r - r\phi_{n,k}(x)(x_{n,k}-x)^{r-1}\}\\ &= (\psi(n,x))^{-1}\Lambda_{n,r+1}(x) - r\Lambda_{n,r-1}(x),\end{aligned}$$

and (8) follows. Choosing here $r = 1$ we obtain (9) and this completes the proof.

We are able to indicate the necessary and sufficient condition which offers to $(L_n)_{n\geq 1}$ the attribute of approximation process.

Theorem 1. *Let $L_n, n \in \mathbb{N}$, be defined by* (6).

(i) *If* $\lim_{n\to\infty}\psi(n,x) = 0$ *uniformly on K then for every $f \in \mathcal{F}$ one has*

$$\lim_{n\to\infty} L_n f = f \ \text{uniformly on } K.$$

(ii) *For every $f \in C_B(\mathbb{R}_+)$, $x \geq 0$ and $\delta > 0$ one has*

$$|(L_nf)(x) - f(x)| \leq \left(1+\delta^{-1}\sqrt{\psi(n,x)}\right)\omega_1(f,\delta), \tag{10}$$

where $\omega_1(f,\cdot)$ represents the modulus of continuity of f.

Proof. The first statement results directly from the theorem of Bohman-Korovkin and relations (7), (9) as well. Note that Lemma 1 guarantees that e_0, e_1, e_2, the Korovkin test functions belong to $\mathcal{F}$. The second statement holds true by virtue of the classical results regarding the rate of convergence, see e.g. the monograph [2; Theorem 5.1.2].

Examples. We take the net Δ_n with equality spaced nodes $x_{n,k} = k/n$.

1° Selecting $\phi_{n,k} = s_{n,k}$ defined by (1), the relations (4) are verified and clearly $xs'_{n,k}(x) = (k-nx)s_{n,k}(x)$. We make the following choice: $\psi(n,x) = x/n$ and (5) is fulfilled. Our operators turn into S_n−the Szász operators.

2° Choosing $\phi_{n,k} = v_{n,k}$ defined by (2) the requirements (4) hold true and we also have $x(1+x)v'_{n,k}(x) = (k-nx)v_{n,k}(x)$. Taking $\psi(n,x) = x(1+x)/n$, (5) is verified and L_n becomes V_n - the Baskakov operator.

In both examples $\mathcal{F}$ may coincide with E_2.

In what follows, with a slight restriction of the generality, for our purpose we specialize the net Δ_n and the function ψ. We consider that a positive sequence $(a_n)_{n\geq 1}$ and the functions $\psi_i \in C(\mathbb{R}_+)$, $i = \overline{1,l}$, l fixed, exist such that for every $n \in \mathbb{N}$ one has

$$(11)\ x_{n,k} = \frac{k}{a_n} \leq k,\ k \in \mathbb{N},\ \text{with} \lim_{n\to\infty} a_n = \infty \text{ and } \psi(n,x) = \sum_{i=1}^{l} \frac{\psi_i(x)}{a_n^i},\ x \geq 0.$$

Under these assumptions, the requirement of Theorem 1 is fulfilled and $(L_n)_{n\geq 1}$ converges to the identity operator. As regards to the local and global rate of convergence, we use (11) in relation to (10) and choosing $\delta = a_n^{-1/2}$ we get

$$|(L_n f)(x) - f(x)| \leq \left(1 + \sqrt{|\psi_1|(x) + \sum_{j=2}^{l} \frac{|\psi_j|(x)}{a_n^{j-1}}}\right) \omega_1\left(f, \frac{1}{\sqrt{a_n}}\right),\ x \geq 0,$$

and consequently, $\|L_n f - f\|_{C(K)} \leq (1 + M(K))\omega_1(f, a_n^{-1/2})$ where $M(K) = \sup_{x\in K} \left(\sum_{j=1}^{l} |\psi_i|(x)\right)^{1/2}$.

Returning to our previous examples we notice that $l = 1$ and $\psi_1 = e_1$ respectively $\psi_1 = e_1 + e_2$. In a concordance with [3; §1.3, page 9] $\psi_1^{1/2}$ will be called the *step-weight function* related to L_n operators.

We mention that a similar representation of the second central moment as given in (11) and (9) (with $a_n = n$) has already been used to investigate a family of summation integral operators [1; Eq. (6)].

3. Modified discrete operators. Starting from (6) under the additional assumptions (11) we define

$$(L_{n,\delta}f)(x) := \sum_{k=0}^{[a_n(x+\delta(n))]} \phi_{n,k}(x) f\left(\frac{k}{a_n}\right), \quad x \geq 0, \quad f \in \mathcal{F}, \tag{12}$$

where $\delta = (\delta(n))_{n\geq 1}$ is a sequence of positive numbers. The study of these operators will be developed in polynomial weighted spaces connected to the weights w_m, $m \in \mathbb{N}_0$, $w_m(x) = (1 + x^{2m})^{-1}$, $x \geq 0$. For every $m \in \mathbb{N}_0$, the spaces $E_m := \{f \in C(\mathbb{R}_+) : \ \|f\|_m := \sup_{x\geq 0} w_m(x)|f(x)| < \infty\}$ endowed with the norm $\|\cdot\|_m$ and the natural order are Banach lattices and they are nested as follows: $C_B(\mathbb{R}_+) = E_0 \subset E_m \subset E_{m+1} \subset C(\mathbb{R}+)$, $m \in \mathbb{N}$.

We need the following lemma which might be of interest in its own right.

Lemma 2. *Let $L_n, n \in \mathbb{N}$, be defined by* (6) *and the assumptions* (11) *are fulfilled. If $\psi_i \in C^{2m-2}(\mathbb{R}_+)$, $i = \overline{1,l}$, then the central moment of $2m$-th order verifies*

$$\Lambda_{n,2m}(x) \leq \frac{C(m,K)}{a_n^m}, \quad x \in K, \tag{13}$$

where $C(m,K)$ is a constant depending only on m and the compact K.

Proof. Relations (9) and (11) imply $\Lambda_{n,2}(x) \leq a_n^{-1} \sum_{i=1}^{l} |\psi_i|(x)$, thus $C(2,K) := \sup_{x\in K} \Big(\sum_{i=1}^{l} |\psi_i|(x)\Big)$. Further on, we can use induction on m and the recursion relation (8) where $\psi(n,\cdot) = \Lambda_{n,2}$, obtaining

$$\Lambda_{n,2m-1}(x) = \mathcal{O}(a_n^{-m}) \ (m \to \infty) \text{ and } \Lambda_{n,2m}(x) = \mathcal{O}(a_n^{-m}) \ (m \to \infty),$$

$m \geq 2$.

Since $\Lambda_{n,2m}$ can be expressed only in terms of $\Lambda_{n,2}$ and its derivatives up to the order $(2m-2)$, we have $\Lambda_{n,2m} \in C(\mathbb{R}_+)$ and the conclusion follows.

Our aim is to prove the main result of this section.

Theorem 2. *Let $L_{n,\delta}, n \in \mathbb{N}$, be defined by* (12). *If $\psi_i \in C^{2m-2}(\mathbb{R}_+)$, $i = \overline{1,l}$, and $\lim_{n\to\infty} \sqrt{a_n}\delta(n) = \infty$ then $L_{n,\delta} f$ converges to f, uniformly on K, for every function f belonging to $E_m \cap \mathcal{F}$.*

Proof. Since $\min_{\lambda\geq 0}(\lambda^{2m} + (1-\lambda)^{2m}) = 2^{1-2m}$, $m \in \mathbb{N}$, the following elementary inequality

$$t^{2m} \leq 2^{2m-1}(x^{2m} + (t-x)^{2m}), \quad t \geq 0, \quad x \geq 0, \quad m \in \mathbb{N}, \tag{14}$$

holds true. On the other hand, for every $f \in E_m$, the positive constants a_f, b_f exist such that $|f| \leq a_f + b_f e_{2m}$ and, consequently, by using (14) we get

$$|f(t)| \leq g_m(x) + 2^{2m-1} b_f (t-x)^{2m}, \quad g_m := a_f + 2^{2m-1} b_f e_{2m}.$$

This implies

$$\left| f\left(\frac{k}{a_n}\right)\right| \leq g_m(x) + 2^{2m-1} b_f \left(\frac{k}{a_n} - x\right)^{2m}, \quad k \in \mathbb{N}_0, \quad x \geq 0. \tag{15}$$

Because $x, \delta(n), a_n$ are positive, if $k \geq [a_n(x+\delta(n))]+1$ then $k/a_n \geq x$ and consequently

$$\{k \in \mathbb{N}_0 : \ k \geq [a_n(x+\delta(n))]+1\} \subset \left\{k \in \mathbb{N}_0 : \left|\frac{k}{a_n} - x\right| > \delta(n)\right\} := I_{n,x,\delta}. \tag{16}$$

Setting $R_n := L_n - L_{n,\delta}$ and taking into account both (16) and (15) we can write

$$\begin{aligned} |(R_n f)(x)| &\leq \sum_{k\in I_{n,x,\delta}} \phi_{n,k}(x) |f(k/a_n)| \\ &\leq \sum_{k\in I_{n,x,\delta}} \phi_{n,k}(x) g_m(x) + 2^{2m-1} b_f \sum_{k\in I_{n,x,\delta}} \phi_{n,k}(x) \left(\frac{k}{a_n} - x\right)^{2m} := \Sigma_1 + \Sigma_2. \end{aligned} \tag{17}$$

Concerning the sums Σ_1 and Σ_2 we have

$$\Sigma_1 \le g_m(x)\delta^{-2m}(n)\sum_{k=0}^{\infty}\phi_{n,k}(x)\left(\frac{k}{a_n}-x\right)^{2m} = g_m(x)\delta^{-2m}\Lambda_{n,2m}(x),$$

respectively $\Sigma_2 \le 2^{2m-1}b_f\Lambda_{n,2m}(x)$. Returning to (17) and using (13) we get

$$|(R_nf)(x)| \le (g_m(x)\delta^{-2m}(n) + 2^{2m-1}b_f)\frac{C(m,K)}{a_n^m} = o(1)\ (n\to\infty),$$

uniformly on K because $\lim_{n\to\infty}\sqrt{a_n}\delta(n)=\infty$. The proof is complete.

For the special operators given by (3), our condition $\lim_{n\to\infty}\sqrt{a_n}\delta(n) = \infty$ becomes $\lim_{n\to\infty}\sqrt{n}\delta(n)=\infty$ and this coincides with the results established by Lehnhoff [5; Theorem 3] respectively J. Wang, S. Zhou [6; Theorem 1, Eq. (1)].

Particular case. We consider $\delta(n)=\lambda>0$ (constant). Denoting our operators by $L_{n,\lambda}$ we deduce that they map $C([0,b+\lambda])$ into $C([0,b])$ for any $b>0$. Since $a_n^{-\lambda}=o(1)$ $(n\to\infty)$ for a given $\lambda>0$, the convergence property ennunciated at Theorem 2 takes place for $K=[0,b+\lambda]$ and every $f\in C(\mathbb{R}_+)\cap\mathcal{F}$.

Next, we analyze the case when δ depends on x, more precisely we put $\delta := M-x$, $M>0$, and the corresponding operators will be briefly denoted by L_n^*, $n\in\mathbb{N}$,

$$(L_n^*f)(x) = \sum_{k=0}^{[Ma_n]}\phi_{n,k}(x)f\left(\frac{k}{a_n}\right),\quad x\ge 0,\quad f\in\mathcal{F}. \tag{18}$$

We can state and prove

Theorem 3. *Let L_n^*, $n\in\mathbb{N}$, be defined by (18). The operators map*

$C([0,M])$ in $C([0,M])$ and have the property

$$\lim_{n\to\infty}(L_n^*f)(x)=f(x) \text{ for all } f\in C([0,M]),$$

uniformly on every compact $K_M\subset[0,M)$.

Proof. The proof is simple and runs as follows. For every function $f\in C([0,M])$ we introduce the function $f_{(M)}\in C(\mathbb{R}_+)$ given by

$$f_{(M)}(x)=\begin{cases} f(x), & 0\le x\le M,\\ f(M), & x>M.\end{cases}$$

If $x\in[0,M)$ then we have

$$(L_n^*f)(x)=(L_nf_{(M)})(x)-f(M)r_n(x), \text{ where } r_n(x)=\sum_{k=[Ma_n]+1}^{\infty}\phi_{n,k}(x).$$

Since $k/a_n>M$ and $0\le x<M$ one obtains $1<(M-x)^{-2}(k/a_n-x)^2$ and consequently

$$\begin{aligned} r_n(x) &\le (M-x)^{-2}\sum_{\left|\frac{k}{a_n}-x\right|>M-x}\phi_{n,k}(x)\left(\frac{k}{a_n}-x\right)^2\\ &\le (M-x)^{-2}\sum_{k=0}^{\infty}\phi_{n,k}(x)\left(\frac{k}{a_n}-x\right)^2\\ &= (M-x)^{-2}\Lambda_{n,2}(x)\le\frac{1}{(M-x)^2}\frac{C(2,K_M)}{a_n}=o(1)\ (n\to\infty), \end{aligned}$$

uniformly on every compact subinterval $K_M\subset[0,M)$. Above, the last upper bound is based on (13). The proof is finished.

We point out that choosing $M=1$, in the particular case when L_n^* turns into modified Szász operator $S_{n,\delta}$, $\delta(x)=1-x$, Theorem 3 encounters a result obtained in [5; Theorem 5].

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Faculty of Mathematics and Computer Science, Babeş-Bolyai University, str. Kogălniceanu, 1, 3400 Cluj-Napoca, Romania.

E-mail: agratini@math.ubbcluj.ro