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COMPLEX EXTENSORS AND LAGRANGIAN SUBMANIFOLDS IN INDEFINITE COMPLEX EUCLIDEAN SPACES

BY

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To Professor Wei-Eihn Kuan on his 70th birthday

Abstract. A general method to construct $SO(n)$-invariant Lagrangian submanifolds in complex Euclidean n-space was introduced in [6]. In this paper we extend the method to construct $SO(k, n-k)$-invariant Lagrangian submanifolds in an indefinite complex Euclidean spaces $\mathbf{C}^n_k$. To do so, we introduce the notion of complex extensors in $\mathbf{C}^n_k$. We show that a complex extensor in $\mathbf{C}^n_k$ is a Lagrangian H-umbilical submanifold. Conversely, we prove that, except the flat cases, Lagrangian H-umbilical submanifolds in $\mathbf{C}^n_k$ are Lagrangian pseudo-Riemannian spheres, Lagrangian pseudo-hyperbolic spaces, complex extensors of a unit pseudo-Riemannian sphere, or complex extensors of a unit pseudo-hyperbolic space.

1. Introduction. The complex number m-space $\mathbf{C}^m$ with complex coordinates $z_1, \ldots, z_m$ endowed with $g_{m,k}$: the real part of the Hermitian form

$$b_{m,k}(z,w) = -\sum_{j=1}^{k} \bar{z}_j w_j + \sum_{j=k+1}^{m} \bar{z}_j w_j, \quad z, w \in \mathbf{C}^m, \tag{1.1}$$

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is a flat indefinite complex space with complex index k. We simply denote the pair $(\mathbf{C}^n, g_{m,k})$ by $\mathbf{C}^m_k$ which is called the *indefinite complex Euclidean m-space* with complex index k.

A submanifold M of $\mathbf{C}^m_k$ is called *totally real* if the almost complex structure J of $\mathbf{C}^m_k$ carries each tangent space of M into its corresponding normal space [14]. It is called *Lagrangian* if the almost complex structure of $\mathbf{C}^n_k$ interchanges the tangent and the normal spaces of M. Lagrangian submanifolds play some important roles in symplectic geometry, Riemannian geometry as well as in mathematical physics (see [10]). (For results on Lagrangian submanifolds from Riemannian geometric point of views, see for examples, [2]-[4], [6]-[15], [18, 19, 22]).

Among examples of Lagrangian submanifolds with large symmetric groups in complex Euclidean n-space $\mathbf{C}^n$, we mention those which are invariant under the standard action of $SO(n)$ on $\mathbf{C}^n$. A general method to construct $SO(n)$-invariant Lagrangian submanifolds in $\mathbf{C}^n$ has been introduced in [6]. It was proved in [6] that one obtains $SO(n)$-invariant submanifolds by constructing the complex extensors of the unit hypersphere of $\mathbf{E}^n$ via a unit speed curve in the complex plane $\mathbf{C}$.

The notion of complex extensors has been applied in [3] to show how to embed a time slice of the Schwarzchild spacetime that models the outer space around a massive star as a $SO(n)$-invariant Lagrangian submanifold.

In this paper we extend the method of [6] to indefinite complex Euclidean spaces which provides us a way to construct $SO(k, n-k)$-invariant Lagrangian submanifolds in $\mathbf{C}^n_k$. Our idea is to extend the notion of complex extensors in $\mathbf{C}^n$ to complex extensors in $\mathbf{C}^n_k$. We show that complex extensors in $\mathbf{C}^n_k$ are Lagrangian H-umbilical submanifolds. Our main result states that, except the flat ones, Lagrangian H-umbilical submanifolds in $\mathbf{C}^n_k$ are Lagrangian pseudo-hyperbolic spaces, Lagrangian pseudo-Riemannian spheres, complex extensors of the unit pseudo-hyperbolic space, or complex extensors of the unit pseudo-Riemannian sphere via unit speed curves in the complex plane. As byproduct, we obtain, for each $k \geq 1$, abundant new examples of $SO(k, n-k)$-invariant Lagrangian submanifolds in $\mathbf{C}_k^n$.

2. Preliminaries. In this section, we briefly recall some facts about indefinite complex space forms. For more details, we refer the reader to [1]. We put $\mathbf{C}^* = \mathbf{C} - \{0\}$.

Let $\tilde{M}_s^n(4c)$ be an indefinite complex space form of complex dimension n and complex index s. The complex index is defined as the (complex) dimension of the largest complex negative definite vector subspace of the tangent space. The curvature tensor $\tilde{R}$ of $\tilde{M}_s^n(4c)$ is given by

$$\begin{aligned} & \tilde{R}(X,Y)Z \\ = \quad & c(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ), \end{aligned}$$

where J denotes the complex structure. We refer to [1] for the construction of the standard models of indefinite complex space forms $CP_s^n(4c)$, when $c > 0$, $CH_s^n(4c)$, when $c < 0$ and $\mathbf{C}_s^n$. For our purposes it is sufficient to know that there exist pseudo-Riemannian submersions, called Hopf fibrations,

$$\pi : \breve{S}_{2s}^{2n+1}(c) \to CP_s^n(4c) : z \mapsto z \cdot \mathbf{C}^\star$$

if $c > 0$ and if $c < 0$ by

$$\pi : \breve{H}_{2s+1}^{2n+1}(c) \to CH_s^n(4c) : z \mapsto z \cdot \mathbf{C}^\star,$$

where

$$\begin{aligned} \breve{S}_{2s}^{2n+1}(c) &= \Big\{ z \in \mathbf{C}^{n+1} | b_{s,n+1}(z,z) = \frac{1}{c} \Big\}, \qquad c > 0, \\ \breve{H}_{2s+1}^{2n+1}(c) &= \Big\{ z \in \mathbf{C}^{n+1} | b_{s+1,n+1}(z,z) = \frac{1}{c} \Big\}, \quad c < 0, \end{aligned}$$

and $b_{p,q}$ is the standard Hermitian form with index p on $\mathbf{C}^q$.

In [1] it is shown that locally any indefinite complex space form is holomorphically isometric to either $\mathbf{C}_s^n$, $CP_s^n(4c)$, or $CH_s^n(4c)$.

Since a submanifold M of a Kähler manifold is Lagrangian if and only if J interchanges the tangent and the normal space, a Lagrangian submanifold of an indefinite complex space form of index s has real index s.

A tangent vector X of a pseudo-Riemannian manifold is called *space-like* (respectively, *time-like* or *light-like*) if $\langle X, X\rangle \geq 0$ (respectively, $\langle X, X\rangle < 0$ or $\langle X, X\rangle = 0$ with $X \neq 0$).

Let M be a submanifold of an indefinite complex space form $\tilde{M}^m_k(4c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on M and $\tilde{M}^m_k(4c)$, respectively. Then the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection. It is well-known that, for each $Y \in T_xM$, the shape operator A_{JY} is a symmetric endomorphism of the tangent space T_xM. The second fundamental form and the shape operator are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle, \tag{2.3}$$

where $<, >$ denotes the indefinite inner product on M as well as on $\tilde{M}^m_k(4c)$. It is known that the shape operator A_ξ is self-adjoint, i.e., $\langle A_\xi X, Y\rangle = \langle A_\xi Y, X\rangle$ for X, Y tangent to M.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X, Y)Z, W\rangle = \left\langle A_{h(Y,Z)}X, W\right\rangle - \left\langle A_{h(X,Z)}Y, W\right\rangle + c\,(\langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle), \tag{2.4}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{2.5}$$

$$\left\langle R^D(X, Y)\xi, \eta\right\rangle = \tilde{R}(X, Y; \xi, \eta) + \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.6}$$

where X, Y, Z, W (respectively, η and ξ) are vector tangent (respectively, normal) to M, $\tilde{R}$ is the curvature tensor of $\tilde{M}_k^m(4c)$, $R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}$, and ∇h is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

If M is a Lagrangian submanifold in $\tilde{M}_k^n(4c)$, then we have

$$D_X JY = J\nabla_X Y, \tag{2.8}$$

$$A_{JY}X = -Jh(X,Y) = A_{JX}Y, \tag{2.9}$$

$$\langle h(X,Y), Z\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle \tag{2.10}$$

for X, Y, Z tangent to M.

We need the following Existence and Uniqueness Theorems for later use.

Existence theorem. *Let (M_k^n, g) be a simply-connected pseudo-Riemannian n-manifold with index k and TM denote the tangent bundle of M_k^n. If h is a TM-valued symmetric bilinear form on M_k^n satisfying*

(1) *$\langle h(X,Y), Z\rangle$ is totally symmetric,*

(2) *$(\nabla h)(X,Y,Z) = \nabla_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ is totally symmetric,*

(3) *$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + h(h(Y,Z),X) - h(h(X,Z),Y)$,*

then there exists a Lagrangian isometric immersion L from (M_k^n, g) into a complete simply-connected indefinite complex space form $\tilde{M}_k^n(4c)$ whose second fundamental form h is given by $h(X,Y) = Jh(X,Y)$.

Uniqueness theorem. *Let $L_1, L_2 \colon M_k^n \to \tilde{M}_k^n(4c)$ be two Lagrangian isometric immersions of a pseudo-Riemannian n-manifold M_k^n with second fundamental forms h^1 and h^2, respectively. If*

$$\left\langle h^1(X,Y), JL_{1\star}Z\right\rangle = \left\langle h^2(X,Y), JL_{2\star}Z\right\rangle, \tag{2.11}$$

for all vector fields X, Y, Z tangent to M_k^n, then there exists an isometry ϕ of $\tilde{M}_k^n(4c)$ such that $L_1 = L_2 \circ \phi$.

These two theorems can be proved in a way similar to the Riemannian case given in [7, 11] (see, also [15]).

3. Complex Extensors. Let $\mathbf{E}_k^m$ denote the pseudo-Euclidean m-space endowed with pseudo-Euclidean metric with index k given by

$$g = -\sum_{j=1}^{k} dx_j^2 + \sum_{\ell=k+1}^{m} dx_\ell^2. \tag{3.1}$$

The group of matrices in $SL(m, \mathbf{R})$ which leave invariant the quadratic form

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2 \tag{3.2}$$

is denoted by $SO(k, m-k)$.

For a real number $r > 0$, we denote by $S_k^{m-1}(r^2)$ the pseudo-Riemannian sphere and by $H_{k-1}^{m-1}(-r^2)$ the pseudo-hyperbolic space defined respectively by

$$S_k^{m-1}(r^2) = \Big\{x \in \mathbf{E}_k^m : \langle x, x\rangle = \frac{1}{r^2}\Big\}, \quad k \geq 0, \tag{3.3}$$

$$H_{k-1}^{m-1}(-r^2) = \Big\{x \in \mathbf{E}_k^m : \langle x, x\rangle = -\frac{1}{r^2}\Big\}, \quad k \geq 2, \tag{3.4}$$

where $<,>$ denotes the indefinite inner product on the pseudo-Euclidean space. If $k = 1$, we put

$$H^{m-1}(-r^2) = \Big\{x \in \mathbf{E}_1^m : \langle x, x\rangle = -\frac{1}{r^2} \text{ and } x_1 > 0\Big\}. \tag{3.5}$$

It is well-known that a complete simply-connected pseudo-Riemannian n-manifold of constant curvature c with index k is isometric to an indefinite Euclidean space $\mathbf{C}_k^n$, a pseudo-Riemannian sphere $S_k^n(c)$, or a pseudo-hyperbolic space $H_k^n(c)$, according to $c = 0, c > 0$ or $c < 0$. Both $S_k^{n-1}(c)$

and $H^{n-1}_{k-1}(c)$ are invariant under the standard action of $SO(k, n-k)$ on $\mathbf{E}^n_k$.

We simply denote $S^{m-1}_k(1)$, $H^{m-1}(-1)$, and $H^{m-1}_{k-1}(1)$ by S^{m-1}_k, H^{m-1}, and H^{m-1}_{k-1}, respectively. S^{m-1}_1 is known as the *de Sitter space-time* and H^{m-1}_1 as the *anti-de Sitter space-time* in the theory of relativity.

Definition 3.1. Let $G : M^{n-1}_t \to \mathbf{E}^m_k$ be an isometric immersion of a semi-Riemannian $(n-1)$-manifold with index t into $\mathbf{E}^m_k$ and let $F : I \to \mathbf{C}^*$ be a unit speed curve in the punctured complex plane $\mathbf{C}^*$. We extend the immersion $G : M^{n-1}_t \to \mathbf{E}^m_k$ to an immersion of $I \times M^{n-1}_t$ into $\mathbf{C}^m_k = \mathbf{C} \otimes \mathbf{E}^m_k$ by

$$\phi = F \otimes G : I \times M^{n-1}_t \to \mathbf{C}^m_k, \tag{3.6}$$

where $F \otimes G$ is the tensor product immersion of F and G defined by

$$(F \otimes G)(s, p) = F(s) \otimes G(p), \quad s \in I,\ p \in M^{n-1}_t. \tag{3.7}$$

We call such an extension $F \otimes G$ of the immersion G a *complex extensor* of G (or of submanifold M^{n-1}_t) via F.

The complex extensor $\phi = F \otimes G : I \times M^{n-1}_t \to \mathbf{C}^m_k$ is called *F-isometric* (respectively, *F-anti-isometric*) if, for each $p \in M^{n-1}_t$,

$$F \otimes G(p) : I \to \mathbf{C}^m_k : s \mapsto F(s) \otimes G(p)$$

carries the unit vector field d/ds into a unit space-like vector field (respectively, a unit time-like vector field). It is called *G-isometric* if, for each $s \in I$,

$$F(s) \otimes G : M^{n-1}_t \to \mathbf{C}^m_1 : p \mapsto F(s) \otimes G(p)$$

is isometric.

Lemma 3.1. *Let $G : M^{n-1}_t \to \mathbf{E}^m_k$ be an isometric immersion and let $F : I \to \mathbf{C}^*$ be a unit speed curve. Then we have*

(1) *The complex extensor $\phi = F \otimes G$ is F-isometric if and only if $G(M^{n-1}_t)$ is contained in the unit pseudo-Riemannian sphere S^{m-1}_k.*

(2) $\phi = F \otimes G$ *is F-anti-isometric if and only if* $G(M_t^{n-1})$ *is contained in the unit pseudo-hyperbolic space* H_{k-1}^{m-1}.

(3) $\phi = F \otimes G$ *is G-isometric if and only if* $F(I)$ *is contained in the unit circle* $S^1 \subset \mathbf{C}$.

(4) $\phi = F \otimes G$ *is totally real if and only if one of the following three cases occurs:*

(4.a) $G(M_t^{n-1})$ *is contained in the unit pseudo-Riemannian sphere* S_k^{m-1}.

(4.b) $G(M_t^{n-1})$ *is contained in the unit pseudo-hyperbolic space* H_{k-1}^{m-1}.

(4.c) $F(s) = cf\varphi(s)$ *for some* $c \in \mathbf{C}$ *and some real-valued function* φ.

Proof. We regard each tangent vector of M_t^{n-1} also as a tangent vector of the product manifold $I \times M_t^{n-1}$ in a natural way. Under the hypothesis we have

$$\phi_s = F'(s) \otimes G, \quad Y\phi = F \otimes Y, \quad \phi_s = \frac{\partial \phi}{\partial s}, \tag{3.8}$$

where Y is a vector tangent to the second component of $I \times M_t^{n-1}$.

From (3.8) we obtain $|\phi_s|^2 = \langle G, G\rangle$, which implies Statements (1) and (2) of the Lemma. Statement (3) follows from the second equation of (3.8).

It follows from a direct computation that the complex extensor $\phi = F \otimes G$ is totally real if and only if, for any $s \in I, p \in M_t^{n-1}$, and $Y \in T_pM_t^{n-1}$, we have

$$\mathrm{Re}(iF(s)\bar{F}'(s))\,\langle G(p), Y\rangle = 0, \tag{3.9}$$

where $\bar{F}'$ denotes the complex conjugate of F' and $\mathrm{Re}(iF\bar{F}')$ the real part of $iF\bar{F}'$. Condition (3.9) implies $\mathrm{Re}(iF(s)\bar{F}'(s)) = 0$ for all $s \in I$ or $\langle G(p), Y\rangle = 0$ for all $p \in M^{n-1}, Y \in T_pM_t^{n-1}$. The first case occurs if and only if $F = c\varphi(s)$ for some $c \in \mathbf{C}$ and real-valued function φ; and the second case occurs if and only if $G(M_t^{(n-1)})$ is contained either in a pseudo-Riemannian sphere $S_k^{m-1}(r^2)$ or in a pseudo-hyperbolic space $H_{k-1}^{m-1}(-r^2)$.

Lemma 3.2. *Let* $G : M_t^{n-1} \to \mathbf{E}_k^m$ *be an isometric immersion and* $F : I \to \mathbf{C}^*$ *a unit speed curve. Then the complex extensor* $\phi = F \otimes G : I \times M_t^{n-1} \to \mathbf{C}_k^m$ *is totally geodesic with respect to the induced metric if and only if one of the following two cases occurs:*

(a) *$G : M_t^{n-1} \to \mathbf{E}_k^m$ is of essential codimension one and $F(s) = (s+a)c$ for some real number a and some unit complex number c.*

(b) *$n = 2$ and G is a line in $\mathbf{E}_k^m$.*

Proof. This is proved exactly in the same way as Proposition 2.2 of [6].

Let $\langle\langle , \rangle\rangle$ denote the standard inner product of the complex plane $\mathbf{C}$. Recall that a local frame $e_1, \dots, e_n$ on a pseudo-Riemannian n-manifold is called *orthonormal* if $\langle e_i, e_j \rangle = \delta_{ij}\varepsilon_j$ where $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$.

Theorem 3.1. *Let $\iota_H : H_{k-1}^{n-1} \to \mathbf{E}_k^n$ be the standard inclusion map of the unit pseudo-hyperbolic space H_{k-1}^{n-1} in $\mathbf{E}_k^n$ and let $F : I \to \mathbf{C}^*$ be a unit speed curve. Then the complex extensor $\phi = F \otimes \iota_H : I \times H_{k-1}^{n-1} \to \mathbf{C}_k^n$ is a Lagrangian submanifold with index k whose second fundamental form satisfying*

$$\begin{aligned} &h(e_1, e_1) = \lambda J e_1, \quad h(e_2, e_2) = \ldots = h(e_k, e_k) = \mu J e_1, \\ &h(e_{k+1}, e_{k+1}) = \ldots = h(e_n, e_n) = -\mu J e_1, \qquad (3.10) \\ &h(e_1, e_j) = \mu J e_j, \quad h(e_j, e_\ell) = 0, \quad 2 \le j \ne \ell \le n \end{aligned}$$

$$\lambda = f'(s), \quad \mu = \frac{\left\langle\left\langle e^{if}, iF \right\rangle\right\rangle}{\langle\langle F, F \rangle\rangle}, \quad F'(s) = e^{if(s)},$$

where $e_1, e_2, \dots, e_n$ is an orthonormal frame on $I \times_{||F||} H_{k-1}^{n-1}$ with $e_1 = \phi_s$ and

(3.11) $\langle e_1, e_1 \rangle = \ldots = \langle e_k, e_k \rangle = -1,\ \langle e_{k+1}, e_{k+1} \rangle = \ldots = \langle e_n, e_n \rangle = 1.$

Proof. Statement (4) of Lemma 3.1 implies that every complex extensor of the unit hyperbolic space H_{k-1}^{n-1} in $\mathbf{E}_k^n$ gives rise to a Lagrangian submanifold of $\mathbf{C}_k^n$.

Since $F : I \to \mathbf{C}^*$ is unit speed, we may put

$$F'(s) = e^{if(s)} \qquad (3.12)$$

for some real-valued function f defined on I. Therefore, F takes the following form:

$$F(s) = \int_a^s e^{if(t)}dt \tag{3.13}$$

for some real number a.

Since ι_H is the inclusion of the unit hyperbolic space H^{n-1}_{k-1} in $\mathbf{E}^n_k$, (3.7) and (3.12) imply

$$\phi_s = e^{if(s)} \otimes \iota_H, \quad Y\phi = F \otimes Y, \tag{3.14}$$

$$\phi_{ss} = if'(s)e^{if(s)} \otimes \iota_H, \quad Y\phi_s = e^{if(s)} \otimes Y, \tag{3.15}$$

$$YZ\phi = F \otimes \nabla_Y Z + \langle Y, Z\rangle\,(F \otimes \iota_H), \tag{3.16}$$

for Y, Z tangent to the second component of $I \times H^{n-1}_{k-1}$.

Since $\langle \iota_H, \iota_H\rangle = -1$, equation (3.14) implies that $e_1 = \phi_s$ is a unit time-like vector field. Moreover, (3.14) implies that the induced metric on $I \times H^{n-1}_{k-1}$ is given by $g = -ds^2 + ||F(s)||^2 g_H$, where g_H is the standard metric on H^{n-1}_{k-1}. Thus, the index of g is k.

Clearly, ϕ_s and $Y\phi$ are orthogonal for Y tangent to the second component of $I \times H^{n-1}_{k-1}$. Therefore, by (3.14)$-$(3.16), we conclude that the second fundamental form of the complex extensor ϕ satisfies (3.10).

Theorem 3.2. *Let $\iota_S : S^{n-1}_k \to \mathbf{E}^n_k$ be the standard inclusion map of the unit pseudo-Riemannian sphere in $\mathbf{E}^n_k$. Then the complex extensor $\phi_S = F \otimes \iota_S$ of ι_S via a unit speed curve F in $\mathbf{C}^*$ is a Lagrangian submanifold with index k in $\mathbf{C}^n_k$ whose second fundamental form satisfies*

$$\begin{aligned} &h(e_1, e_1) = \lambda J e_1, \quad h(e_2, e_2) = \ldots = h(e_{k+1}, e_{k+1}) = -\mu J e_1, \\ &h(e_{k+2}, e_{k+2}) = \ldots = h(e_n, e_n) = \mu J e_1, \\ &h(e_1, e_\ell) = \mu J e_\ell, \quad h(e_t, e_\ell) = 0, \quad 2 \le t \ne \ell \le n, \end{aligned} \tag{3.17}$$

$$\lambda = f'(s), \quad \mu = \frac{\left\langle\left\langle e^{if}, iF\right\rangle\right\rangle}{\langle\langle F, F\rangle\rangle}, \quad F'(s) = e^{if(s)},$$

where $e_1, e_2, \ldots, e_n$ *is an orthonormal frame on* $I \times_{||F||} S_k^{n-1}$ *with* $e_1 = \phi_s$ *and*

$$\text{(3.18)} \qquad \begin{aligned} \langle e_2, e_2\rangle &= \ldots = \langle e_{k+1}, e_{k+1}\rangle = -1, \\ \langle e_1, e_1\rangle &= \langle e_{k+2}, e_{k+2}\rangle = \ldots = \langle e_n, e_n\rangle = 1. \end{aligned}$$

Proof. This can be done in a way similar to Theorem 3.1.

Remark 3.1. The complex extensors of H_{k-1}^{n-1} and S_k^{n-1} via a unit speed curve given in Theorems 3.1 and 3.2 are invariant under the standard action of $SO(k, n-k)$ on $\mathbf{C}_k^n$.

Example 3.1. (*Lagrangian pseudo-hyperbolic spaces*). For a positive real number b, let $F_b : I_b = (-\frac{\pi}{b}, \frac{\pi}{b}) \to \mathbf{C}^*$ be the unit speed curve defined by

$$\text{(3.19)} \qquad F_b(s) = \frac{e^{2bsi} + 1}{2bi}.$$

Then, with respect to the induced metric, the complex extensor:

$$\text{(3.20)} \qquad \phi_H = F_b \otimes \iota_H : I_b \times H_{k-1}^{n-1} \to \mathbf{C}_k^n$$

is a Lagrangian isometric immersion of an open portion of $H_k^n(-b^2)$ of constant negative curvature $-b^2$ into $\mathbf{C}_k^n$. The induced metric on $I_b \times H_{k-1}^{n-1}$ via ϕ_H is

$$\text{(3.21)} \qquad g = -ds^2 + \frac{\cos^2(bs)}{b^2} g_H,$$

where g_H is the standard metric on H^{n-1}_{k-1} given by

$$(3.22)\quad g_H = -\cosh^2 u_{k+1}\left(du_2^2+\cos^2 u_2 du_3^2+\cdots+\prod_{j=2}^{k-1}\cos^2 u_j du_k^2\right)+du_{k+1}^2$$
$$+\sinh^2 u_{k+1}\left(du_{k+2}^2+\cos^2 u_{k+2}du_{k+3}^2+\cdots+\prod_{j=k+2}^{n-1}\cos^2 u_j du_n^2\right).$$

The coordinate system $\{u_2,\ldots,u_n\}$ on H^{n-1}_{k-1} in $\mathbf{E}^n_k$ is defined by

$$(3.23)\quad \begin{aligned} x_1 &= \sin u_2 \cosh u_{k+1},\\ &\vdots\\ x_{k-1} &= \cos u_2 \ldots \cos u_{k-1} \sin u_k \cosh u_{k+1},\\ x_k &= \cos u_2 \ldots \cos u_k \cosh u_{k+1},\\ x_{k+1} &= \sin u_{k+2} \sinh u_{k+1},\\ &\vdots\\ x_{n-1} &= \cos u_{k+2} \ldots \cos u_{n-1} \sin u_n \sinh u_{k+1},\\ x_n &= \cos u_{k+2} \ldots \cos u_{n-1} \sinh u_{k+1}. \end{aligned}$$

We call such a submanifold a *Lagrangian pseudo-hyperbolic space*. The second fundamental form of the Lagrangian pseudo-hyperbolic space is given by (3.10) with $\lambda = 2b$ and $\mu = b$.

Example 3.2. (*Lagrangian pseudo-Riemannian spheres*). The complex extensor:

$$(3.24)\quad \phi_S = F_b \otimes \iota_S : I_b \times S^{n-1}_k \to \mathbf{C}^n_k$$

is a Lagrangian isometric immersion of an open part of $S^n_k(b^2)$ of constant curvature b^2 into $\mathbf{C}^n_k$. The induced metric on $I_b \times S^{n-1}_k$ via ϕ_S is

$$(3.25)\quad g = ds^2 + \frac{\cos^2(bs)}{b^2} g_S,$$

where g_S is the standard metric on S_k^{n-1} given by

$$\begin{aligned} g_S = &-du_2^2 - \sinh^2 u_2\left(du_3^2 + \cos^2 u_3 du_4^2 + \cdots + \prod_{j=3}^{k} \cos^2 u_j du_{k+1}^2\right) \\ &+ \cosh^2 u_2\left(du_{k+2}^2 + \cos^2 u_{k+2} du_{k+3}^2 + \cdots + \prod_{j=k+2}^{n-1} \cos^2 u_j du_n^2\right) \end{aligned} \tag{3.26}$$

The coordinate system $\{u_2, \ldots, u_n\}$ on S_k^n in $\mathbf{E}_k^n$ is defined by

$$\begin{aligned} x_1 &= \sinh u_2 \sin u_3, \\ &\;\;\vdots \\ x_{k-1} &= \sinh u_2 \cos u_3 \ldots \cos u_k \sin u_{k+1}, \\ x_k &= \sinh u_2 \cos u_3 \ldots \cos u_{k+1}, \\ x_{k+1} &= \cosh u_2 \sin u_{k+2}, \\ &\;\;\vdots \\ x_{n-1} &= \cosh u_2 \cos u_{k+2} \ldots \cos u_{n-1} \sin u_n, \\ x_n &= \cosh u_2 \cos u_{k+2} \ldots \cos u_n, \end{aligned} \tag{3.27}$$

We call such a submanifold a *Lagrangian pseudo-Riemannian sphere.* The second fundamental form of the Lagrangian pseudo-Riemannian sphere is given by (3.17) with $\lambda = 2b$ and $\mu = b$.

4. Lagrangian H-umbilical Submanifolds. In views of Theorems 3.1 and 3.2, we define a Lagrangian H-umbilical submanifold in an indefinite complex Euclidean space $\mathbf{C}_k^n$ as follows.

Definition 4.1. A Lagrangian submanifold M in an indefinite complex Euclidean space $\mathbf{C}_k^n$ is called *Lagrangian H-umbilical* if its second fundamental form satisfies

$$(4.1)\qquad \begin{aligned} h(e_1,e_1) &= \lambda Je_1, \quad h(e_1,e_t)=\mu Je_t, \\ h(e_t,e_t) &= \mu\delta_t Je_1, \;\; \delta_t\in\{-1,1\}, \quad t=2,\ldots,n, \\ h(e_\ell,e_t) &= 0, \quad 2\le \ell\ne t\le n, \end{aligned}$$

for some functions λ and μ with respect to some orthonormal local frame $e_1,\ldots,e_n$.

Since the second fundamental form h of a Lagrangian submanifold satisfies Condition (2.10), Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds which satisfy the following two conditions:

(a) JH is an eigenvector of the shape operator A_H and
(b) the restriction of A_H to $(JH)^{\perp}$ is proportional to the identity map.

In this way, we can regard Lagrangian H-umbilical submanifolds as the simplest Lagrangian submanifolds next to the totally geodesic ones.

The main result of this section is the following classification theorem.

Theorem 4.1. *Let $L: M\to \mathbf{C}^n_k$ be a Lagrangian H-umbilical submanifold in $\mathbf{C}^n_k$ with $n\ge 3$ and index $k>0$.*

- (i) *If M has constant sectional curvature, then, up to rigid motions of $\mathbf{C}^n_k$, one of the following three cases occurs:*
  - (i-a) *M is a flat pseudo-Riemannian manifold.*
  - (i-b) *M is an open portion of a pseudo-hyperbolic space $H^n_k(-b^2)$ and L is locally a Lagrangian pseudo-hyperbolic space in $\mathbf{C}^n_k$.*
  - (i-c) *M is an open portion of a pseudo-Riemannian sphere $S^n_k(b^2)$ and L is locally a Lagrangian pseudo-Riemannian sphere in $\mathbf{C}^n_k$.*
- (ii) *If M contains no open subset of constant sectional curvature, then, up to rigid motions, one of the following two cases occurs:*
  - (ii-a) *L is an open portion of a complex extensor of the unit pseudo-hyperbolic space H^{n-1}_{k-1} via a unit speed curve in $\mathbf{C}^*$.*

(ii-b) *L is an open portion of a complex extensor of the unit pseudo-Riemannian sphere S^{n-1}_k via a unit speed curve in $\mathbf{C}^*$.*

Proof. Let $n \geq 3$, $k > 0$, and $L : M \to \mathbf{C}^n_k$ be a Lagrangian H-umbilical isometric immersion whose second fundamental form satisfies (4.1) for some functions λ and μ with respect to some orthonormal local frame field $e_1, \ldots, e_n$.

If we put

$$\tilde{\nabla}_X e_A = \sum_{B=1}^{n} \epsilon_B \omega_A^B(X) e_B + \sum_{B=1}^{n} \epsilon_B \omega_A^{B^*}(X) J e_B, \;\; \epsilon_B = \langle e_B, e_B \rangle \tag{4.2}$$

$$\tilde{\nabla}_X (J e_A) = \sum_{B=1}^{n} \epsilon_B \omega_{A^*}^B(X) e_B + \sum_{B=1}^{n} \epsilon_B \omega_{A^*}^{B^*}(X) J e_B, \tag{4.3}$$

for $A, B = 1, \ldots, n$, then we have

$$\omega_A^B = -\omega_B^A, \quad \omega_{B^*}^A = \omega_A^{B^*}, \quad \omega_{B^*}^{A^*} = \omega_B^A. \tag{4.4}$$

Let $\omega^1, \ldots, \omega^n$ denote the dual 1-forms of $e_1, \ldots, e_n$ defined by

$$\omega^A(e_B) = \delta_{AB} = \begin{cases} 0, & \text{if } A \neq B, \\ 1, & \text{if } A = B. \end{cases} \tag{4.5}$$

Then we have

$$\omega_A^{B^*} = \sum_{C=1}^{n} \epsilon_C h_{AC}^{B^*} \omega^C, \quad h_{AC}^{B^*} = \langle h(e_A, e_C), J e_B \rangle. \tag{4.6}$$

The Cartan's structure equations are given by

$$d\omega^A = \sum_{B=1}^{n} \epsilon_B \omega^B \wedge \omega_B^A, \tag{4.7}$$

$$d\omega_A^B = \sum_{C=1}^{n} \epsilon_C \omega_A^C \wedge \omega_C^B + \sum_{C=1}^{n} \epsilon_C \omega_A^{C^*} \wedge \omega_{C^*}^B. \tag{4.8}$$

Case (1): e_1 *is time-like*. In this case, we may assume

(4.9) $\langle e_1, e_1\rangle = \ldots = \langle e_k, e_k\rangle = -1, \quad \langle e_{k+1}, e_{k+1}\rangle = \ldots = \langle e_n, e_n\rangle = 1,$

so, we have

$$\epsilon_1 = \ldots = \epsilon_k = -1, \;\; \epsilon_{k+1} = \ldots = \epsilon_n = 1. \tag{4.10}$$

From (2.3), (2.9), (4.1), and (4.9)–(4.10), we find $\delta_t = -\epsilon_t$ for $t = 2, \ldots, n$. Hence (4.1) becomes

$$\begin{aligned} &h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_t) = \mu J e_t, \\ &h(e_j, e_j) = \mu J e_1, \quad h(e_\alpha, e_\alpha) = -\mu J e_1, \quad h(e_\ell, e_t) = 0, \\ &2 \le j \le k, \quad k+1 \le \alpha \le n, \quad 2 \le \ell \ne t \le n. \end{aligned} \tag{4.11}$$

From (4.2)–(4.4), (4.11), and Codazzi's equation, we obtain

$$e_1\mu = (\lambda - 2\mu)\epsilon_t \omega_1^t(e_t), \tag{4.12}$$
$$e_t\lambda = (2\mu - \lambda)\omega_1^t(e_1), \tag{4.13}$$
$$(\lambda - 2\mu)\omega_1^\ell(e_t) = 0, \quad 2 \le \ell \ne t \le n, \tag{4.14}$$
$$e_t\mu = 3\mu\omega_t^1(e_1), \tag{4.15}$$
$$\mu\omega_1^t(e_1) = 0, \tag{4.16}$$

for $2 \le t \le n$.

Since the ambient space $\mathbf{C}^n_k$ is flat, (4.10)–(4.11) and Gauss' equation imply that the sectional curvature function K of M satisfies

$$K(e_1, e_t) = \mu(\mu - \lambda), \quad t = 2, \ldots, n, \tag{4.17}$$
$$K(e_\ell, e_t) = -\mu^2, \quad 2 \le \ell \ne t \le n. \tag{4.18}$$

Case (1-a): *M is of constant curvature*. In this case, (4.17) and (4.18) imply $\mu(\lambda - 2\mu) = 0$.

Case (1-a.1): $\mu = 0$ *identically.* In this case, M is a flat pseudo-Riemannian manifold with index k. Hence, we obtain Case (i-a) of Theorem 4.1.

Case (1-a.2): $\mu \neq 0$. In this case, $\lambda = 2\mu \neq 0$ on a nonempty open subset V of M. Thus, (4.12) and (4.13) imply that μ is a nonzero constant, say $b \neq 0$. Thus, by continuity, we have $V = M$. Hence, the equation of Gauss implies that M is a pseudo-Riemannian manifold of constant negative curvature $-b^2$. Therefore, M is locally isometric to the warped product $I_b \times_{\cos(bs)/b} H^{n-1}_{k-1}$, $I_b = (-\pi/2b, \pi/2b)$, whose metric is given by

$$g = -ds^2 + \frac{\cos^2(bs)}{b^2} g_H. \tag{4.19}$$

Therefore, the Uniqueness Theorem implies that, up to rigid motions of $\mathbf{C}^n_k$, the Lagrangian immersion is given by (3.20). This gives Case (i-b) of Theorem 4.1.

Case (1-b): *M contains no open subset of constant curvature.* In this case, the set $U := \{\, p \in M \,:\, \mu(\lambda - 2\mu) \neq 0 \text{ at } p \,\}$ is an open dense subset of M.

Equations (4.13)–(4.16) imply

$$e_t\lambda = e_t\mu = 0, \quad t = 2, \ldots, n. \tag{4.20}$$

$$\omega_1^\ell(e_t) = 0, \quad 2 \leq \ell \neq t \leq n, \quad \text{on } U. \tag{4.21}$$

Moreover, (4.12), (4.16) and (4.21) yield

$$\omega_1^t = \kappa\epsilon_t\omega^t, \quad \kappa = \tfrac{e_1\mu}{\lambda - 2\mu}, \quad t = 2, \ldots, n, \quad \text{on } U. \tag{4.22}$$

For $2 \leq \ell, t \leq n$, (4.21) gives $\langle [e_\ell, e_t], e_1 \rangle = \omega_t^1(e_\ell) - \omega_\ell^1(e_t) = 0$. Thus, the distribution $\mathcal{D}^\perp =: \mathrm{Span}\,\{e_2, \ldots, e_n\}$ is integrable. Let $\mathcal{D}$ denote the distribution spanned by e_1. Then $\mathcal{D}$ is also integrable, since $\mathcal{D}$ is one-dimensional. Thus, there is a local coordinate system $\{s, x_2, \ldots, x_n\}$ such that (a) $\mathcal{D}$ is spanned by $\{\partial/\partial s\}$, (b) $\mathcal{D}^{\perp}$ is spanned by $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ and (c) $e_1 = \partial/\partial s$, $\omega^1 = ds$.

From (4.20) we know that λ and μ depend only on s. Hence, the function κ defined in (4.22) depends only on s, too. From (4.2), (4.21) and (4.22), we find

$$\langle \nabla_{e_\ell} e_t, e_1 \rangle = -\kappa \delta_{\ell t} \langle e_\ell, e_t \rangle, \quad 2 \leq \ell, t \leq n, \tag{4.23}$$

which implies that $\mathcal{D}^{\perp}$ is a spherical distribution, *i.e.*, $\mathcal{D}^{\perp}$ is an integrable distribution whose leaves are extrinsic spheres in M. By an extrinsic sphere, we mean a totally umbilical submanifolds with parallel mean curvature vector. Moreover, by (4.11), (4.23), and Gauss' equation, we know that each leaf of $\mathcal{D}^{\perp}$ is of constant sectional curvature $-(\mu^2 + \kappa^2)$. Furthermore, from (4.22), we have $\nabla_{e_1} e_1 = 0$. Thus, integral curves of e_1 are geodesics. Consequently, by applying a result of [17, 21], we conclude that U is locally a warped product $\mathbf{E}^1_1 \times_{f(s)} H^{n-1}_{k-1}$ of a time-like line and the unit pseudo-hyperbolic space H^{n-1}_{k-1} for some positive function $f(s)$. Hence, there is a local coordinate system on M such that the metric tensor is given by

$$g = -ds^2 + f(s)^2 g_H \tag{4.24}$$

where g_H is the metric on H^{n-1}_{k-1} defined by (3.22).

Equations (3.22) and (4.24) and a direct long computation yield

$$\begin{aligned}
&\nabla_{\partial/\partial s} \frac{\partial}{\partial s} = 0, \quad \nabla_{\partial/\partial s} \frac{\partial}{\partial u_t} = \frac{f'}{f} \frac{\partial}{\partial u_t}, \quad t = 2, \ldots, n, \\
&\nabla_{\partial/\partial u_i} \frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \leq i < j \leq k, \\
&\nabla_{\partial/\partial u_2} \frac{\partial}{\partial u_2} = -ff' \cosh^2 u_{k+1} \frac{\partial}{\partial s} + \frac{\sinh(2u_{k+1})}{2} \frac{\partial}{\partial u_{k+1}}, \\
&\nabla_{\partial/\partial u_j} \frac{\partial}{\partial u_j} = \prod_{\ell=2}^{j-1} \cos^2 u_\ell \left\{ \frac{\sinh(2u_{k+1})}{2} \frac{\partial}{\partial u_{k+1}} - ff' \cosh^2 u_{k+1} \frac{\partial}{\partial s} \right\} \\
&\qquad + \sum_{\ell=2}^{j-1} \left(\frac{\sin 2u_\ell}{2} \prod_{i=\ell+1}^{j-1} \cos^2 u_i \right) \frac{\partial}{\partial u_\ell}, \quad j = 3, \ldots, k,
\end{aligned}$$

$$\nabla_{\partial/\partial u_{k+1}}\frac{\partial}{\partial u_{k+1}} = ff'\frac{\partial}{\partial s},$$

$$\nabla_{\partial/\partial u_j}\frac{\partial}{\partial u_{k+1}} = \tanh u_{k+1}\frac{\partial}{\partial u_j}, \quad 2 \le j \le k, \tag{4.25}$$

$$\nabla_{\partial/\partial u_j}\frac{\partial}{\partial u_\beta} = 0, \ \ 2 \le j \le k; \ \ k+2 \le \beta \le n,$$

$$\nabla_{\partial/\partial u_\alpha}\frac{\partial}{\partial u_\beta} = -\tan u_\alpha\frac{\partial}{\partial u_\beta}, \quad k+2 \le \alpha < \beta \le n,$$

$$\nabla_{\partial/\partial u_{k+2}}\frac{\partial}{\partial u_{k+2}} = ff'\sinh^2 u_{k+1}\frac{\partial}{\partial s} - \frac{\sinh(2u_{k+1})}{2}\frac{\partial}{\partial u_{k+1}},$$

$$\nabla_{\partial/\partial u_\alpha}\frac{\partial}{\partial u_\alpha} = \prod_{\ell=2}^{\alpha-1}\cos^2 u_\ell\left\{ff'\sinh^2 u_{k+1}\frac{\partial}{\partial s} - \frac{\sinh(2u_{k+1})}{2}\frac{\partial}{\partial u_{k+1}}\right\}$$
$$+\sum_{\beta=k+2}^{\alpha-1}\left(\frac{\sin 2u_\beta}{2}\prod_{l=\beta+1}^{\alpha-1}\cos^2 u_l\right)\frac{\partial}{\partial u_\beta},$$

$$\nabla_{\partial/\partial u_\alpha}\frac{\partial}{\partial u_{k+1}} = \coth u_{k+1}\frac{\partial}{\partial u_\alpha}, \quad k+2 \le \alpha \le n.$$

By applying (4.11), (4.25) and Gauss' formula, we find

$$L_{ss} = i\lambda L_s, \quad i = \sqrt{-1}, \tag{4.26}$$

$$L_{su_t} = \Big(\frac{f'}{f} + i\mu\Big)L_{u_t}, \ \ t = 2,\ldots,n, \tag{4.27}$$

$$L_{u_iu_j} = -\tan u_i L_{u_j}, \quad 2 \le i < j \le k, \tag{4.28}$$

$$L_{u_2u_2} = (i\mu f^2 - ff')\cosh^2 u_{k+1}L_s + \frac{\sinh 2u_{k+1}}{2}L_{u_{k+1}}, \tag{4.29}$$

$$L_{u_ju_j} = \prod_{\ell=2}^{j-1}\cos^2 u_\ell\left\{(i\mu f^2 - ff')\cosh^2 u_{k+1}L_s + \frac{\sinh 2u_{k+1}}{2}L_{u_{k+1}}\right\}$$
$$+\sum_{\ell=2}^{j-1}\left(\frac{\sin 2u_\ell}{2}\prod_{i=\ell+1}^{j-1}\cos^2 u_i\right)L_{u_\ell}, \ \ j = 3,\ldots,k, \tag{4.30}$$

$$L_{u_{k+1}u_{k+1}} = (ff' - i\mu f^2)L_s, \tag{4.31}$$

$$L_{u_ju_{k+1}} = \tanh u_{k+1}L_{u_j}, \quad 2 \le j \le k, \tag{4.32}$$

$$L_{u_ju_\beta} = 0, \ \ 2 \le j \le k; \ \ k+2 \le \beta \le n, \tag{4.33}$$

$$L_{u_\alpha u_\beta} = -\tan u_\alpha L_{u_\beta}, \quad k+2 \le \alpha < \beta \le n, \tag{4.34}$$

(4.35) $L_{u_{k+2}u_{k+2}} = (f' - i\mu f)f\sinh^2 u_{k+1}L_s - \dfrac{\sinh(2u_{k+1})}{2}L_{u_{k+1}},$

(4.36) $L_{u_\alpha u_\alpha} = \prod_{\ell=k+2}^{\alpha-1}\cos^2 u_\ell\left\{(f'-i\mu f)f\sinh^2 u_{k+1}L_s - \dfrac{\sinh(2u_{k+1})}{2}L_{u_{k+1}}\right\}$

$$+\sum_{\beta=k+2}^{\alpha-1}\left(\frac{\sin 2u_\beta}{2}\prod_{l=\beta+1}^{\alpha-1}\cos^2 u_l\right)L_{u_\beta},$$

(4.37) $L_{u_{k+1}u_\alpha} = \coth u_{k+1}L_{u_\alpha}, \quad k+2\le\alpha\le n.$

Since $L_{ssu_t} = L_{su_ts}$, (4.26) and (4.27) imply

$$\kappa' + \kappa^2 = \mu^2 - \lambda\mu, \quad \kappa = \tfrac{\mu'}{\lambda-2\mu}, \tag{4.38}$$

where $\kappa = f_s/f$. Also, from $L_{u_2u_{k+1}u_{k+1}} = L_{u_{k+1}u_{k+1}u_2}$, (4.31) and (4.32), we find $f^2 = 1/(\kappa^2+\mu^2)$. Therefore, we get

$$f = c\exp\left(\int\kappa(s)dx\right) = \frac{1}{\sqrt{\kappa^2+\mu^2}}, \tag{4.39}$$

for some integration constant $c\neq 0$.

Solving the equation (4.26) yields

$$L = A(u_2,\dots,u_n)\int^s e^{i\int^s\lambda(t)dt}ds + B(u_2,\dots,u_n) \tag{4.40}$$

for some $\mathbf{C}^n_k$-valued functions A and B, where $\int^s\lambda(t)dt$ is an antiderivative of $\lambda(s)$.

By (4.27) and (4.40), we find

$$(\kappa+i\mu)B_{u_t} = \left(e^{i\int^s\lambda(t)dt} - (\kappa+i\mu)\int^s e^{-i\int^x\lambda(t)dt}dx\right)A_{u_t} \tag{4.41}$$

for $t=2,\dots,n$. Since A and B are independent of s, (4.41) implies

$$e^{i\int^s\lambda(t)dt} - (\kappa+i\mu)\int^s e^{-i\int^x\lambda(t)dt}dx = \alpha(\kappa+i\mu) \tag{4.42}$$

for some $\alpha\in\mathbf{C}$. Thus, (4.41) gives $B=\alpha A+C$ for some $\alpha\in\mathbf{C}$ and

$C \in \mathbf{C}^n_k$. Thus, after applying a suitable translation on $\mathbf{C}^n_k$, we obtain from (4.40) that

$$L(s,u_2,\dots,u_n)=F(s)A(u_2,\dots,u_n), \quad F(s)=\alpha+\int^s e^{i\int^s \lambda(t)dt}ds. \tag{4.43}$$

From (4.27) and (4.43), we find

$$F'(s) = (\kappa + i\mu)F(s). \tag{4.44}$$

Since $||F'(s)|| = 1$, (4.39) and (4.44) imply

$$||F(s)|| = f(s). \tag{4.45}$$

Equation (4.43) gives

$$L_s = F'(s)A, \quad L_{u_{k+1}u_{k+1}} = F(s)A_{u_{k+1}u_{k+1}}. \tag{4.46}$$

On the other hand, by (4.31), (4.39), (4.44) and (4.46), we find

$$L_{u_{k+1}u_{k+1}} = (ff' - i\mu f^2)F'A = (ff' - i\mu f^2)\left(\frac{f'}{f} + i\mu\right)FA = FA. \tag{4.47}$$

Combining (4.46), and (4.47) yields $A_{u_{k+1}u_{k+1}} = A$. Thus, we obtain

$$A = b_1 \sinh u_{k+1} + b_2 \cosh u_{k+1} \tag{4.48}$$

for some $\mathbf{C}^n_k$-valued functions b_1, b_2 of $u_2, \dots, u_k, u_{k+1}, \dots, u_n$.

By applying (4.32) with $j = 2$ and (4.48), we find

$$b_1 = b_1(u_3, \dots, u_k, u_{k+2}, \dots, u_n),$$

$$b_2 = b_3(u_3, \dots, u_k, u_{k+2}, \dots, u_n) \sin u_2 + b_4(u_3, \dots, u_k, u_{k+2}, \dots, u_n) \cos u_2.$$

Continuing such procedure $(k-1)$-times with the help of (4.28)$-$(4.32), we obtain

$$b_1 = b_1(u_{k+2}, \dots, u_n)$$

$$b_2 = c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \cdots + c_{k-1} \sin u_k \prod_{j=2}^{k-1} \cos u_j + c_k \prod_{j=2}^{k} \cos u_j \tag{4.49}$$

for some $\mathbf{C}^n_k$-valued functions $c_1, \dots, c_k$ of $u_{k+2}, \dots, u_n$.

Similarly, by (4.33)−(4.37), we know that $c_1, \dots, c_k$ are constant vectors and

$$b_1 = c_{k+1} \sin u_{k+2} + c_{k+2} \sin u_{k+3} \cos u_{k+2} + \cdots + c_{n-1} \sin u_n \prod_{\alpha=k+2}^{n-1} \cos u_\alpha + c_n \prod_{\alpha=2}^{n} \cos u_\alpha \tag{4.50}$$

for some constant vectors $c_1, \dots, c_k$ in $\mathbf{C}^n_k$. Therefore, by combining (4.43) and (4.48)−(4.50), we obtain

$$\begin{aligned} L = {} & F(s)\Big\{ c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \cdots \\ & + c_{k-1} \sin u_k \prod_{j=2}^{k-1} \cos u_j + c_k \prod_{j=2}^{k} \cos u_j \Big\} \cosh u_{k+1} \\ & + F(s)\Big\{ c_{k+1} \sin u_{k+2} + c_{k+2} \sin u_{k+3} \cos u_{k+2} + \cdots \\ & + c_{n-1} \sin u_n \prod_{\alpha=k+2}^{n-1} \cos u_\alpha + c_n \prod_{\alpha=k+2}^{n} \cos u_\alpha \Big\} \sinh u_{k+1} \end{aligned} \tag{4.51}$$

for some constant vectors $c_1, \dots, c_n$ in $\mathbf{C}^n_k$.

Because M is a Lagrangian submanifold in $\mathbf{C}^n_k$, we may choose the following initial conditions:

$$\begin{aligned} L_s(0, \dots, 0) &= (1, 0, \dots, 0), \\ L_{u_2}(0, \dots, 0) &= \left(0, \frac{1}{f(0)}, \dots, 0\right), \end{aligned} \tag{4.52}$$

$$\vdots$$
$$L_{u_n}(0,\dots,0) = \left(0,\dots,0,\frac{1}{f(0)}\right),$$

in view of (3.22) and (4.51). By using (4.51) and (4.52) we obtain

$$(4.53)\quad Lz = F(s)\Bigg(\sin u_2 \cosh u_{k+1}, \sin u_3 \cos u_2 \cosh u_{k+1}, \dots, \sin u_k \cosh u_{k+1} \prod_{j=2}^{k-1} \cos u_j, \cosh u_{k+1} \prod_{j=2}^{k} \cos u_j, \sinh u_{k+1} \sin u_{k+2}, \sinh u_{k+1} \sin u_{k+3} \cos u_{k+2}, \dots, \sinh u_{k+1} \sin u_n \prod_{\alpha=k+2}^{n-1} \cos u_\alpha, \sinh u_{k+1} \prod_{\alpha=k+2}^{n} \cos u_\alpha \Bigg)$$

which implies that, up to rigid motions of $\mathbf{C}^n_k$, M is the complex extensor of the unit pseudo-hyperbolic space via the unit speed curve F. Thus, we obtain Case (ii-a) of Theorem 4.1.

Case (2): e_1 *is space-like*. In this case, we may assume

$$(4.54)\quad \begin{aligned} &\langle e_2, e_2\rangle = \dots = \langle e_{k+1}, e_{k+1}\rangle = -1, \\ &\langle e_1, e_1\rangle = \langle e_{k+2}, e_{k+2}\rangle = \dots = \langle e_n, e_n\rangle = 1, \end{aligned}$$

so, we have $\epsilon_2 = \dots = \epsilon_{k+1} = -1$, $\epsilon_1 = \epsilon_{k+2} = \dots = \epsilon_n = 1$.

From (2.3), (2.9), (4.1), and (4.54) we find $\delta_t = \epsilon_t$ for $t = 2, \dots, n$. Hence (4.1) becomes

$$(4.55)\quad \begin{aligned} &h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_t) = \mu J e_t, \\ &h(e_j, e_j) = -\mu J e_1, \quad h(e_\alpha, e_\alpha) = \mu J e_1, \quad h(e_\ell, e_t) = 0, \\ &2 \le j \le k+1, \quad k+2 \le \alpha \le n, \quad 2 \le \ell \ne t \le n. \end{aligned}$$

From (4.2)−(4.4), (4.55), and Codazzi's equation, we find

$$e_1\mu = (\lambda - 2\mu)\epsilon_t\omega_1^t(e_t), \tag{4.56}$$

$$e_t\lambda = (\lambda - 2\mu)\omega_1^t(e_1), \tag{4.57}$$

$$(\lambda - 2\mu)\omega_1^\ell(e_t) = 0, \tag{4.58}$$

$$e_t\mu = -3\mu\omega_t^1(e_1), \tag{4.59}$$

$$\mu\omega_1^t(e_1) = 0, \tag{4.60}$$

for $2 \leq \ell \neq t \leq n$.

Since the ambient space is flat, the equation of Gauss and (4.10)−(4.11) imply

$$K(e_1, e_t) = \mu(\lambda - \mu), \quad t = 2, \ldots, n, \tag{4.61}$$

$$K(e_\ell, e_t) = \mu^2, \quad 2 \leq \ell \neq t \leq n. \tag{4.62}$$

Case (2-a): *M is of constant curvature.* In this case, (4.61) and (4.62) imply $\mu(\lambda - 2\mu) = 0$.

Case (2-a.1): $\mu = 0$ *identically.* In this case, M is a flat pseudo-Riemannian manifold with index k.

Case (2-a.2): $\mu \neq 0$. In this case, $\lambda = 2\mu \neq 0$ on a nonempty open subset V of M. Thus, (4.57) and (4.60) imply that μ is a nonzero constant, say $b \neq 0$. Hence, by continuity, we obtain $V = M$. Therefore M is a pseudo-Riemannian manifold of constant curvature b^2. Hence, M is locally isometric to the warped product $I_b \times_{\cos(bs)/b} S_k^{n-1}$, $I_b = (-\pi/2b, \pi/2b)$. Thus, by applying the Uniqueness Theorem, we obtain Case (i-c) of Theorem 4.1.

Case (2-b): *M contains no open subset of constant curvature.* In this case, the set $U := \{\, p \in M \,:\, \mu(\lambda - 2\mu) \neq 0 \text{ at } p \,\}$ is an open dense subset of M.

As Case (1-b), Equations (4.56)−(4.60) imply that the distribution $\mathcal{D}^{\perp}$ spanned by $\{e_2, \ldots, e_n\}$ is integrable whose leaves are extrinsic spheres in M and integral curves of e_1 are geodesics. Thus, there is a local coordinate system $\{s, x_2, \ldots, x_n\}$ such that (a) $\mathcal{D}$ is spanned by $\{\partial/\partial s\}$, (b) $\mathcal{D}^{\perp}$ is spanned by $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ and (c) $e_1 = \partial/\partial s$, $\omega^1 = ds$. Furthermore, in this case we know that U is locally a warped product $\mathbf{E}^1 \times_{f(s)} S_k^{n-1}$ of a space-like line and the unit pseudo-Riemannian sphere S_k^{n-1} for some positive function f. Therefore, there is a local coordinate system on M such that the metric tensor is given by

$$g = ds^2 + f(s)^2 g_S \tag{4.63}$$

where g_S is the metric on S_k^{n-1} defined by (3.26).

After computing Christoffel symbols of g, we obtain from (4.55) and the formula of Gauss that

$$L_{ss} = i\lambda L_s, \quad i = \sqrt{-1}, \tag{4.64}$$

$$L_{su_t} = \left(\frac{f'}{f} + i\mu\right) L_{u_t}, \;\; t = 2, \ldots, n, \tag{4.65}$$

$$L_{u_2u_j} = \coth u_2 L_{u_j}, \quad 3 \le j \le k+1, \tag{4.66}$$

$$L_{u_2u_\alpha} = \tanh u_2 L_{u_\alpha}, \quad k+2 \le \alpha \le n. \tag{4.67}$$

$$L_{u_iu_j} = -\tan u_i L_{u_j}, \quad 3 \le i < j \le k+1, \tag{4.68}$$

$$L_{u_2u_2} = (ff' - i\mu f^2) L_s, \tag{4.69}$$

$$L_{u_3u_3} = (ff' - i\mu f^2)\sinh^2 u_2 L_s - \frac{\sinh 2u_2}{2} L_{u_2}, \tag{4.70}$$

$$\begin{aligned} L_{u_ju_j} = {} & \prod_{\ell=3}^{j-1} \cos^2 u_\ell \left\{ (ff' - i\mu f^2)\sinh^2 u_2 L_s + \frac{\sinh 2u_2}{2} L_{u_2} \right\} \\ & + \sum_{\ell=3}^{j-1} \left(\frac{\sin 2u_\ell}{2} \prod_{i=\ell+1}^{j-1} \cos^2 u_i \right) L_{u_\ell}, \;\; j = 4, \ldots, k+1, \end{aligned} \tag{4.71}$$

$$L_{u_ju_\beta} = 0, \;\; 3 \le j \le k+1; \;\; k+2 \le \beta \le n, \tag{4.72}$$

$$L_{u_\alpha u_\beta} = -\tan u_\alpha L_{u_\beta}, \quad k+2 \le \alpha < \beta \le n, \tag{4.73}$$

$$(4.74)\quad L_{u_{k+2}u_{k+2}} = (i\mu f^2 - ff')\cosh^2 u_{k+1} L_s + \frac{\sinh(2u_{k+1})}{2} L_{u_2},$$

$$(4.75)\quad L_{u_\alpha u_\alpha} = \prod_{\ell=k+2}^{\alpha-1} \cos^2 u_\ell \left\{ (i\mu f^2 - ff')\cosh^2 u_2 L_s + \frac{\sinh(2u_{k+1})}{2} L_{u_{k+1}} \right\} + \sum_{\beta=k+2}^{\alpha-1} \left(\frac{\sin 2u_\beta}{2} \prod_{l=\beta+1}^{\alpha-1} \cos^2 u_l \right) L_{u_\beta},$$

Since $L_{ssu_t} = L_{su_ts}$ and $L_{u_2u_2u_3} = L_{u_2u_3u_2}$, (4.64)−(4.66) and (4.69) imply

$$(4.76)\qquad \kappa' + \kappa^2 = \mu^2 - \lambda\mu, \quad \kappa = \frac{f'}{f} = \frac{\mu'}{\lambda - 2\mu}, \quad f^2 = 1/(\kappa^2 + \mu^2).$$

After solving the system (4.64)−(4.75) with the help of (4.76) as in Case (1-b), we obtain

$$(4.77)\quad \begin{aligned} L = {} & F(s)\Big\{ c_1 \sin u_3 + c_2 \sin u_4 \cos u_3 + \cdots \\ & + c_k \sin u_{k+1} \prod_{j=3}^{k} \cos u_j + c_k \prod_{j=3}^{k+1} \cos u_j \Big\} \sinh u_2 \\ & + F(s)\Big\{ c_{k+1} \sin u_{k+2} + c_{k+2} \sin u_{k+3} \cos u_{k+2} + \cdots \\ & + c_{n-1} \sin u_n \prod_{\alpha=k+2}^{n-1} \cos u_\alpha + c_n \prod_{\alpha=2}^{n} \cos u_\alpha \Big\} \cosh u_2 \end{aligned}$$

for some constant vectors $c_1, \ldots, c_n$ in $\mathbf{C}^n_k$, where $F(s)$ is the unit speed curve defined by (4.43). By choosing the same initial conditions (4.52) as Case (1-b), we obtain

$$(4.78)\quad L = F(s)\Big(\sinh u_2 \sin u_3, \sinh u_2 \sin u_4 \cos u_3, \ldots, \sinh u_2 \sin u_{k+1} \prod_{j=3}^{k} \cos u_j, \sinh u_2 \prod_{j=3}^{k+1} \cos u_j, \cosh u_2 \sin u_{k+2},$$

$$\cosh u_2 \sin u_{k+3} \cos u_{k+2}, \ldots, \cosh u_2 \sin u_n \prod_{\alpha=k+2}^{n-1} \cos u_\alpha,$$
$$\cosh u_2 \prod_{\alpha=k+2}^{n} \cos u_\alpha \Bigg).$$

This shows that, up to rigid motions, the Lagrangian submanifold is the complex extensor of the unit pseudo-Riemannian sphere via the unit speed curve F. Hence, we obtain Case (ii-b) of Theorem 4.1.

The converse is easy to verified.

Theorem 4.1 implies immediately the following.

Corollary 4.1. *Let M be a Lagrangian submanifold of $\mathbf{C}^n_k$ with $n \geq 3$ and $k \geq 1$. Then, up to rigid motions, M is an open portion of a Lagrangian pseudo-Riemannian sphere or of a Lagrangian pseudo-hyperbolic space if and only if M is a Lagrangian H-umbilical submanifold with nonzero constant curvature.*

Corollary 4.2. *Let $L : M \to \mathbf{C}^n_1$ be a Lagrangian H-umbilical submanifold in the Lorentzian complex Euclidean n-space with $n \geq 3$.*

(i) *If M is of constant curvature, then, up to rigid motions of $\mathbf{C}^n_1$, one of the following three cases occurs:*
   - (i-a) *M is a flat Lorentzian n-manifold.*
   - (i-b) *M is an open portion of a Lagrangian hyperbolic space in $\mathbf{C}^n_1$.*
   - (i-c) *M is an open portion of a Lagrangian de Sitter spacetime in $\mathbf{C}^n_1$.*

(ii) *If M contains no open subset of constant curvature, then, up to rigid motions, L is locally one of the following two Lagrangian submanifolds:*
   - (ii-a) *L is a complex extensor of the unit hyperbolic space H^{n-1} via a unit speed curve in $\mathbf{C}^*$.*
   - (ii-b) *L is a complex extensor of the unit de Sitter spacetime S^{n-1}_1 via a unit speed curve in $\mathbf{C}^*$.*

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