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EXACT LAWS FOR SUMS OF ORDER STATISTICS FROM THE PARETO DISTRIBUTION

BY

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Abstract. Consider independent and identically distributed random variables $\{X, X_{nj}, 1 \leq j \leq m, n \geq 1\}$ with density $f(x) = px^{-p-1}I(x \geq 1)$, where $p > 0$. We select the k^{th} order statistic from each row $1 \leq k \leq m$. Then we test to see whether or not Laws of Large Numbers with nonzero limits exist.

1. Introduction. In this paper we observe weighted sums of order statistics taken from small samples. We look at m observations from the Pareto distribution, i.e., $f(x) = px^{-p-1}I(x \geq 1)$. Then we select the k^{th} order statistic, $1 \leq k \leq m$. We denote these order statistics as $X_{j(k)}$, which will have the density

$$f_{j(k)}(x) = \frac{p \cdot m!}{(m-k)!(k-1)!}(1-x^{-p})^{k-1}x^{-p(m-k+1)}I(x \geq 1).$$

Our goal is to determine whether or not there exist positive constants a_j and b_n such the $\sum_{j=1}^{n} a_j X_{j(k)}/b_n$ converges to a nonzero constant. These are called Exact Laws of Large Numbers since they create a fair game situation where the $a_n X_{n(k)}$ represent the amount a player wins on the n^{th} play of

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some game and $b_n - b_{n-1}$ represents the corresponding fair entrance fee for the participant.

As usual, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. We use the constant C to denote a generic real number that is not necessarily the same in each appearance.

2. The nonexistence of exact weak laws when $p(m-k+1)<1$. Under mild conditions we show that even Weak Laws with nonzero limits cannot exist.

Theorem 1. *If $p(m-k+1) < 1$ and $\max_{1\le j\le n} a_j = o(b_n)$, then an Exact Weak Law cannot hold.*

Proof. We claim that the only finite limit of $\sum_{j=1}^n a_j X_{j(k)}/b_n$ is zero. Assume that a Weak Law does hold. Then since there are only a finite number of terms in each row

$$\max_{1\le j\le n} \operatorname{med}\left\{\frac{a_j X_{j(k)}}{b_n}\right\} = \frac{\operatorname{med}\{X_{j(k)}\} \max\limits_{1\le j\le n} a_j}{b_n} \to 0.$$

Thus, by the Degenerate Convergence Theorem, which can be found on page 356 of [1], we can claim that

$$\frac{\sum\limits_{j=1}^n a_j^{p(m-k+1)}}{b_n^{p(m-k+1)}} \to 0.$$

We take care of the case $k = 1$ first. From the Weak Law we have

$$0 \leftarrow \sum_{j=1}^n P\{X_{j(k)} > b_n/a_j\} = \sum_{j=1}^n pm \int_{b_n/a_j}^\infty x^{-pm-1}dx = \sum_{j=1}^n \frac{a_j^{pm}}{b_n^{pm}}.$$

On the other hand if $k \geq 2$, then

$$
\begin{aligned}
0 &\leftarrow \sum_{j=1}^{n} P\{X_{j(k)} > b_n/a_j\} \\
&= \sum_{j=1}^{n} \frac{p\cdot m!}{(m-k)!(k-1)!}\int_{b_n/a_j}^{\infty}(1-x^{-p})^{k-1}x^{-p(m-k+1)-1}dx \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\sum_{j=1}^{n}\int_{b_n/a_j}^{\infty}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i x^{-p(m-k+i+1)-1}dx \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\sum_{j=1}^{n}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i\int_{b_n/a_j}^{\infty}x^{-p(m-k+i+1)-1}dx \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\sum_{j=1}^{n}\sum_{i=0}^{k-1}\frac{\binom{k-1}{i}(-1)^i(b_n/a_j)^{-p(m-k+i+1)}}{p(m-k+i+1)} \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\frac{\sum_{j=1}^{n}a_j^{p(m-k+1)}}{b_n^{p(m-k+1)}}\sum_{i=0}^{k-1}\frac{\binom{k-1}{i}(-1)^i a_j^{pi}}{p(m-k+i+1)b_n^{pi}} \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\frac{\sum_{j=1}^{n}a_j^{p(m-k+1)}}{b_n^{p(m-k+1)}}\Big[\frac{1}{p(m-k+1)} \\
&\quad +\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^i a_j^{pi}}{p(m-k+i+1)b_n^{pi}}\Big] \\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\frac{\sum_{j=1}^{n}a_j^{p(m-k+1)}}{b_n^{p(m-k+1)}}\Big[\frac{1}{p(m-k+1)}+o(1)\Big]
\end{aligned}
$$

implying that $\sum_{j=1}^{n} a_j^{p(m-k+1)} = o(b_n^{p(m-k+1)})$ since $\max_{1\leq j\leq n} a_j = o(b_n)$.

Next, we investigate the truncated mean. We have

$$
\begin{aligned}
0 &< \sum_{j=1}^{n}\frac{a_j}{b_n}EX_{j(k)}I(X_{j(k)} < b_n/a_j) \\
&= \sum_{j=1}^{n}\frac{p\cdot m!a_j}{(k-1)!(m-k)!b_n}\int_{1}^{b_n/a_j}(1-x^{-p})^{k-1}x^{-p(m-k+1)}dx \\
&< \frac{C}{b_n}\sum_{j=1}^{n}a_j\int_{1}^{b_n/a_j}x^{-p(m-k+1)}dx \\
&< \frac{C}{b_n}\sum_{j=1}^{n}a_j(b_n/a_j)^{-p(m-k+1)+1}
\end{aligned}
$$

$$= \frac{C\sum_{j=1}^{n} a_j^{p(m-k+1)}}{b_n^{p(m-k+1)}}$$
$$\to 0.$$

Thus the only Weak Law we can have is one with a zero limit.

3. Unusual strong laws when $p(m-k+1)=1$.

In this situation we can get an Exact Strong Law, but only if we select our coefficients and norming sequences properly.

Theorem 2. *If $p(m-k+1)=1$, then for all $\beta>0$ we have*

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}\frac{(\lg j)^{\beta-2}}{j}X_{j(k)}}{(\lg n)^{\beta}}=\frac{p\cdot m!}{\beta(k-1)!(m-k)!}\ \textit{almost surely.}$$

Proof. Let $a_n=(\lg n)^{\beta-2}/n$, $b_n=(\lg n)^{\beta}$ and $c_n=b_n/a_n=n(\lg n)^2$. We use the partition

$$\begin{aligned}\frac{1}{b_n}\sum_{j=1}^{n}a_jX_{j(k)} &= \frac{1}{b_n}\sum_{j=1}^{n}a_j[X_{j(k)}I(1\le X_{j(k)}\le c_j)-EX_{j(k)}I(1\le X_{j(k)}\le c_j)]\\ &+ \frac{1}{b_n}\sum_{j=1}^{n}a_jX_{j(k)}I(X_{j(k)}>c_j)\\ &+ \frac{1}{b_n}\sum_{j=1}^{n}a_jEX_{j(k)}I(1\le X_{j(k)}\le c_j).\end{aligned}$$

The first term vanishes almost surely by the Khintchine-Kolmogorov Convergence Theorem, see page 113 of [1], and Kronecker's lemma since

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}EX_{n(k)}^2I(1\le X_{n(k)}\le c_n)<C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}dx<C\sum_{n=1}^{\infty}\frac{1}{c_n}=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.$$

The second term vanishes, with probability one, by the Borel-Cantelli lemma since

$$\sum_{n=1}^{\infty} P\{X_{n(k)} > c_n\} < C\sum_{n=1}^{\infty}\int_{c_n}^{\infty} x^{-2}dx = C\sum_{n=1}^{\infty}\frac{1}{c_n} < \infty.$$

The real work is in obtaining the limit of the third term. Since $c_j \to \infty$

$$\begin{aligned}
& EX_{j(k)}I(1 \le X_{j(k)} \le c_j) \\
= \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\int_1^{c_j}(1-x^{-p})^{k-1}x^{-1}dx \\
= \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i\int_1^{c_j}x^{-pi-1}dx \\
= \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\left[\int_1^{c_j}x^{-1}dx+\sum_{i=1}^{k-1}\binom{k-1}{i}(-1)^i\int_1^{c_j}x^{-pi-1}dx\right] \\
= \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\left[\lg c_j+\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i+1}}{pic_j^{pi}}+\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^i}{pi}\right] \\
\sim \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\lg c_j \\
\sim \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\lg j.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{\sum_{j=1}^n a_j EX_{j(k)}I(1\le X_{j(k)}\le c_j)}{b_n} \\
\sim \; & \frac{p\cdot m!}{(k-1)!(m-k)!}\cdot\frac{\sum_{j=1}^n\frac{(\lg j)^{\beta-1}}{j}}{(\lg n)^\beta}\to\frac{p\cdot m!}{\beta(k-1)!(m-k)!}
\end{aligned}$$

which completes the proof.

4. Typical strong laws when $p(m-k+1)>1$. In this situation we can obtain all kinds of Strong Laws since the first moment does exist. In order to obtain Exact Strong Laws we can define a_n and b_n as any pair of positive sequences as long as $b_n \uparrow \infty$, $\sum_{j=1}^n a_j/b_n \to L$, where $L \neq 0$, and the condition involving $c_n = b_n/a_n$ in each theroem is satisfied. If $L = 0$, then the limit theorems still holds, however the limit is zero, which is not that interesting.

This section is broken down into three cases, each has different conditions as to whether the Strong Law exists. The calculation of $EX_{j(k)}$ follows in the ensuing lemma.

Lemma. *If* $p(m-k+1)>1$, *then*

$$EX_{j(k)}=\frac{m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)}.$$

Proof. Let $Y_{j(k)}=X_{j(k)}^{-p}$. From the density of $X_{j(k)}$ We see that the density of $Y_{j(k)}$ is $\frac{m!}{(m-k)!(k-1)!}y^{m-k}(1-y)^{k-1}I(0<y<1)$, which is the distribution of a $\beta(m-k+1,k)$ random variable. Therefore

$$\begin{aligned}EX_{j(k)} &= EY_{j(k)}^{-1/p}=\frac{m!}{(k-1)!(m-k)!}\int_0^1 y^{-1/p}y^{m-k}(1-y)^{k-1}dy\\ &= \frac{m!}{(k-1)!(m-k)!}\int_0^1 y^{m-k-1/p}(1-y)^{k-1}dy\\ &= \frac{m!}{(k-1)!(m-k)!}\cdot\frac{\Gamma(m-k+1-1/p)\Gamma(k)}{\Gamma(m+1-1/p)}\\ &= \frac{m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)}\end{aligned}$$

which completes the proof of the lemma.

In all three ensuing theorems we use the partition

$$\begin{aligned}\frac{1}{b_n}\sum_{j=1}^n a_jX_{j(k)} &= \frac{1}{b_n}\sum_{j=1}^n a_j[X_{j(k)}I(1\le X_{j(k)}\le c_j)-EX_{j(k)}I(1\le X_{j(k)}\le c_j)]\\ &\quad+\frac{1}{b_n}\sum_{j=1}^n a_jX_{j(k)}I(X_{j(k)}>c_j)\\ &\quad+\frac{1}{b_n}\sum_{j=1}^n a_jEX_{j(k)}I(1\le X_{j(k)}\le c_j)\end{aligned}$$

where the selection of a_n, b_n and $c_n = b_n/a_n$ must satisfy the assumption of each theorem. These three hypotheses are slightly different and are dependent on how large a first moment the random variable $X_{j(k)}$ possesses. The difference in the these theorems is the condition involving the sequence c_n.

Theorem 3. *If* $1 < p(m-k+1) < 2$ *and* $\sum_{n=1}^{\infty} c_n^{-p(m-k+1)} < \infty$, *then*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_{j(k)}}{b_n} = \frac{L \cdot m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)} \quad \textit{almost surely.}$$

Proof. The first term in our partition goes to zero, with probability one, since

$$\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_{n(k)}^2 I(1 \le X_{n(k)} \le c_n) < C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} x^{-p(m-k+1)+1} dx$$
$$< C \sum_{n=1}^{\infty} \frac{1}{c_n^2} c_n^{-p(m-k+1)+2} = C \sum_{n=1}^{\infty} c_n^{-p(m-k+1)} < \infty.$$

As for the second term

$$\sum_{n=1}^{\infty} P\{X_{n(k)} > c_n\} < C \sum_{n=1}^{\infty} \int_{c_n}^{\infty} x^{-p(m-k+1)-1} dx < C \sum_{n=1}^{\infty} c_n^{-p(m-k+1)} < \infty.$$

Then, from our lemma and $\sum_{j=1}^{n} a_j \sim L b_n$ we have

$$\frac{\sum_{j=1}^{n} a_j E X_{j(k)} I(1 \le X_{j(k)} \le c_j)}{b_n} \to \frac{L \cdot m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)}$$

which completes this proof.

Theorem 4. *If* $p(m-k+1) = 2$ *and* $\sum_{n=1}^{\infty} \lg(c_n)/c_n^2 < \infty$, *then*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_{j(k)}}{b_n} = \frac{L \cdot m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)} \quad \textit{almost surely.}$$

Proof. The first term goes to zero, almost surely, since

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}EX_{n(k)}^2 I(1\le X_{n(k)}\le c_n) < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}x^{-1}dx = C\sum_{n=1}^{\infty}\frac{\lg c_n}{c_n^2}<\infty.$$

Likewise, the second term disappears, with probability one, since

$$\sum_{n=1}^{\infty}P\{X_{n(k)}>c_n\} < C\sum_{n=1}^{\infty}\int_{c_n}^{\infty}x^{-3}dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^2} < C\sum_{n=1}^{\infty}\frac{\lg c_n}{c_n^2}<\infty.$$

As in the last proof, the calculation for the truncated mean is exactly the same, which leads us to the same limit.

Theorem 5. *If* $p(m-k+1)>2$ *and* $\sum_{n=1}^{\infty}c_n^{-2}<\infty$, *then*

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}a_jX_{j(k)}}{b_n}=\frac{L\cdot m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)} \quad \textit{almost surely.}$$

Proof. The first term goes to zero, with probability one, since

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{c_n^2}EX_{n(k)}^2 I(1\le X_{n(k)}\le c_n)\\ <\ & C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}x^{-p(m-k+1)+1}dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}<\infty.\end{aligned}$$

As for the second term

$$\begin{aligned}&\sum_{n=1}^{\infty}P\{X_{n(k)}>c_n\}\\ <\ & C\sum_{n=1}^{\infty}\int_{c_n}^{\infty}x^{-p(m-k+1)-1}dx < C\sum_{n=1}^{\infty}c_n^{-p(m-k+1)} < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}<\infty.\end{aligned}$$

Then as in the last two theorems

$$\frac{\sum_{j=1}^{n}a_jEX_{j(k)}I(1\le X_{j(k)}\le c_j)}{b_n}\to\frac{L\cdot m!\Gamma(m-k+1-1/p)}{(m-k)!\Gamma(m+1-1/p)}$$

which completes this proof.

Clearly, in all of these three theorems the situation of $a_n = 1$ and $b_n = n = c_n$ is easily satisfied. Whenever $p(m-k+1) > 1$ we have tremendous freedom in selecting our constants. That is certainly not true when $p(m-k+1) = 1$.

5. More unusual results when $p(m-k+1)=1$. We saw in Section 3 that unusual results occur when $p(m-k+1) = 1$. In order to establish an Exact Strong Law when $p(m-k+1) = 1$ one is forced to set a_n to be some slowly varying function divided by n, while b_n must also be slowly varying. If one wants to try more conventional constants such as $a_n = 1$ and $b_n = n$ we will have to set our sights a bit lower and settle for Exact Weak Laws. The Weak Law can be found in Theorem 6. Then we use that Weak Law to obtain the almost sure behavior of our normalized partial sums. That result is known as a Generalized Law of the Iterated Logarithm.

Theorem 6. *If* $p(m-k+1) = 1$ *and* $\alpha > -1$, *then*

$$\frac{\sum_{j=1}^n j^\alpha X_{j(k)}}{n^{\alpha+1}\lg n} \xrightarrow{P} \frac{p \cdot m!}{(\alpha+1)(k-1)!(m-k)!}.$$

Proof. This proof is a consequence of the Degenerate Convergence Theorem. As usual, set $a_j = j^\alpha$ and $b_n = n^{\alpha+1}\lg n$. Thus

$$\sum_{j=1}^n P\{X_{j(k)} > \frac{b_n}{a_j}\} < C\sum_{j=1}^n \int_{b_n/a_j}^\infty x^{-2}dx < \frac{C}{b_n}\sum_{j=1}^n a_j = \frac{C}{n^{\alpha+1}\lg n}\sum_{j=1}^n j^\alpha < \frac{C}{\lg n} \to 0$$

and

$$\sum_{j=1}^n \mathrm{Var}\left(\frac{a_j}{b_n} X_{j(k)} I(1 \le X_{j(k)} \le b_n/a_j)\right) < C\sum_{j=1}^n \frac{a_j^2}{b_n^2}\int_1^{b_n/a_j} dx < \frac{C}{b_n}\sum_{j=1}^n a_j \to 0.$$

As for our truncated expectation, using Theorem 1 from [2], page 281, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{a_j}{b_n}EX_{j(k)}I(1\le X_{j(k)}\le b_n/a_j)\\
=&\sum_{j=1}^{n}\frac{p\cdot m!a_j}{(k-1)!(m-k)!b_n}\int_1^{b_n/a_j}(1-x^{-p})^{k-1}x^{-1}dx\\
=&\frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^{n}a_j\int_1^{b_n/a_j}(1-x^{-p})^{k-1}x^{-1}dx\\
=&\frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^{n}a_j\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i\int_1^{b_n/a_j}x^{-pi-1}dx\\
=&\frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^{n}a_j\left[\lg(\frac{b_n}{a_j})+\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i+1}a_j^{pi}}{pib_n^{pi}}+\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^i}{pi}\right]\\
\sim&\frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^{n}a_j\lg(b_n/a_j)\\
=&\frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^{n}j^\alpha\lg(n^{\alpha+1}\lg n/j^\alpha)\\
=&\frac{p\cdot m!}{(k-1)!(m-k)!}\left[\frac{(\alpha+1)\sum_{j=1}^{n}j^\alpha}{n^{\alpha+1}}+\frac{\lg_2 n\sum_{j=1}^{n}j^\alpha}{n^{\alpha+1}\lg n}-\frac{\alpha\sum_{j=1}^{n}j^\alpha\lg j}{n^{\alpha+1}\lg n}\right]\\
\to&\frac{p\cdot m!}{(k-1)!(m-k)!}\left[1+0-\frac{\alpha}{\alpha+1}\right]\\
=&\frac{p\cdot m!}{(\alpha+1)(k-1)!(m-k)!}
\end{aligned}$$

which completes this proof.

Using our Exact Weak Law we conclude this paper with a Generalized Law of the Iterated Logarithm.

Theorem 7. *If* $p(m-k+1)=1$ *and* $\alpha>-1$, *then*

$$\liminf_{n\to\infty}\frac{\sum_{j=1}^{n}j^\alpha X_{j(k)}}{n^{\alpha+1}\lg n}=\frac{p\cdot m!}{(\alpha+1)(k-1)!(m-k)!}\quad \textit{almost surely}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n j^\alpha X_{j(k)}}{n^{\alpha+1}\lg n} = \infty \quad \textit{almost surely.}$$

Proof. From Theorem 6 we have

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^n j^\alpha X_{j(k)}}{n^{\alpha+1}\lg n} \le \frac{p\cdot m!}{(\alpha+1)(k-1)!(m-k)!} \quad \text{almost surely.}$$

Once again set $a_j = j^\alpha$, $b_n = n^{\alpha+1}\lg n$ and $c_n = b_n/a_n = n\lg n$. In order to obtain the opposite inequality we use the following partition

$$\begin{aligned}
\frac{1}{b_n}\sum_{j=1}^n a_j X_{j(k)} &\ge \frac{1}{b_n}\sum_{j=1}^n a_j X_{j(k)} I(1 \le X_{j(k)} \le j) \\
&= \frac{1}{b_n}\sum_{j=1}^n a_j [X_{j(k)} I(1 \le X_{j(k)} \le j) - EX_{j(k)} I(1 \le X_{j(k)} \le j)] \\
&\quad + \frac{1}{b_n}\sum_{j=1}^n a_j EX_{j(k)} I(1 \le X_{j(k)} \le j).
\end{aligned}$$

Note that this partition differs from the others used in this paper.

The first term goes to zero, almost surely, since $b_n \uparrow$ and

$$\sum_{n=1}^\infty c_n^{-2} EX_{n(k)}^2 I(1 \le X_{n(k)} \le n) < C\sum_{n=1}^\infty c_n^{-2}\int_1^n dx < C\sum_{n=1}^\infty \frac{1}{n(\lg n)^2} < \infty.$$

As for the second term we need to expand the terms for a final time. Using Theorem 1 from [2], page 281, we have

$$\begin{aligned}
&\frac{1}{b_n}\sum_{j=1}^n a_j EX_{j(k)} I(1 \le X_{j(k)} \le j) \\
&= \frac{p\cdot m!}{(k-1)!(m-k)!b_n}\sum_{j=1}^n a_j \int_1^j (1-x^{-p})^{k-1} x^{-1} dx \\
&= \frac{p\cdot m!}{(k-1)!(m-k)!n^{\alpha+1}\lg n}\sum_{j=1}^n j^\alpha \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i \int_1^j x^{-pi-1} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{p\cdot m!}{(k-1)!(m-k)!n^{\alpha+1}\lg n}\sum_{j=1}^{n} j^{\alpha}\left[\lg j + \sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i+1}}{pij^{pi}} + \sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i}}{pi}\right]\\
&\sim \frac{p\cdot m!}{(k-1)!(m-k)!n^{\alpha+1}\lg n}\sum_{j=1}^{n} j^{\alpha}\lg j\\
&\sim \frac{p\cdot m!}{(k-1)!(m-k)!n^{\alpha+1}\lg n}\cdot\frac{n^{\alpha+1}\lg n}{\alpha+1}\\
&= \frac{p\cdot m!}{(\alpha+1)(k-1)!(m-k)!}
\end{aligned}$$

which completes the lower limit. As for the upper limit, set $M > 0$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\{X_{n(k)} > Mc_n\}\\
&= \sum_{n=1}^{\infty}\int_{Mc_n}^{\infty}\frac{p\cdot m!}{(m-k)!(k-1)!}(1-x^{-p})^{k-1}x^{-2}dx\\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\sum_{n=1}^{\infty}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}\int_{Mc_n}^{\infty}x^{-pi-2}dx\\
&= \frac{p\cdot m!}{(m-k)!(k-1)!}\sum_{n=1}^{\infty}\sum_{i=0}^{k-1}\frac{\binom{k-1}{i}(-1)^{i}}{(pi+1)(Mc_n)^{pi+1}}\\
&= \frac{p\cdot m!}{M(m-k)!(k-1)!}\sum_{n=1}^{\infty}\frac{1}{n\lg n}\sum_{i=0}^{k-1}\frac{\binom{k-1}{i}(-1)^{i}}{(pi+1)(Mn\lg n)^{pi}}\\
&= \frac{p\cdot m!}{M(m-k)!(k-1)!}\sum_{n=1}^{\infty}\frac{1}{n\lg n}\left[1+\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i}}{(pi+1)(Mn\lg n)^{pi}}\right]\\
&= \infty
\end{aligned}$$

since

$$\left|\sum_{i=1}^{k-1}\frac{\binom{k-1}{i}(-1)^{i}}{(pi+1)(Mn\lg n)^{pi}}\right| < \sum_{i=1}^{k-1}\frac{\binom{k-1}{i}}{(Mn\lg n)^{pi}} \to 0$$

as $n\to\infty$. This implies that

$$\limsup_{n\to\infty}\frac{a_n X_{n(k)}}{b_n} = \infty \quad \text{almost surely}$$

which in turn allows us to conclude that

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_{j(k)}}{b_n} = \infty \quad \text{almost surely}$$

which completes this proof.

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