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BAIRE ONE FUNCTIONS

BY

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Abstract. We obtain a new set of characterizations of Baire one functions and use it to show that finite one-sided preponderant limits and derivatives are in the Baire class one.

1. Introduction and definitions. In current literature, a function is called Baire one if it is the pointwise limit of a sequence of continuous functions. Several useful characterizations of Baire one functions are known; see Natanson [5], Gleyzal [3], Preiss [7]. In this paper, Section 2, we obtain a new set of characterizations of Baire one finite functions and show some of its applications.

Denjoy [2] showed that an approximately continuous function is Baire one; and its one-sided version and other generalizations appear in Sinharoy [11], Bruckner et al [1] and Sarkhel and Seth [10]. Tolstoff [13] showed that a finite approximate derivative is Baire one, but the proof is complicated. Various other proofs of this result and its generalizations appear in Goffman and Neugebauer [4], Snyder [12], Preiss [7], O'Malley [6], Bruckner et al [1] and Sarkhel and Seth [10]. Zajiček [14, Theorem 3, p.558] proved that a finite one-sided approximate derivative is also Baire one, and Sinharoy [11, Theorem 3, p.319] proved a more general result (here, the finiteness condition is essential [14, Example 1, p.559]). In [1, p.113], it is indicated

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that finite preponderant derivatives [2] are also Baire one. Applying our characterization, we quickly prove that finite right-hand (similarly, left-hand) preponderant limits and derivatives also share this property. We consider a closed interval $[a, b]$ on the real line $\mathbb{R}$ and adopt the following generalized definition.

Definition 1. A function $F : [a, b] \to \mathbb{R}$ is said to have finite right preponderant limit p [resp. derivative q] at a point $c \in [a, b)$, if there is a number $r \in [0, \frac{1}{2})$ so that for each $\varepsilon > 0$ the set

$$\left\{ x \in (c, b) : |F(x) - p| \geq \varepsilon \left[\text{resp.} \left| \frac{F(x) - F(c)}{x - c} - q \right| \geq \varepsilon \right] \right\}$$

has its outer upper density [8] at c not exceeding r. Requiring $r = 0$, these define the approximate limit and derivative [8, p.220].

Given $E \subset \mathbb{R}$, $|E|$ will denote its outer Lebesgue measure and $\overline{E}$ its closure. By a *perfect portion* of E we shall mean a section $E \cap [c, d]$ that is prefect and contains both c and d. We also need

Definition 2.(Sarkhel and Kar [9]) Let $X \subset \mathbb{R}$. An increasing sequence of sets $(E_n)_{n=1}^{\infty}$ whose union is X is called an X-chain with parts E_n; if further each part E_n is closed in X, then the X-chain is said to be closed.

2. Results. Put $I = [a, b]$ and consider functions F, $f : I \to \mathbb{R}$.

The idea of the following theorem emerged from the notions of (PAC) [9, p.337] and $\delta-$decomposition [1, p.104] and the proofs of [9, Theorem 3.3, p.338] and [1, Theorem 5.2, p.110].

Theorem 1. *The following conditions are mutually equivalent*:

(i) *The finite function f is Baire one on I.*

(ii) *For each* $\varepsilon > 0$*, there exist numbers* $\delta_x > 0$ *for all* $x \in I$ *so that for each pair of points* $x, y \in I$ *with* $|x - y| < \min\{\delta_x, \delta_y\}$ *we have* $|f(x) - f(y)| < \varepsilon$*.*

(iii) *For each* $\varepsilon > 0$*, there is a closed* I*-chain* $(E_n)_{n=1}^{\infty}$ *such that for each* n *and each pair of points* $x, y \in E_n$ *with* $|x - y| < 1/n$ *we have* $|f(x) - f(y)| < \varepsilon$*.*

Proof. (i) $\Rightarrow$ (ii). Assuming (i), let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions on I such that $f_n(x) \to f(x)$ for each $x \in I$. Then, given $\varepsilon > 0$, for each $x \in I$ there is a positive integer $m(x)$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geq m(x)$. Also there is a positive integer $k(x)$ such that $|f_{m(x)}(x) - f_{m(x)}(y)| < \varepsilon/3$ for all $y \in I$ with $|x - y| < 1/k(x)$.

Let $\delta_x = \min\{1/m(x), 1/k(x)\}$. Now suppose $x, y \in I$ and $|x - y| < \min\{\delta_x, \delta_y\}$. Due to symmetry, we can assume that $m(x) \geq m(y)$. Since also $|x - y| < \delta_x \leq 1/k(x)$, we have then $|f(x) - f_{m(x)}(x)| < \varepsilon/3$, $|f_{m(x)}(x) - f_{m(x)}(y)| < \varepsilon/3$ and $|f_{m(x)}(y) - f(y)| < \varepsilon/3$. Hence $|f(x) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, which proves (ii).

(ii) $\Rightarrow$ (iii). Assuming (ii), given $\varepsilon > 0$ there exist numbers $\delta_x > 0$ for all $x \in I$ so that for each pair of points $x, y \in I$ with $|x - y| < \min\{\delta_x, \delta_y\}$ we have $|f(x) - f(y)| < \varepsilon/3$.

Let $A_n = \{x \in I : \delta_x > 1/n\}$, then $(A_n)_{n=1}^{\infty}$ is an I-chain. Now suppose $x, y \in \overline{A}_m$ and $|x - y| < 1/m$. Since $(A_n)_{n=1}^{\infty}$ is an I-chain, we can find $k > m$ so that both $x, y \in A_k$. Then select x_1, $y_1 \in A_m$ with $0 \leq |x - x_1| < 1/k$, $0 \leq |y - y_1| < 1/k$ and $|x_1 - y_1| < 1/m$. Since $A_m \subset A_k$ so $x_1, y_1 \in A_k$. Evidently then $|f(x) - f(x_1)| < \varepsilon/3$, $|f(x_1) - f(y_1)| < \varepsilon/3$ and $|f(y_1) - f(y)| < \varepsilon/3$. Hence $|f(x) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, which proves (iii) with $E_n = \overline{A}_n$.

(iii) $\Rightarrow$ (i). Assume (iii). Suppose for a contradiction that f is not Baire one on I. Then by a well known result [5, p.146], there must exist a nonempty perfect set $P \subset I$ such that $f \mid P$ is discontinuous at each point of P. Then for all $t \in P$, we have $\omega(t) > 0$ where

$$\omega(t) = \inf_{h>0} \sup\{|f(x) - f(y)| : x, y \in P \cap (t-h, t+h)\}.$$

Let $P_n = \{t \in P : \omega(t) \geq 2/n\}$, then $(P_n)_{n=1}^{\infty}$ is a closed P-chain. So by Baire category theorem [8, p.54] some P_k must contain some perfect portion $P \cap [c, d]$ of P. By (iii), there is a closed I-chain $(E_n)_{n=1}^{\infty}$ such that for each n and each pair of points x, $y \in E_n$ with $|x - y| < 1/n$ we have $|f(x) - f(y)| < 1/k$. Now $(P \cap [c, d] \cap E_n)_{n=1}^{\infty}$ is a closed $P \cap [c, d]$-chain, and so again by Baire category theorem some $P \cap [c, d] \cap E_m$ must contain some perfect portion $P \cap [u, v]$ of $P \cap [c, d]$.

Pick $t \in P \cap (u, v)$. Since $P \cap (u, v) \subset E_m$, evidently $\omega(t) \leq 1/k$, which contradicts that $\omega(t) \geq 2/k$ since $t \in P \cap (u, v) \subset P \cap [c, d] \subset P_k$. Hence (iii) implies (i), and this completes the proof of the theorem.

Corollary. *If f is Baire one on I, then f is bounded on each part of some closed I-chain.*

The next two theorems are major applications of Theorem 1.

Theorem 2. *If F has finite right preponderant limit $f(x)$ at each point x of $[a, b)$, then f is Baire one on $[a, b]$.*

Proof. Let $F(x) = f(b)$ for $x > b$. Then for each $x \in I$ we can find $r_x \in (0, \frac{1}{2})$ so that, given any $\varepsilon > 0$, the set

$$I_x = \{t > x : |F(t) - f(x)| \geq \varepsilon/2\}$$

has outer upper density at x less than r_x. Then select $h_x \in (0, \frac{1}{2} - r_x)$ so that

$$|I_x \cap [x, v]| < r_x(v - x) \quad \text{for all} \quad v \in (x, x + h_x).$$

Let $\delta_x = \frac{1}{2}h_x^2$. Now suppose $x, y \in I$ and $|x - y| < \min\{\delta_x, \delta_y\}$. Due to symmetry, we can assume that $h_x \le h_y$. Then put

$$u = x + \delta_x \quad \text{and} \quad v = x - \delta_x + h_x.$$

Since $0 < h_x < \frac{1}{2} - r_x < 1$, clearly $x < u < v < x + h_x$ with

$$v - u = h_x - 2\delta_x = (1 - h_x)h_x > \Big(\frac{1}{2} + r_x\Big)(v - x) > 0.$$

Since $|x - y| < \delta_x$ and $h_x \le h_y < \frac{1}{2} - r_y$, so $y < u < v < y + h_y$ with

$$v - u = (1 - h_x)h_x > \Big(\frac{1}{2} + r_y\Big)(v + \delta_x - x) > \Big(\frac{1}{2} + r_y\Big)(v - y).$$

Hence, using the definitions of h_x and h_y we obtain

$$\begin{aligned}
|(u, v)\backslash(I_x \cup I_y)| &\ge |(u, v)| - |I_x \cap [x, v]| - |I_y \cap [y, v]| \\
&> (v - u) - r_x(v - x) - r_y(v - y) \\
&> \Big[1 - \frac{2r_x}{1 + 2r_x} - \frac{2r_y}{1 + 2r_y}\Big](v - u) \\
&= \Big[\frac{1 - 2r_x}{2(1 + 2r_x)} + \frac{1 - 2r_y}{2(1 + 2r_y)}\Big](v - u) > 0.
\end{aligned}$$

So there are points $t \in (u, v)\backslash(I_x \cup I_y)$, which therefore satisfy both $|F(t) - f(x)| < \varepsilon/2$ and $|F(t) - f(y)| < \varepsilon/2$. Therefore $|f(x) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence by Theorem 1((ii)$\Rightarrow$(i)), f is Baire one on I.

Theorem 3. *If F has finite right preponderant derivative $f(x)$ at each point x of $[a, b)$, then f is Baire one on $[a, b]$.*

Proof. Let $F(x) = F(b) + f(b)(x - b)$ for $x > b$. Then for each $x \in I$ we can find $r_x \in (0, \frac{1}{2})$ so that, given any $\varepsilon > 0$, the set

$$I_x = \Big\{t > x : \Big|\frac{F(t) - F(x)}{t - x} - f(x)\Big| \ge \frac{\varepsilon(1 - 2r_x)}{16}\Big\}$$

has outer upper density at x less than r_x. Then select $h_x \in (0, \frac{1}{2} - r_x)$ so that

$$|I_x \cap [x, v]| < r_x(v - x) \quad \text{for all} \quad v \in (x, x + h_x).$$

Let $\delta_x = \frac{1}{2}h_x^2$. Now suppose $x, y \in I$ and $|x - y| < \min\{\delta_x, \delta_y\}$. Due to symmetry, we can assume that $h_x \leq h_y$. Then put

$$u = x + \delta_x \quad \text{and} \quad v = x - \delta_x + h_x.$$

From the proof of the preceding theorem, $x < u < v < x + h_x$, $y < u < v < y + h_y$ and

$$v - u > \max\left\{\left(\frac{1}{2} + r_x\right)(v - x), \left(\frac{1}{2} + r_y\right)(v - y)\right\}, \tag{1}$$

$$\frac{|(u, v)\backslash(I_x \cup I_y)|}{v - u} > \frac{1 - 2r_x}{2(1 + 2r_x)} + \frac{1 - 2r_y}{2(1 + 2r_y)} > 0. \tag{2}$$

Since the diameter of a set E is no less than $|E|$, so by (2) there exist $c, d \in (u, v)\backslash(I_x \cup I_y)$ such that

$$\frac{d - c}{v - u} > \max\left\{\frac{1 - 2r_x}{2(1 + 2r_x)}, \frac{1 - 2r_y}{2(1 + 2r_y)}\right\}. \tag{3}$$

Now, since $c, d \in (u, v)\backslash I_x$, using (1) and (3) we have

$$\begin{aligned}
& \left|\frac{F(d) - F(c)}{d - c} - f(x)\right| \\
= & \left|\left(\frac{F(d) - F(x)}{d - x} - f(x)\right)\frac{d - x}{d - c} - \left(\frac{F(c) - F(x)}{c - x} - f(x)\right)\frac{c - x}{d - c}\right| \\
< & \frac{\varepsilon(1 - 2r_x)}{16} \cdot \frac{d - x}{d - c} + \frac{\varepsilon(1 - 2r_x)}{16} \cdot \frac{c - x}{d - c} \\
< & \frac{\varepsilon(1 - 2r_x)}{16} \cdot \frac{2(v - x)}{d - c} \\
< & \frac{\varepsilon(1 - 2r_x)}{8} \cdot \frac{2}{1 + 2r_x} \cdot \frac{v - u}{d - c} \\
< & \frac{\varepsilon}{2}.
\end{aligned}$$

Similarly, since $c, d \in (u,v)\backslash I_y$, using (1) and (3) we get

$$\left|\frac{F(d)-F(c)}{d-c}-f(y)\right|<\frac{\varepsilon(1-2r_y)}{16}\left(\frac{d-y}{d-c}+\frac{c-y}{d-c}\right)<\frac{\varepsilon}{2}.$$

It therefore follows that $|f(x)-f(y)|<\varepsilon/2+\varepsilon/2=\varepsilon$. Hence by Theorem 1 ((ii)$\Rightarrow$(i)), f is Baire one on I.

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References

1. A. M. Bruckner, R. J. O'Malley and B. S. Thomson, *Path derivatives: a unified view of certain generalized derivatives,* Trans. Amer. Math. Soc., **283**(1984), 97-125.

2. A. Denjoy, *Sur les fonctions dérivées sommables,* Bull. Soc. Math. France, **43**(1915), 161-248.

3. A. Gleyzal, *Interval functions,* Duke Math. J., **8**(1941), 223-230.

4. C. Goffman and C. J. Neugebauer, *On approximate derivatives,* Proc. Amer. Math. Soc., **11**(1960), 962-966.

5. I. P. Natanson, *Theory of Functions of a Real Variable,* vol.II, Ungar, New York, 1967.

6. R. J. O'Malley, *Decomposition of approximate derivatives,* Proc. Amer. Math. Soc., **69**(1978), 243-247.

7. D. Preiss, *Approximate derivatives and Baire classes,* Czech. Math. J., **96**(21) (1971), 373-382.

8. S. Saks, *Theory of the Integral,* Dover, New York, 1964.

9. D. N. Sarkhel and A. B. Kar, (PVB) *functions and integration,* J. Austral. Math. Soc., Ser. A., **36**(1984), 335-353.

10. D. N. Sarkhel and P. K. Seth, *On some generalized approximative relative derivatives,* Rendi. Circ. Mat. Palermo (2), **35**(1986), 5-21.

11. M. Sinharoy, *Remarks on Darboux and mean value properties of approximate derivatives,* Comment. Math. (Prace Mat.), **23**(1983), 315-324.

12. L. E. Snyder, *Approximate Stolz angle limits,* Proc. Amer. Math. Soc., **17**(1966), 416-422.

13. G. P. Tolstoff, *Sur la dérivée approximative exacte,* Rec. Math. (Mat. Sbornik), N. S., **4**(1938), 499-504.

14. L. Zajiček, *On approximate Dini derivates and one-sided approximate derivatives of arbitrary functions,* Comment. Math. Univ. Carolinae, **22**(1981), 549-560.

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