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PERIODIC SOLUTION OF TWO-PATCHES PREDATOR-PREY DISPERSION-DELAY MODELS WITH FUNCTIONAL RESPONSE*

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Abstract. In this paper, a nonautonomous predator-prey dispersion delay model with functional response is studied, where all parameters are time dependent. In this system, which consists of two-patches, the prey species can disperse between two-patches, but the two predator species are confined to one patch and cannot disperse. By using the continuation theorem of coincidence degree theory, the existence of a positive periodic solution for above system is established.

1. Introduction. For many species spatial factors are important in population dynamics, as discussed by many authors. The theoretical study of spatial distribution can traced back at least as far as Skellem [17], and has been extensively studied in many papers (for example in [6, 9, 10, 11, 13, 14, 18] and references cited therein). Most of the previous papers focused on the the coexistence of populations modelled by systems of ordinary differential equations and stability (local and global) of equilibria. Many existing models deal with a single population dispersing among patches. Some of them deal with competition and predator-prey interactions in patchy environments.

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On the other hand, the effect of the past history on the systems' stability is also an important problem in population biology. Recently persistence and stability of a population dynamical system involving time delays have been discussed by some authors (for example [7, 8, 12] and references cited therein). They obtained some sufficient conditions that guarantee permanence of population or stability of positive equilibria or positive periodic solution. Song and Chen [15, 16] extended the autonomous Lotka-Volterra system to a two species nonautonomous dispersion Lotka-Volterra system, and they investigated persistence of the populations and periodic behavior of the system.

In this paper, we consider a nonautonomous predator-prey dispersion state dependent delay model with functional response.

$$(1.1)\quad \begin{cases} x_1' = x_1\Big(a_1(t) - b_1(t)x_1 - \dfrac{c_1(t)y}{1+\alpha_1(t)x_1} - \dfrac{c_2(t)z}{1+\alpha_2(t)x_1}\Big) \\ \qquad\qquad + D_1(t)(x_2 - x_1), \\ x_2' = x_2(a_2(t) - b_2(t)x_2) + D_2(t)(x_1 - x_2), \\ y' = y\Big(-d_1(t) + \dfrac{e_1(t)x_1(t-\tau_1(t,x_1(t),x_2(t)))}{1+\alpha_1(t)x_1(t-\tau_1(t,x_1(t),x_2(t)))} - q_1(t)y\Big), \\ z' = z\Big(-d_2(t) + \dfrac{e_2(t)x_1(t-\tau_1(t,x_1(t),x_2(t)))}{1+\alpha_2(t)x_1(t-\tau_1(t,x_1(t),x_2(t)))} - q_2(t)y\Big), \end{cases}$$

where $x_1(t)$, $y(t)$ and $z(t)$ are the densities of prey species x and predators species y and z in patch I at the time t respectively; $x_2(t)$ is the density of prey species x in patch II in the time t. Predators species y and z are both confined to patch 1, while the prey species x can disperse between two-patches. $D_i (i = 1, 2)$ are dispersion coefficients of species x.

Our purpose in this paper is, by using the continuation theorem which was proposed in [2, 3], to study the existence of positive periodic solution of system (1.1). Moreover, since, at present, there are only a few papers which have been published on the existence of periodic solutions of state dependent delay differential equations (see, [19] and references cited therein). We also use the same method to study the existence of periodic solutions of system (1.1). For the work concerning the existence of periodic solutions of delay differential equations which was done by using coincidence degree theory, we refer to [4,5] and reference cited therein.

2. Main result. In this section, based on the Mawhin's continuation theorem we shall study the existence of at least one positive periodic solution of system (1.1). First, we shall make some preparations.

Let X, Y be real Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ a Fredholm mapping of index zero and $P : X \to X$, $Q : Y \to Y$ continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$, $KerQ = \mathrm{Im}L$ and $X = \mathrm{Ker}L \oplus \mathrm{Ker}P, Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Denote by L_p the restriction of L $\mathrm{Dom}L \cap \mathrm{Ker}P$, $K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ the inverse (to L_p) and $J : \mathrm{Im}Q \to \mathrm{Ker}L$ and isomorphism of $\mathrm{Im}Q$ onto $\mathrm{Ker}L$.

For convenience of use, we introduce the continuation theorem [2, p.40] as follows.

Theorem A. *Let $\Omega \subset X$ be open bounded set and $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$(i.e. $QN : \overline{\Omega} \to Y$ and $K_p(I - Q)N : \overline{\Omega} \to Y$ are compact).*

Asume

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$;*

(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} = 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$.

In what follows we shall use the notation

$$\overline{f}=\frac{1}{w}\int_0^w f(t)dt,\quad f^l=\min_{t\in[0,w]}|f(t)|,\quad f^M=\max_{t\in[0,w]}|f(t)|,$$

where f is a continuous w-periodic function.

In system (1.1), we always assume the following.

(H_1) *$a_i(t)$, $b_i(t)$, $c_i(t)$, $\alpha_i(t)$, $D_i(t)$, $d_i(t)$, $e_i(t)$, $q_i(t)$ $(i=1,2)$, and $\beta_{1i}(t)$ $(i=1,2,3,4)$ are positive periodic continuous functions with period $w>0$.*

(H_2) *$\tau_1(t,x_1(t),x_2(t))$ is continuous and w-periodic with respect to t.*

We are now in a position to state and prove our main result.

Theorem 2.1. *In addition to (H_1) and (H_2), assume the following:*

(H_3) $(a_1-D_1)^l>\frac{c_1^M}{q_1^l}\left(\frac{e_1}{q_1^l}\right)^M+\frac{c_2^M}{q_2^l}\left(\frac{e_2}{\alpha_2}\right)^M$;

(H_4) $(\alpha_2-D_2)^l>0$;

(H_5) $(e_i^l-d_i^M\alpha_i^M)\Big[(\alpha_1-D_1)-\frac{c_1^M}{q_1^l}\left(\frac{e_1}{\alpha_1}\right)^M-\frac{c_2^M}{q_2^l}\left(\frac{e_2}{\alpha_2}\right)^M\Big]>d_i^Mb_1^M,\quad i=1,2.$

Then system (1.1) has at least one positive w-periodic solution.

Proof. Consider the following equations:

$$(2.1)\quad\begin{cases}\dfrac{du_1(t)}{dt}=a_1(t)-D_1(t)-b_1(t)e^{u_1(t)}-\dfrac{e_1(t)e^{u_3}(t)}{1+\alpha_1(t)e^{u_1(t)}}\\ \qquad\qquad -\dfrac{c_2(t)e^{u_4(t)}}{1+\alpha_2(t)e^{u_1(t)}}+D_1(t)e^{u_2(t)-u_1(t)},\\ \dfrac{du_2(t)}{dt}=a_2(t)-D_2(t)-b_2(t)e^{u_2(t)}+D_2(t)e^{u_1(t)-u_2(t)},\\ \dfrac{du_3(t)}{dt}=-d_1(t)+\dfrac{e_1(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}{1+\alpha_1(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}-q_1(t)e^{u_3}(t)\\ \dfrac{du_4(t)}{dt}=-d_2(t)+\dfrac{e_2(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}{1+\alpha_2(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}-q_2(t)e^{u_4}(t),\end{cases}$$

where $a_i(t)$, $b_i(t)$, $c_i(t)$, $\alpha_i(t)$, $D_i(t)$, $d_i(t)$, $e_i(t)$, $q_i(t)$ $(i = 1, 2)$, and $\beta_{1i}(t)$ $(i = 1, 2, 3, 4)$ are the same as those in (H1), and τ, and $\tau_1(t, e^{u_1(t)}, e^{u_2}(t))$ are the same as those in (H2). It is easy to see that system (2.1) has one w-periodic solution $(u_1^*(t),\ u_2^*(t),\ u_3^*(t),\ u_4^*(t))^T$, then

$$(x_1^*(t), x_2^*(t), y^*(t), z^*(t))^T = (\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)], \exp[u_4^*(t)])$$

is a positive w-perodic solution of system (1.1). So, to complete the proof, it suffices to show that system (2.1) has one w-periodic solution.

In order to apply the continuation theorem of coincidence degree theory to establish the existence of w-periodic solution of system (2.1), we take $X = Y = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in C(R, R^4) : u_i(t + w) = u_i(t), i = 1, 2, 3, 4$ and $\|(u_1(t), u_2(t), u_3(t), u_4(t))^T\| = \max_{t\in[0,w]} |u_1(t)| + \max_{t\in[0,w]} |u_2(t)| + \max_{t\in[0,w]} |u_3(t)| + \max_{t\in[0,w]} |u_4(t)|$, here, $|\cdot|$ denotes the Euclidean norm. With this norm $\|\cdot\|$, X is a Banach space. Set

$$L : \mathrm{Dom}L \cap X,\ L(u_1(t), u_2(t), u_3(t), u_4(t))^T = (u_1'(t), u_2'(t), u_3'(t), u_4'(t))^T,$$

where $\mathrm{Dom}L = \{(u_1(t, u_2(t), u_3(t), u_4(t)))^T \in C^1(R, R^4)\}$,and $N : X \to X$,

$$N\begin{bmatrix} u_1 \\ \\ u_2 \\ \\ u_3 \\ \\ u_4 \end{bmatrix} = \begin{bmatrix} a_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - \dfrac{c_1(t)e^{u_3(t)}}{1+\alpha_1(t)e^{u_1}(t)} \\ \qquad -\dfrac{c_2(t)e^{u_4(t)}}{1+\alpha_2(t)e^{u_1}(t)} + D_1(t)e^{u_2(t)-u_1(t)} \\ a_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} \\ -d_1(t) + \dfrac{e_1(t)e^{u_1(t-r_1(t,e^{u_1}(t),e^{u_2}(t)))}}{1+\alpha_1(t)e^{u_1(t-\tau_1(t,e^{u_1}(t),e^{u_2}(t)))}} - q_1(t)e^{u_3(t)} \\ -d_2(t) + \dfrac{e_2(t)e^{u_1(t-r_1(t,e^{u_1}(t),e^{u_2}(t)))}}{1+\alpha_2(t)e^{u_1(t-\tau_1(t,e^{u_1}(t),e^{u_2}(t)))}} - q_2(t)e^{u_4(t)} \end{bmatrix}.$$

Define two projectores P and Q as

$$P\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = Q\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{w}\int_0^w u_1(t)dt \\ \frac{1}{w}\int_0^w u_2(t)dt \\ \frac{1}{w}\int_0^w u_3(t)dt \\ \frac{1}{w}\int_0^w u_4(t)dt \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in X.$$

Cleary, $\mathrm{Ker}L = R^4$, $\mathrm{Im}L = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in X : \int_0^w u_i(t)dt = 0, \quad i = 1, 2, 3, 4\}$ is closed in X and $\mathrm{dimKer}L = \mathrm{codimImL} = 4$. Therefore, L is a Fredholm mapping of index zero. Furthermore, through an easy computation we find that the inverse K_p of L_p has the form

$$K_p : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P,$$

$$K_p\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_0^t u_1(s)ds - \frac{1}{w}\int_0^w \int_0^\eta u_1(t)dtd\eta \\ \int_0^t u_2(s)ds - \frac{1}{w}\int_0^w \int_0^\eta u_2(t)dtd\eta \\ \int_0^t u_3(s)ds - \frac{1}{w}\int_0^w \int_0^\eta u_3(t)dtd\eta \\ \int_0^t u_4(s)ds - \frac{1}{w}\int_0^w \int_0^\eta u_4(t)dtd\eta \end{bmatrix}$$

Therefore

$$K_p(I - Q)f : X \to X,$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \to \begin{bmatrix} \int_0^t f_1(u_1(t), u_2(t), u_3(t), u_4(t))dt] \\ \int_0^t f_2(u_1(t), u_2(t), u_3(t), u_4(t))dt] \\ \int_0^t f_3(u_1(t), u_2(t), u_3(t), u_4(t))dt] \\ \int_0^t f_4(u_1(t), u_2(t), u_3(t), u_4(t))dt] \end{bmatrix}$$

$$- \begin{bmatrix} \frac{1}{w}\int_0^w \int_0^\eta f_1(u_1(t), u_2(t), u_3(t), u_4(t))dtd\eta \\ \frac{1}{w}\int_0^w \int_0^\eta f_2(u_1(t), u_2(t), u_3(t), u_4(t))dtd\eta \\ \frac{1}{w}\int_0^w \int_0^\eta f_3(u_1(t), u_2(t), u_3(t), u_4(t))dtd\eta \\ \frac{1}{w}\int_0^w \int_0^\eta f_4(u_1(t), u_2(t), u_3(t), u_4(t))dtd\eta \end{bmatrix}$$

$$+ \begin{bmatrix} \left(\frac{1}{2}-\frac{t}{w}\right)\int_0^w f_1(u_1(t),u_2(t),u_3(t),u_4(t))dt \\ \left(\frac{1}{2}-\frac{t}{w}\right)\int_0^w f_2(u_1(t),u_2(t),u_3(t),u_4(t))dt \\ \left(\frac{1}{2}-\frac{t}{w}\right)\int_0^w f_3(u_1(t),u_2(t),u_3(t),u_4(t))dt \\ \left(\frac{1}{2}-\frac{t}{w}\right)\int_0^w f_4(u_1(t),u_2(t),u_3(t),u_4(t))dt \end{bmatrix},$$

where $f=(f_1,f_2,f_3,f_4)^T$, $f_i(i=1,2,3,4)$ are continuous and w-periodic functions.

Evidently, QN and $K_p(I-Q)N$ are continuous by Lebesgue convergence theorem and moreover, by Arzela-Ascoli theorem, $QN(\overline{\Omega})$, $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega\subset X$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$. Corresponding to the operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, we have

$$(2.2)\begin{cases} \dfrac{du_1(t)}{dt}=\lambda\Big[a_1(t)-D_1(t)-b_1(t)e^{u_1(t)}-\dfrac{c_1(t)e^{u_3(t)}}{1+\alpha_1(t)e^{u_1(t)}} \\ \qquad\qquad -\dfrac{c_2(t)e^{u_4(t)}}{1+\alpha_2(t)e^{u_1(t)}}+D_1(t)e^{u_2(t)-u_1(t)}\Big], \\ \dfrac{du_2(t)}{dt}=\lambda\Big[a_2(t)-D_2(t)-b_2(t)e^{u_2(t)}+D_2(t)e^{u_1(t)-u_2(t)}\Big], \\ \dfrac{du_3(t)}{dt}=\lambda\Big[-d_1(t)+\dfrac{e_1(t)e^{u_1(t-\tau_2(t,e^{u_1(t)},e^{u_2(t)}))}}{1+\alpha_1(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}-q_1(t)e^{u_3(t)}\Big], \\ \dfrac{du_4(t)}{dt}=\lambda\Big[-d_2(t)+\dfrac{e_2(t)e^{u_1(t-\tau_2(t,e^{u_1(t)},e^{u_2}(t)))}}{1+\alpha_2(t)e^{u_1(t-\tau_1(t,e^{u_1(t)},e^{u_2(t)}))}}-q_2(t)e^{u_4(t)}\Big]. \end{cases}$$

Suppose that $(u_1(t),u_2(t),u_3(t),u_4(t))^T\in X$ is a solution of of system (2.2) for some $\lambda\in(0,1)$.

Choose $t_i\in[0,w]$ such that

$$u_i(t_i)=\max_{t\in[0,w]}u_i(t),\quad i=1,2,3,4.$$

From this and system (2.2), we obtain

$$a_1(t_1) - D_1(t_1) - b_1(t_1)e^{u_1(t_1)} - \frac{c_1(t_1)e^{u_3(t_1)}}{1+\alpha_1(t_1)e^{u_1(t_1)}}$$

$$-\frac{c_2(t_1)e^{u_4(t_1)}}{1+\alpha_2(t_1)e^{u_1(t_1)}} + D_1(t_1)e^{u_2(t_1-u_1(t_1))} = 0, \tag{2.3}$$

$$a_2(t_2) - D_2(t_2) - b_2(t_2)\, e^{u_2(t_2)} + D_2(t_2)\, e^{u_1(t_2)-u_2(t_2)} = 0, \tag{2.4}$$

$$-d_1(t_3) + \frac{e_1(t_3)e^{u_1(t_3-\tau_1(t_3,\, e^{u_1(t_3)},\, e^{u_2(t_3)}))}}{1+\alpha_1(t_3)\, e^{u_1(t_3-\tau_1(t_3,\, e^{u_1(t_3)},\, e^{u_2(t_3)}))}} - q_1(t_3)e^{u_3(t_3)} = 0 \tag{2.5}$$

and

$$-d_2(t_4) + \frac{e_2(t_4)e^{u_1(t_4-\tau_1(t_4,\, e^{u_1(t_4)},\, e^{u_2(t_4)}))}}{1+\alpha_2(t_4)\, e^{u_1(t_4-\tau_1(t_4,\, e^{u_1(t_4)},\, e^{u_2(t_4)}))}} - q_2(t_4)e^{u_4(t_4)} = 0. \tag{2.6}$$

(2.3) implies that

$$b_2^M e^{u_2(t_2)} > (a_2 - D_2)^l.$$

That is

$$e^{u_2(t_2)} > \frac{(a_2-D_2)^l}{b_2^M}. \tag{2.7}$$

From (2.4) and (2.7), it follows that

$$b_2^l e^{u_2(t_2)} < (a_2 - D_2)^M + \frac{D_2^M}{(a_2-D_2)^l} b_2^M e^{u_1(t_2)}.$$

That is

$$(a_2-D_2)^l b_2^l e^{u_2(t_2)} < (a_2-D_2)^M (a_2-D_2)^l + D_2^M b_2^M e^{u_1(t_1)}. \tag{2.8}$$

Multipling (2.3) by $e^{u_1(t_1)}$ gives

$$b_1^l e^{2u_1(t_1)} < (a_1 - D_1)^M e^{u_1(t_1)} + D_1^M e^{u_2(t_2)}, \tag{2.9}$$

from which, together with (2.8), it follows that

$$
\begin{aligned}
(2.10) \qquad & (a_2 - D_2)^l b_2^l b_1^l e^{2u_1(t_1)} \\
< \ & [(a_2 - D_2)^l (a_2 - D_1)^M b_2^l + D_1^M D_2^M b_2^M] e^{u_1(t_1)} \\
& + D_1^M (a_2 - D_2)^M (a_2 - D_2)^l.
\end{aligned}
$$

Thus there exists a positive constant ρ_1 such that

$$
(2.11) \qquad e^{u_1(t_1)} < \rho_1.
$$

Substituting (2.11) into (2.18), it follows that there exists a positive constant ρ_2 such that

$$
(2.12) \qquad e^{u_2(t_2)} < \rho_2.
$$

From (2.5) and (2.6), we have

$$
(2.13) \qquad e^{u_3(t_3)} < \frac{(\frac{e_1}{\alpha_1})^M}{q_1^l}
$$

and

$$
(2.14) \qquad e^{u_4(t_4)} < \frac{(\frac{e_2}{\alpha_2})^M}{q_2^l}.
$$

From (2.11)$-$(2.14), we have for $\forall t \in [0, w]$,

$$
(2.15) \qquad e^{u_1(t)} < \rho_1,
$$

$$
(2.16) \qquad e^{u_2(t)} < \rho_2,
$$

$$
(2.17) \qquad e^{u_3(t)} < \frac{(\frac{e_1}{\alpha_1})^M}{q_1^l}
$$

and

$$
(2.18) \qquad e^{u_4(t)} < \frac{(\frac{e_2}{\alpha_2})^M}{q_2^l}.
$$

Choose $\xi_i \in [0, w]$ such that

$$u_i(\xi_i) = \min_{t\in[0,w]} u_i(t), \quad i = 1, 2, 3, 4.$$

From this and system (2.2), we obtain

$$a_1(\xi_1) - D_1(\xi_1) - b_1(\xi_1)e^{u_1(\xi_1)} - \frac{c_1(\xi_1)e^{u_3(\xi_1)}}{1+\alpha_1(\xi_1)e^{u_1(\xi_1)}} - \frac{c_2(\xi_1)e^{u_4(\xi_1)}}{1+\alpha_2(\xi_1)e^{u_1(\xi_1)}} + D_1(\xi_1)e^{u_2(\xi_1)-u_1(\xi_1)} = 0, \tag{2.19}$$

$$a_2(\xi_2) - D_2(\xi_2) - b_2(\xi_2)e^{u_2(\xi_2)} + D_2(\xi_2)e^{u_1(\xi_2)-u_2(\xi_2)} = 0, \tag{2.20}$$

$$-d_1(\xi_3) + \frac{e_1(\xi_3 e^{u_1(\xi_3-\tau_1(\xi_3,e^{u_1(\xi_3)},e^{u_2(\xi_3)})}}{1+\alpha_1(\xi_3)e^{u_1(\xi_3-\tau_1(\xi_3,e^{u_1(\xi_3)},e^{u_2(\xi_3)}))}} - q_1(\xi_3)e^{u_3(\xi_3)} = 0 \tag{2.21}$$

and

$$-d_2(\xi_4) + \frac{e_2(\xi_4)e^{u_1(\xi_4)-\tau_1(\xi_4,e^{u_1(\xi_4)},e^{u_2(\xi_4)})}}{1+\alpha_1(\xi_4)e^{u_1(\xi_4-\tau_1(\xi_4,e^{u_1(\xi_4)},e^{u_2(\xi_4)}))}} - q_1(\xi_4)e^{u_4(\xi_4)} = 0. \tag{2.22}$$

From (2.19) and (2.20), we obtain

$$b_1(\xi)e^{u_1(\xi_1)} > a_1(\xi_1) - D_1(\xi_1) - c_1(\xi_1)e^{u_3(\xi_1)} - c_2(\xi_1)e^{u_4(\xi_1)}$$

and

$$b_2(\xi_2)e^{u_2(\xi_2)} > a_2(\xi_2) - D_2(\xi_2).$$

That is

$$e^{u_1(\xi_1)} > \frac{(a_1 - D_1)^l - c_1^M \frac{\left(\frac{e_1}{\alpha_1}\right)^M}{q_1^l} - c_2^M \frac{\left(\frac{e_2}{\alpha_2}\right)^M}{q_2^l}}{b_1^M} \tag{2.23}$$

and

$$e^{u_2(\xi_2)} > \frac{(a_2 - D_2)^l}{b_2^M}. \tag{2.24}$$

From (2.21) and (2.22), we have

$$(2.25)\quad q_1^M e^{u_3(\xi_3)} > q_1(\xi_3)e^{u_3(\xi_3)} > \frac{e_1^l e^{u_1(\xi_3-\tau_1(\xi_3,e^{u_1(\xi_3)},e^{u_2(\xi_3)}))}}{1+\alpha_1^M e^{u_1(\xi_3-\tau_1(\xi_3,e^{u_1(\xi_3)},e^{u_2(\xi_3)}))}} - d_1^M$$

and

$$(2.26)\quad q_2^M e^{u_4(\xi_4)} > q_2(\xi_4)e^{u_4(\xi_4)} > \frac{e_2^l e^{u_1(\xi_4-\tau_1(\xi_4,e^{u_1(\xi_4)},e^{u_2(\xi_4)}))}}{1+\alpha_2^M e^{u_1(\xi_4-\tau_1(\xi_4,e^{u_1(\xi_4)},e^{u_2(\xi_4)}))}} - d_2^M.$$

Since $f(x) = \frac{e_i^l x}{1+\alpha_i^M x}(i = 1, 2)$ is increasing in $x \in (0,\infty)$, then from (2.23), (2.25) and (2.26), it follows that

$$q_1^M e^{u_3(\xi_3)} > \frac{e_1^l\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_2}{\alpha_2})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]}{b_1^M + \alpha_1^M\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_1}{\alpha_1})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]} - d_1^M$$

and

$$q_2^M e^{u_4(\xi_4)} > \frac{e_2^l\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_2}{\alpha_2})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]}{b_1^M + \alpha_2^M\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_2}{\alpha_2})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]} - d_2^M.$$

By the conditions (H_3) and (H_5) in Theorem 2.1, we have

$$(2.27)\, q_1^M e^{u_3(\xi_3)} > \frac{(e_1^l - d_1^M\alpha_1^M)\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_1}{\alpha_1})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big] - d_1^M b_1^M}{b_1^M + \alpha_1^M\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_1}{\alpha_1})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]} > 0$$

and

$$(2.28)\, q_2^M e^{u_4(\xi_4)} > \frac{(e_2^l - d_2^M\alpha_2^M)\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_1}{\alpha_1})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big] - d_2^M b_1^M}{b_1^M + \alpha_2^M\Big[(a_1-D_1)^l - \frac{c_1^M}{q_1^l}(\frac{e_1}{\alpha_1})^M - \frac{c_2^M}{q_2^l}(\frac{e_2}{\alpha_2})^M\Big]} > 0.$$

Therefore, from (2.23), (2.24), (2.27) and (2.28), we have for $\forall t \in [0, w]$

$$(2.29)\qquad e^{u_i(t)} > \delta_i, \quad i = 1, 2, 3, 4,$$

where δ_i are some positive constants.

From (2.15)−(2.18) and (2.29), we obtain

$$\begin{aligned}
&|u_i(t)| < \max\{|\ln\delta_i|, |\ln\rho_i|\} \overset{def}{=} R_i, \quad i=1,2,\\
&|u_3(t)| < \max\Big\{|\ln\delta_3|, \Big|\ln\Big(\frac{e_1}{\alpha_1}\Big)^M\Big|\Big\} \overset{def}{=} R_3,\\
&|u_4(t)| < \max\Big\{|\ln\delta_4|, \Big|\ln\frac{1}{q_2^l}\Big(\frac{e_2}{\alpha_2}\Big)^M\Big|\Big\} \overset{def}{=} R_4.
\end{aligned}$$

Claery, $R_i (i = 1, 2, 3, 4)$ are independent of λ. Using the differential mean valued theorem, it follows that there exist some points $t_i^* \in [0, w] (i = 5, 6, 7, 8)$ such that

$$QN\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \overline{(a_1-D_1)} - \overline{b_i}e^{u_1} - \dfrac{\overline{c_1}e^{u_3}}{1+\alpha_1(t_3^*)e^{u_1}} - \dfrac{\overline{c_2}e^{u_4}}{1+\alpha_2(t_6^*)e^{u_1}} + \overline{D_1}e^{u_2-u_1} \\ \overline{(a_2-D_2)} - \overline{b_2}e^{u_2} + \overline{D}_2e^{u_1-u_2} \\ -\overline{d_1} + \dfrac{\overline{e_1}e^{u_1}}{1+\alpha_1(t_7^*)e^{u_1}} - \overline{q_1}e^{u_3} \\ -\overline{d_2} + \dfrac{\overline{e_2}e^{u_1}}{1+\alpha_2(t_8^*)e^{u_1}} - \overline{q_2}e^{u_4} \end{bmatrix}.$$

Denote $M = R_1 + R_2 + R_3 + R_0$; here R_0 is taken sufficiently large such that each solution $(\alpha^*, \beta^*, \gamma^*, v^*)^T$ of the following system:

$$(2.30)\begin{cases}
\overline{a_1 - D_1} - \overline{b_1}e^{\alpha} - \dfrac{\overline{c_1}e^{r}}{1+\alpha_1(t_5^*)e^{\alpha}} - \dfrac{\overline{c_2}e^{v}}{1+\alpha_2(t_6^*)e^{\alpha}} + \overline{D_1}e^{\beta-\alpha} = 0,\\
\overline{a_2 - D_2} - \overline{b_2}e^{\beta} + \overline{D_2}e^{\alpha-\beta} = 0,\\
-\overline{d_1} + \dfrac{\overline{e_1}e^{\alpha}}{1+\alpha_1(t_7^*)e^{\alpha}} - \overline{q}_1e^{\gamma} = 0,\\
-\overline{d_2} + \dfrac{\overline{e_2}e^{\alpha}}{1+\alpha_2(t_8^*)e^{\alpha}} - \overline{q}_2e^{\gamma} = 0,
\end{cases}$$

satisfies $\|(\alpha^*, \beta^*, r^*, v^*)^T\| = |\alpha^*| + |\beta^*| + |r^*| + |v^*| < M$, provided that system (2.30) has a solution or a number of solutions. Now we take $\Omega = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in X : \|(u_1, u_2, u_3, u_4)^T\| < M\}$. This satisfies condition (a) in Theorem A. When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap R^4$,

$(u_1, u_2, u_3, u_4)^T$ is a constant vector in R^4 with $|u_1|+|u_2|+|u_3|+|u_4| = M$. If system (2.30) has a solution or a number of solutions, then

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If system (2.30) does not have a solution, then naturally

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This prove that condition (b) in Theorem A is satisfied.

Finally we will prove that condition (c) in Theorem A is satisfied. To this end, we define $\phi : \mathrm{Dom}L \times [0,1] \to X$ by

$$\begin{aligned} \phi(u_1, u_2, u_3, u_4) \quad &= \quad \begin{bmatrix} \overline{(a_1 - D_1)} - \overline{b}_1 e^{u_1} \\ \overline{(a_2 - D_1)} - \overline{b}_2 e^{u_2} \\ -\overline{d_1} + \dfrac{\overline{e_1} e^{u_1}}{1+\alpha_1(t_7^*)e^{u_1}} - \overline{q_1} e^{u_3} \\ -\overline{d_2} + \dfrac{\overline{e_2} e^{u_1}}{1+\alpha_2(t_8^*)e^{u_1}} - \overline{q_2} e^{u_4} \end{bmatrix} \\ &\quad + \mu \begin{bmatrix} \dfrac{-\overline{c_1} e^{u_3}}{1+\alpha_1(t_5^*)e^{u_1}} - \dfrac{\overline{e_2} e^{u_4}}{1+\alpha_2(t_6^*)e^{u_1}} + \overline{D_1} e^{u_2 - u_1} \\ \overline{D_2} e^{u_1 - u_2} \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

where $\mu \in [0,1]$ is a parameter. When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap R^4$, $(u_1, u_2, u_3, u_4)^T$ is a constant vector in R^4 with $|u_1| + |u_2| + |u_3| + |u_4| = M$. We will show that when $(u_1,u_2,u_3,u_4)^T \in \partial\Omega\cap \mathrm{Ker}L$, $\phi(u_1,u_2,u_3,u_4,\mu)\neq 0$. If the conclusion is not true, i.e., constant vector $(u_1,u_2,u_3,u_4)^T$ with $|u_1|+|u_2|+|u_3|+|u_4|=M$ satisfies $\phi(u_1,u_2,u_3,u_4,\mu)=0$, then form

$$\begin{cases} \overline{(a_1-D_1)}-\overline{b}_1e^{u_1}-\mu\dfrac{\overline{c_1}e^{u_3}}{1+\alpha_1(t_5^*)e^{u_1}}-\mu\dfrac{\overline{c_2}e^{u_4}}{1+\alpha_2(t_6^*)e^{u_1}}+\mu\overline{D_1}e^{\mu_2-\mu_1}=0,\\ \overline{(a_2-D_2)}-\overline{b}_2e^{u_2}+\mu\overline{D_2}e^{u_1-u_2}=0,\\ -\overline{d_1}+\dfrac{\overline{e_1}e^{u_1}}{1+\alpha_1(t_7^*)e^{u_1}}-\overline{q}_1e^{u_3}=0,\\ -\overline{d_2}+\dfrac{\overline{e_2}e^{u_1}}{1+\alpha_2(t_8^*)e^{u_1}}-\overline{q}_2e^{u_4}=0, \end{cases}$$

by following the arguments of (2.11)−(2.14), (2.27) and (2.28) and magnifying $\overline{f}$ into f^M and reducing $\overline{f}$ into f^l, here f denotes every function in (H_1), we obtain

$$|u_1|<\max\{|\ln\delta_i|,|\ln\rho_i|\},\quad i=1,2,$$

$$|u_3|<\max\Big\{|\ln\delta_3|,\Big|\ln\frac{1}{q_1^l}\Big(\frac{e_1}{\alpha_1}\Big)^M\Big|\Big\}$$

$$|u_4|<\max\Big\{|\ln\delta_4|,\Big|\ln\frac{1}{q_2^l}\Big(\frac{e_2}{\alpha_2}\Big)\Big|\Big\}.$$

Thus

$$\begin{aligned}|u_1|+|u_2|+|u_3|+|u_4|<&\sum_{i=1}^{2}\max\{|\ln\delta_i|,|\ln\rho_i|\}\\ &+\max\Big\{|\ln\delta_3|,\Big|\ln\frac{1}{q_1}\Big(\frac{e_1}{\alpha_1}\Big)^M\Big|\Big\}+\max\Big\{|\ln\delta_4|,\Big|\ln\frac{1}{q_2^l}\Big(\frac{e_2}{\alpha_2}\Big)^M\Big|\Big\}<M,\end{aligned}$$

which contradicts the fact that constant vector $(u_1,u_2,u_3,u_4)^T$ satisfies $|u_1|+|u_2|+|u_3|+|u_4|=M$. Using the property of topological degree and taking $J=I:\mathrm{Im}L\to\mathrm{Ker}L$, $(u_1,u_2,u_3,u_4)^T\to(u_1,u_2,u_3,u_4)^T$, we have

$$\begin{aligned}
&\deg(JQN(u_1,u_2,u_3,u_4)^T,\Omega\cap\text{Ker}L,(0,0,0,0)^T)\\
=\;&\deg(\phi(u_1,u_2,u_3,u_4),\Omega\cap\text{Ker}L,(0,0,0,0)^T)\\
=\;&\deg(\phi(u_1,u_2,u_3,u_4,0),\Omega\cap\text{Ker}L,(0,0,0,0)^T)\\
=\;&\deg\Big\{\Big[\overline{(a_1-D_1)}-\overline{b}_1e^{u_1},\overline{(a_2-D_2)}-\overline{b}_2e^{u_2},\\
&-\overline{d_1}+\frac{\overline{e_1}e^{u_1}}{1+\alpha_1(t_7^*)e^{u_1}}-\overline{q}_1e^{u_3},\\
&-\overline{d_2}+\frac{\overline{e_2}e^{u_1}}{1+\alpha_2(t_8^*)e^{u_1}}-\overline{q}_2e^{u_4},\Big]^T,\Omega\cap\text{Ker}L,(0,0,0,0)^T\Big\}.
\end{aligned}$$

Because of the conditions of Theorem 2.1, then the system of algebraic equations:

$$\begin{cases}
\overline{(a_1-D_1)}-\overline{b_1}u=0,\\
\overline{(a_2-D_2)}-\overline{b_2}u=0,\\
-\overline{d_1}+\dfrac{\overline{e_1}u}{1+\alpha_1(t_7^*)u}-\overline{q_1}m=0,\\
-\overline{d_2}+\dfrac{\overline{e_2}u}{1+\alpha_2(t_8^*)u}-\overline{q_2}n=0
\end{cases}$$

has a unique solution (u^*,v^*,m^*,n^*) which satisfies:

$$u^*=\frac{\overline{(a_1-D_1)}}{\overline{b_1}}>0$$

$$v^*=\frac{\overline{(a_2-D_2)}}{\overline{b_2}}>0$$

$$m^*=\frac{(\overline{e_1}-\overline{d_1}\alpha_1(t_7^*))\overline{(a_1-D_1)}-\overline{d_1}\,\overline{b_1}}{\overline{q_1}\Big[\overline{b_1}+\alpha_1(t_7^*)\overline{(a_1-D_1)}\Big]}>0,$$

and

$$n^*=\frac{(\overline{e_1}-\overline{d_2}\alpha_2(t_8^*))\overline{(a_2-D_2)}-\overline{d_2}\,\overline{b_1}}{\overline{q_2}\Big[\overline{b_1}+\alpha_2(t_8^*)\overline{(a_2-D_2)}\Big]}>0.$$

Hence

$$\begin{aligned}
&\deg(JQN(u_1,u_2,u_3,u_4)^T,\Omega\cap \mathrm{Ker}L,(0,0,0,0)^T)\\
=&\begin{vmatrix}
-\overline{b_1}u^* & 0 & 0 & 0\\
0 & -\overline{b_2}v^* & 0 & 0\\
\left(\frac{\overline{e_1}u^*}{1+\alpha_1(t_7^*)u^*}\right)^2 & 0 & -\overline{q_1}m^* & 0\\
\left(\frac{\overline{e_2}u^*}{1+\alpha_2(t_8^*)u^*}\right)^2 & 0 & 0 & -\overline{q_2}n^*
\end{vmatrix}\\
=&\ \mathrm{sign}(\overline{b_1}\,\overline{b_2}\,\overline{q_1}\,\overline{q_2}u^*v^*m^*n^*)\\
=&\ 1.
\end{aligned}$$

This completes the proof of condition (c) in Theorem A and the proof of Theorem 2.1 is completed.

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