BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 31, Number 3, September 2003

ON PROXIMAL LIMITS

BY

D. N. SARKHEL

Abstract. We prove that the general approximate limit process of Saks [7] for an arbitrary function of one real variable is completely equivalent to a path limit process [2], but the proximal limit process of Sarkhel and De [8] is not. We also show by example that, while the approximate Dini derivates of a monotonic function are in fact its Dini derivates [6], even an absolutely continuous monotonic function can have a finite proximal derivative at a point without having an ordinary derivative there. Further, we obtain a Cauchy criterion for the existence of a finite proximal limit, and prove a monotonicity theorem using the notion of proximal limit alone.

1. Introduction and preliminaries. Generalizing the notion of approximate limit of a function [7, p.218], Sarkhel and De [8, §4] introduced the notion of proximal limit which has important applications in differentiation and integration theories [1, 8, 9]. The definition of proximal limit is based on the notion of a sparse set at a point [8, §3], which generalizes the notion of dispersion point of a set. Filipczak [3] studied an interesting abstract category analogue of sparse sets. The purpose of the present paper is embodied in the abstract, which also suggests that the title of the paper might as well be *proximal limit versus approximate limit.*

By a set E we shall mean a subset of the real line $\boldsymbol{R}$, $|E|$ will denote

---

Received by the editors December 31, 2001.

AMS 2000 Subject Classification: 26A03, 26A24, 26A48.

Key words and phrases: Proximal limit, approximate limit, path limit, sparse set, monotonic function, proximal derivative.

its outer Lebesgue measure and $d^+(E,c)$, $d_+(E,c)$ [$d^-(E,c)$, $d_-(E,c)$] will denote its upper and lower right [left] densities at a point $c \in \boldsymbol{R}$. Thus

$$d^+(E,c) = \limsup_{x \to c^+} \frac{|E \cap [c,x]|}{x-c}, \quad d_+(E,c) = \liminf_{x \to c^+} \frac{|E \cap [c,x]|}{x-c}.$$

Definition 1.1[8]. The set E is said to be sparse at c on the right, written $E \in S(c^+)$, if for each $\varepsilon > 0$ there exists $h > 0$ so that every interval $(a,b) \subseteq (c, c+h)$, with $a - c < h(b-c)$, contains at least one point x such that $|E \cap [c,x]| < \varepsilon(x-c)$.

The main theorem concerning sparseness is the following.

Theorem 1.2 [8, Theorem 3.1]. *The following conditions are equivalent.*

(i) *The set E is sparse at c on the right.*
(ii) *For each $A \subset \boldsymbol{R}$, $d^+(A,c) < 1$ implies $d^+(E \cup A, c) < 1$.*
(iii) *For each $A \subset \boldsymbol{R}$, $d^+(A,c) < 1$ and $d_+(A,c) = 0$ together implies $d^+(E \cup A, c) < 1$ and $d_+(E \cup A, c) = 0$.*
(iv) *For each $A \subset \boldsymbol{R}$, $d_+(A,c) = 0$ implies $d_+(E \cup A, c) = 0$.*

A simple consequence of this theorem is [8, Corollary 3.1.1]:

Theorem 1.3. *If $E \in S(c^+)$, then $d^+(E,c) < 1$ and $d_+(E,c) = 0$ and every subset of E belongs to $S(c^+)$. If E, $F \in S(c^+)$, then $E \cup F \in S(c^+)$. If $d^+(E,c) = 0$, then $E \in S(c^+)$.*

There are similar definition, notation and results on the left of c. In the sequel we shall need the following two lemmas.

Lemma 1.4. *Given a measurable set $M \subseteq R$ and a point $c \in R$, there is a closed set $F \subseteq M \cup \{c\}$ with $c \in F$ so that $d^+(M \backslash F, c) = 0$, $d^+(F,c) = d^+(M,c)$ and $d_+(F,c) = d_+(M,c)$.*

Proof. Choose a strictly decreasing sequence $c_n \to c$. Since the set M is measurable, there are closed sets $F_n \subseteq \boldsymbol{M} \cap (c_{n+1}, c_n]$ with

$$|M \cap (c_{n+1}, c_n] \backslash F_n| < \frac{1}{n}(c_{n+1} - c_{n+2}), \quad n = 1, 2, \ldots.$$

Clearly the set $F = \{c\} \cup \cup_{n=1}^{\infty} F_n$ is closed, $F \subseteq M \cup \{c\}$ and $c \in F$.

Now, for all n and all $x \in (c_{n+1}, c_n]$ we have

$$\begin{aligned} |(M\backslash F) \cap [c, x]| &\leq \sum_{i=n}^{\infty} |M \cap (c_{i+1}, c_i] \backslash F_i| \\ &< \sum_{i=n}^{\infty} \frac{1}{i}(c_{i+1} - c_{i+2}) \\ &< \frac{1}{n}(x - c). \end{aligned}$$

Hence plainly $d^+(M\backslash F, c) = 0$. The rest follows from the relation

$$\frac{|M \cap [c, x]|}{x - c} = \frac{|F \cap [c, x]|}{x - c} + \frac{|(M\backslash F) \cap [c, x]|}{x - c}.$$

Lemma 1.5. (Sarkhel and De [8, lemma 2.3]). *Let $A \subseteq [a, b]$ be such that $a \in A$, $d^-(A, y) < 1$ for all $y \in B = [a, b]\backslash A$ and $d^+(B, x) < 1$ for all $x \in A$. Then $B = \phi$.*

This lemma proved useful in [8] and subsequently Filipczak [4] found further applications. This has a counterpart.

Lemma 1.6. *Let $A \subseteq [a, b]$ be such that $b \in A$, $d^+(A, y) < 1$ for all $y \in B = [a, b]\backslash A$, and $d^-(B, x) < 1$ for all $x \in A$. Then $B = \phi$.*

Proof. This can be proved directly as Lemma 1.5. But we can deduce it from Lemma 1.5 by reflections in the origin as follows. We have $(-A) \subseteq [-b, -a]$, $-b \in (-A)$, $d^-(-A, y) = d^+(A, -y) < 1$ for all $y \in (-B) =$

$[-b, -a] \backslash (-A)$ and $d^+(-B, x) = d^-(B, -x) < 1$ for all $x \in (-A)$. Hence by Lemma 1.5, $-B = \phi$ and so $B = \phi$.

In what follows we consider an arbitrary function $f : \boldsymbol{R} \to \boldsymbol{R}$.

2. Proximal limit versus approximate limit. In the literature there are three different definitions of approximate limit which are in fact equivalent, but we are unable to trace back any ready reference of a neat and complete proof of this. In this section we make a more detailed study of this and show that the proximal analogue of this is not true. We also show that the proximal process is effective even for derivatives of monotonic functions, while the approximate process is not.

Definition 2.1 [7, p.220]. The right upper [lower] approximate limit of f at the point c, denoted by $A^+f(c)[A_+f(c)]$, is the infimum [supremum] of the extended real numbers r for which $d^+(E, c) = 0$, where

$$E = \{x : f(x) > r\} \, [E = \{x : f(x) < r\}].$$

If $A^+f(c) = A_+f(c)$, then this common value is called the right approximate limit of f at c and is denoted by $f_{ap(c^+)}$. There are similar definitions and notations on the left of c.

Definition 2.2 [8, p.33]. The right upper [lower] proximal limt of f at the point c, denoted by $P^+f(c)[P_+f(c)]$, is the infimum [supremum] of the extended real numbers r for which $E \in S(c^+)$, where

$$E = \{x : f(x) > r\} \, [E = \{x : f(x) < r\}].$$

If $P^+f(c) = P_+f(c)$, then this common value is called the right proximal limit of f at c and is denoted by $f_{pr}(c_+)$. There are similar definitions and notations on the left of c.

Note. We always have $A_+f(c) \le P_+f(c) \le P^+f(c) \le A^+f(c)$.

Theorem 2.3. *There is a closed set F with $d_+(F,c)=1$ so that*

$$A_+f(c) = \liminf_{\substack{x\to c^+\\ x\in F}} f(x) \le \limsup_{\substack{x\to c^+\\ x\in F}} f(x) = A^+f(c).$$

Proof. If $A^+f(c) < +\infty$, choose a strictly decreasing sequence $r_n \to A^+f(c)$; but if $A^+f(c) = +\infty$ then let $r_n = +\infty$ for all n. If $A_+f(c) > -\infty$, choose a strictly increasing sequence $s_n \to A_+f(c)$; but if $A_+f(c) = -\infty$ then let $s_n = -\infty$ for all n. Then for all n we have $d^+(E_n,c) = d^+(F_n,c) = 0$, where

$$E_n = \{x : f(x) > r_n\}, \quad F_n = \{x : f(x) < s_n\}.$$

Then there is a strictly decreasing sequence $c_n \to c$ so that

$$|(E_n \cup F_n) \cap [c,x]| < \frac{1}{n}(x-c) \quad \text{for all} \quad x \in (c, c_n]$$

and $c_n - c < 2(c_n - c_{n+1}), \quad n = 1, 2, \ldots$. Put

$$E = \cup_{n=1}^{\infty}((E_n \cup F_n) \cap (c_{n+1}, c_n]).$$

Then for all n and all $x \in (c_{n+1}, c_n]$ we have

$$\begin{aligned}
|E \cap [c,x]| &\le \sum_{i=n+1}^{\infty} |(E_i \cup F_i) \cap [c, c_i]| + |(E_n \cup F_n) \cap [c,x]| \\
&< \sum_{i=n+1}^{\infty} \frac{1}{i}(c_i - c) + \frac{1}{n}(x-c) \\
&< \frac{1}{n} \sum_{i=n+1}^{\infty} 2(c_i - c_{i+1}) + \frac{1}{n}(x-c) \\
&< \frac{3}{n}(x-c).
\end{aligned}$$

Hence it follows at once that $d^+(E,c) = 0$.

Now, taking a measurable cover H of E we have $d^+(H,c) = d^+(E,c) = 0$

[8, pp.27-28]. Then the set $M = \boldsymbol{R}\backslash H$ is measurable and $d_+(M, c) = 1$ [8, p.27]. Also, since $H \supseteq E$ so that $M \subseteq R\backslash E$, clearly $s_n \le f(x) \le r_n$ for all $x \in M \cap (c, c_n)$, $n = 1, 2, \ldots$. Since $\lim r_n = A^+ f(c)$ and $\lim s_n = A_+ f(c)$, it follows that

$$A_+ f(c) \le \liminf_{\substack{x \to c^+ \\ x \in M}} f(x) \le \limsup_{\substack{x \to c^+ \\ x \in M}} f(x) \le A^+ f(c).$$

By Lemma 1.4, there is a closed set $F \subseteq M \cup \{c\}$ such that $d_+(F, c) = d_+(M, c) = 1$. Since $F \backslash \{c\} \subseteq M$, it follows from above that

$$A_+ f(c) \le \liminf_{\substack{x \to c^+ \\ x \in F}} f(x) \quad \text{and} \quad \limsup_{\substack{x \to c^+ \\ x \in F}} f(x) \le A^+ f(c).$$

Since $d^+(R\backslash F, c) = 0$, obviously both these inequalities must be equalities and this completes the proof.

The following immediate corollary shows the equivalence of three familiar definitions of unique approximate limit.

Corollary 2.4. *The following conditions are equivalent.*

(i) $f_{ap}(c^+)$ *exists and has the value* $\alpha \in [-\infty, +\infty]$.

(ii) $\lim_{\substack{x \to c^+ \\ x \in F}} f(x) = \alpha$ *for some closed set* F *with* $d_+(F, c) = 1$.

(iii) $\lim_{\substack{x \to c^+ \\ x \in E}} f(x) = \alpha$ *for some measurable set* E *with* $d_+(E, c) = 1$.

Example 2.5. We now show the great generality of the proximal limit process by showing that the proximal analogue of neither Theorem 2.3. nor even of its Corollary 2.4 is true. By [8, Example 3.1], for each positive integer n there is an open set $G_n \subset (0, +\infty)$ with $d^+(G_n, 0) > 1 - 1/n$ such that $G_n \in S(0+)$. Define

$$\begin{aligned} g(x) &= \frac{1}{n} \quad \text{if} \quad x \in G_n \backslash \bigcup_{1 \le i < n} G_i, \quad n = 1, 2, \ldots, \\ &= 0 \quad \text{if} \quad x \in \boldsymbol{R} \backslash \bigcup_{n=1}^{\infty} G_n. \end{aligned}$$

Note that $g \geq 0$ and $g(0) = 0$. Given $\varepsilon > 0$, put

$$A = \{x : |g(x) - g(0)| > \varepsilon\} = \{x : g(x) > \varepsilon\}.$$

Clearly $A \subseteq \bigcup_{i=1}^{m} G_i$ where m is an integer exceeding $1/\varepsilon$. So, by Theorem 1.3, $A \in S(0+)$. Hence $g_{pr}(0+) = 0$. The existence of a function like g follows also from [3, Theorem 15(a), p.164].

We show, however, that there is no $E \in S(0+)$ so that $g(x) \to 0$ as $x \to 0+$ over $\boldsymbol{R}\backslash E$. Suppose such E exists, then by Theorem 1.3 $d^+(E, 0) < 1$ and we shall derive a contradiction from this. Choose a positive integer k so that $d^+(E, 0) < 1 - 2/k$. Since $g(x) \to 0$ as $x \to 0+$ over $\boldsymbol{R}\backslash E$, there is $h > 0$ so that $g(x) < 1/k$ for all $x \in (0, h)\backslash E$. Then $G_k \cap (0, h) \subseteq E$, by definition of g, and so $d^+(G_k, 0) \leq d^+(E, 0)$, yielding $1 - 1/k < 1 - 2/k$, which is the desired contradiction.

However, here is a partial proximal analogue of Theorem 2.3.

Theorem 2.6. *There is a closed set F with $d^+(F, c) = 1$ so that*

$$P_+ f(c) \leq \liminf_{\substack{x \to c^+ \\ x \in F}} f(x) \leq \limsup_{\substack{x \to c^+ \\ x \in F}} f(x) \leq P^+ f(c).$$

Consequently, if $f_{pr}(c^+)$ exists then $\lim_{\substack{x \to c^+ \\ x \in F}} f(x) = f_{pr}(c^+)$.

Proof. If $P^+ f(c) < +\infty$, choose a strictly decreasing sequence $r_n \to P^+ f(c)$; but if $P^+ f(c) = +\infty$ then let $r_n = +\infty$ for all n. If $P_+ f(c) > -\infty$, choose a strictly increasing sequence $s_n \to P_+ f(c)$; but if $P_+ f(c) = -\infty$ then let $s_n = -\infty$ for all n. Then E_n, $F_n \in S(c^+)$, where

$$E_n = \{x : f(x) > r_n\}, \quad F_n = \{x : f(x) < s_n\}.$$

So, by Theorem 1.3, $E_n \cup F_n \in S(c^+)$ and, hence, further $d_+(E_n \cup F_n, c) = 0$.

Then there is a strictly decreasing sequence $c_n \to c$ so that

$$\frac{|(E_n \cup F_n) \cap [c, c_n]|}{c_n - c} < \frac{1}{n} \quad \text{and} \quad \frac{c_n - c}{c_n - c_{n+1}} < 2, \quad n = 1, 2 \ldots .$$

Put $E = \bigcup_{n=1}^{\infty}((E_n \cup F_n) \cap (c_{n+1}, c_n])$. Then for all n we have

$$\begin{aligned} |E \cap [c, c_n]| &\leq \sum_{i=n}^{\infty} |(E_i \cup F_i) \cap [c, c_i]| \\ &< \sum_{i=n}^{\infty} \frac{1}{i}(c_i - c) \\ &< \frac{1}{n} \sum_{i=n}^{\infty} 2(c_i - c_{i+1}) \\ &= \frac{2}{n}(c_n - c). \end{aligned}$$

Hence it follows at once that $d_+(E, c) = 0$.

Now, taking a measurable cover H of E, we have $d_+(H, c) = d_+(E, c) = 0$. Then the set $M = \boldsymbol{R} \backslash H$ is measurable and $d^+(M, c) = 1[8, p.27]$. Also, since $H \supseteq E$ so that $M \subseteq \boldsymbol{R} \backslash E$, clearly $s_n \leq f(x) \leq r_n$ for all $x \in M \cap (c, c_n)$, $n = 1, 2, \ldots$. Since lim $r_n = P^+ f(c)$ and lim $s_n = P_+ f(c)$, it follows that

$$P_+ f(c) \leq \liminf_{\substack{x \to c^+ \\ x \in M}} f(x) \leq \limsup_{\substack{x \to c^+ \\ x \in M}} f(x) \leq P^+ f(c).$$

By Lemma 1.4, there is a closed set $F \subseteq M \cup \{c\}$ scuh that $d^+(F, c) = d^+(M, c) = 1$. Since $F \backslash \{c\} \subseteq M$, it follows from above that

$$P_+ f(c) \leq \liminf_{\substack{x \to c^+ \\ x \in F}} f(x) \leq \limsup_{\substack{x \to c^+ \\ x \in F}} f(x) \leq P^+ f(c),$$

which proves the theorem together with its consequence.

Khintchine [5, p.242] showed that if a monotonic function has a finite approximate derivative at a point, then it has actually an ordinary derivative at that point. Subsequently, Mišik [6] discovered that the approximate Dini derivates and the Dini derivates of a monotonic function are in fact identical in corresponding pairs. For further elaborations on this point, see [2, p.106] and [10, §65].

In sharp contrast to this, we show that even an absolutely continuous monotonic function can have a finite proximal derivative at a point without having an ordinary derivative there.

Given a function F, let $f(x) = (F(x) - F(c))/(x - c)$ for $x \neq c$. If $f_{pr}(c^+)$ and $f_{pr}(c^-)$ exist and are equal, then this common value is called the proximal derivative, PDF(c), of F at c.

Example 2.7. By [8, Example 3.1.], there is a set $A \subset (0, +\infty)$ such that $A \in S(0+)$ but $d^+(A, 0) > 0$. Consider the measure function $F(x) = |A \cap (-\infty, x]|$. We will show that PDF$(0) = 0$. Given $\eta > 0$, put

$$E = \left\{ x : x > 0 \quad \text{and} \quad \left| \frac{F(x) - F(0)}{x - 0} \right| = \frac{|A \cap [0, x]|}{x} > \eta \right\}.$$

If 0 is not a limit point of E on the right, then plainly $E \in S(0+)$. Suppose now 0 is a limit point of E on the right. For any $x > 0$, let $x' = \sup E \cap [0, x]$. Then $0 < x' \leq x$, and, hence, by definitions of E and x',

$$|A \cap [0, x]| \geq |A \cap [0, x']| \geq \eta x' \geq \eta |E \cap [0, x']| = \eta |E \cap [0, x]|.$$

Now, since $A \in S(0+)$, given $\varepsilon > 0$ there is $h > 0$ so that each interval $(a, b) \subseteq (0, h)$, with $a < hb$, contains at least one point x such that $|A \cap [0, x]| < \eta \varepsilon x$, which by above gives that $|E \cap [0, x]| < \varepsilon x$. This shows that $E \in S(0+)$.

Hence, noting that $F(x) = 0$ for $x \leq 0$, we have PDF$(0) = 0$. Thus, clearly though F is absolutely continuous and nondecreasing, we find that it has proximal derivative 0 at the origin and yet the ordinary derivative does not exist there, as $D^+F(0) = d^+(A, 0) > 0 = d_+(A, 0) = D_+F(0)$.

3. Cauchy criterion and monotonicity criterion

Theorem 3.1. *The right proximal limit $f_{pr(c^+)}$ exists finitely if and only if, for each $\varepsilon > 0$, there is a point $b > c$ so that the set $\{x : |f(x) - f(b)| > \varepsilon\}$ is sparse at c on the right.*

Note. The result is also true for limits with respect to local systems of Thomson [10].

Proof. Assuming the stated condition, both the sets

$$\{x : f(x) < f(b) - \varepsilon\} \quad \text{and} \quad \{x : f(x) > f(b) + \varepsilon\}$$

are sparse at c on the right. Therefore,

$$-\infty < f(b) - \varepsilon \leq P_+ f(c) \leq P^+ f(c) \leq f(b) + \varepsilon < +\infty.$$

Thus $0 \leq P^+ f(c) - P_+ f(c) \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $P^+ f(c) = P_+ f(c) \neq \pm\infty$. Hence $f_{pr}(c^+)$ exists finitely.

Conversely, assume that $f_{pr}(c^+)$ exists and is a finite number α. Then for each $\varepsilon > 0$ both the sets

$$\{x : f(x) > \alpha + \varepsilon/2\} \quad \text{and} \quad \{x : f(x) < \alpha - \varepsilon/2\}$$

are sparse at c on the right. So by Theorem 1.3 the set

$$\{x : |f(x) - \alpha| > \varepsilon/2\}$$

is sparse at c on the right. Certainly then there exists a point $b > c$ satisfying $|f(b) - \alpha| \leq \varepsilon/2$. Since $|f(x) - f(b)| > \varepsilon$ implies

$$|f(x) - \alpha| \geq |f(x) - f(b)| - |f(b) - \alpha| > \varepsilon - \varepsilon/2 = \varepsilon/2,$$

it follows that the set $\{x : |f(x) - f(b)| > \varepsilon\}$ is sparse at c on the right, and this completes the proof.

The monotonicity theorem below greatly extends Theorem 4.3 of [8]. Here we need to use the full strength of Theorem 1.2.

Theorem 3.2. *Suppose $P^- f(x) \leq f(x) \leq P_+ f(x)$ for all x, and, furthere, $f(D)$ has void interior when*

$$D = \{x : d^-(A_{\varepsilon,x}, x) = d^+(B_{\varepsilon,x}, x) = 1 \quad \textit{for all} \quad \varepsilon > 0\},$$

where $A_{\varepsilon,x} = f^{-1}([f(x), f(x) + \varepsilon])$, $B_{\varepsilon,x} = f^{-1}([f(x) - \varepsilon, f(x)])$.

Then the function f is nondecreasing.

Proof. Suppose, for a contradiction, that $f(b) < f(a)$ for some $a < b$. Then, since $f(D)$ has void interior, we can find $r \notin f(D)$ so that $f(b) < r < f(a)$. Put $B = [a, b] \backslash A$, where A denotes the set of points x of $[a, b]$ such that either $f(x) > r$ or, else, $f(x) = r$ and $d^+(B_{\varepsilon,x}, x) < 1$ for some $\varepsilon > 0$. We will verify the hypotheses of Lemma 1.5.

Consider any $y \in B$. If $f(y) = r$, then the condition $r \notin f(D)$ gives that $y \notin D$, and the condition $y \notin A$ gives that $d^+(B_{\varepsilon,y}, y) = 1$ for all $\varepsilon > 0$, which jointly implies that for some $\varepsilon > 0$, $\varepsilon = \eta$ say, $d^-(A_{\eta,y}, y) < 1$. But, since $f(y) = r$ so

$$A \subseteq E \cup A_{\eta,y} \quad \text{where} \quad E = \{t : f(t) > f(y) + \eta\}.$$

Since $P^- f(y) \leq f(y)$, so $E \in S(y^-)$. Hence by the left hand analogue of Theorem 1.2 $d^-(A, y) \leq d^-(E \cup A_{\eta,y}, y) < 1$.

If $f(y) < r$, then $P^- f(y) \leq f(y) < r$ and so $A \in S(y^-)$ because $f(x) \geq r$ for all $x \in A$. So again $d^-(A, y) < 1$, by the left hand analogue of Theorem 1.3.

Consider next any $x \in A$. If $f(x) = r$, then $d^+(B_{\varepsilon,x}, x) < 1$ for some $\varepsilon > 0$. But, since $f(x) = r$ so

$$B \subseteq F \cup B_{\varepsilon,x} \quad \text{where} \quad F = \{t : f(t) < f(x) - \varepsilon\}.$$

Since $P_+f(x) \geq f(x)$, so $F \in S(x^+)$. Hence by Theorem 1.2 $d^+(B,x) \leq d^+(F \cup B_{\varepsilon,x}, x) < 1$.

If $f(x) > r$, then $P_+f(x) \geq f(x) > r$ and so $B \in S(x^+)$ because $f(y) \leq r$ for all $y \in B$. So again $d^+(B,x) < 1$ by Theorem 1.3.

Since $a \in A$ and $b \in B$, we thus arrive at a contradiction to Lemma 1.5, which in fact proves the theorem.

The author wishes to thank the referee for his suggestions.

References

1. M. K. Bose and R. K. Nath, *The proximal Cesàro-Perron integral,* Math. Japonica, **30**(3) (1985), 357-369.

2. A. M. Bruckner, R. J. O'Malley and B. S. Thomson, *Path derivatives: a unified view of certain generalized derivatives,* Trans. Amer. Math. Soc., **283**(1) (1984), 97-125.

3. T. Filipczak, *On some abstract density topologies,* Real Analysis Exchange, **14**(1) (1988-89), 140-166.

4. T. Filipczak, *Monotonicity theorems for some local systems,* Real Analysis Exchange, **19**(1) (1993-94), 114-120.

5. A. Khintchine, *Recherches sur la structure des fonctions mesurables,* Fund. Math., **9**(1927), 212-279.

6. L. Mišik, *Über approximative derivierte Zahlen monotoner Funktionen,* Czechoslovak Math. J., **26**(101) (4) (1976), 579-583.

7. S. Saks, *Theory of the Integral,* Dover, New York, 1964.

8. D. N. Sarkhel and A. K. De, *The proximally continuous integrals,* J. Austral. Math. Soc. (Series A), **31**(1981), 26-45.

9. D. N. Sarkhel and A. B. Kar, (PVB) *functions and integration,* J. Austral. Math. Soc. (Series A), **36** (1984), 335-353.

10. B. S. Thomson, *Real Functions,* Lecture Notes in Math., **1170**, Springer-Verlag, 1985.

Department of Mathematics, University of Kalyani, Kalyani−741235, West Bengal, India.