

LUSIN APPROXIMATION OF SOBOLEV FUNCTIONS BY HÖLDER CONTINUOUS FUNCTIONS

BY

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Abstract. The idea of Lusin approximation is applied to the case of functions in the Sobolev spaces, which are shown to be Hölder continuous except over small sets. The result matches in a natural way with the Sobolev imbedding theorem and is well suited to get extensions of some classical interior regularity results for Poisson's equation.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a regular open set and let μ be the Lebesgue measure in \mathbb{R}^N . For a given real-valued measurable function u defined on Ω , the classical Lusin theorem gives the existence of a continuous approximant of u that can be chosen so to have the measure of the set where the approximant differs from u being arbitrary small.

There has been interest in using this type of approximation for functions in the Sobolev spaces $W^{1,p}$, by authors working either in Real or Harmonic Analysis or in Calculus of Variations, see for example [4], [6], [11], [12], [13], [14] and [16]. Taking inspiration from [12], E. Acerbi and N. Fusco proved Lemma I.11 in [2], which later on led them to the following result:

Theorem 1.1. *Let be $p \geq 1$. There exists a constant C , which only depends on N and p , such that for any $u \in W^{1,p}(\Omega)$ and any positive number*

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K , there exists a function $v \in W^{1,\infty}(\Omega)$ such that:

- (i) $\|v\|_{W^{1,\infty}(\Omega)} \leq K$ and
- (ii) $\mu(\{x : v(x) \neq u(x)\}) \leq C(\|u\|_{W^{1,p}}/K)^p$.

The details of the proof can be found in [4]. This result can also be proved changing slightly the statement of Theorem 3 and combining it with Theorem 2, both in [12]. Theorem 1.1, or its precursor Lemma I.11 in [2], have been used by several authors in the Calculus of Variations, see [2], [3] and [18]. We prove here the following extension of Theorem 1.1:

Theorem 1.2. *Let be $p \geq 1$ and α a positive real number such that $1 - N/p < \alpha < 1$. Then there exists a constant C , depending only on N , p and α , such that for any $u \in W^{1,p}(\Omega)$ and any positive number K , there exists a function $v \in C^{0,\alpha}(\overline{\Omega})$ satisfying:*

- (i) $\|v\|_{C^{0,\alpha}(\overline{\Omega})} \leq K$ and
- (ii) $\mu(\{x : v(x) \neq u(x)\}) \leq C(\|u\|_{W^{1,p}}/K)^{p+\delta}$, with $\delta = \frac{p^2(1-\alpha)}{N-p(1-\alpha)}$.

Due to the improvement in (ii) and the weaker space in which we look for the approximating functions, it is no longer possible to obtain this result by combining Theorem 2 and the modified version of Theorem 3 in [12].

We proved a much weaker version of Theorem 1.2 in [10]. A somewhat similar result was proved using Bessel potentials by Malý in [14], where instead of controlling the growth of the Hölder norm of the approximants, he makes the Sobolev norm of the difference to go to zero. Hajlasz and Kinunen proved a result very similar to ours, see Theorem 5.3 in [11], actually stronger as they can control both norms and it is valid for Ω in a metric space having a doubling measure. Their proof, however, uses two fractional maximal functions, it relies heavily on a Whitney type covering theorem, it does not cover the case $p = 1$ and gives the approximants being Hölder continuous only on bounded sets. Therefore we think that an alternative,

and seemingly simpler, proof of the smoothing part of the result might be of interest.

To prove Theorem 1.2 we follow the general idea of the proof of Theorem 1.1 in [4], however we can not use the Hardy-Littlewood maximal function, which has to be replaced by a new maximal function, whose properties have to be proved starting from scratch. In Section 2 we prove several preliminary lemmas. In Section 3 we can then easily prove Theorem 1.2. Section 4 is devoted to discuss the connection between our result and the Sobolev imbedding theorem. In Section 5 we show how Theorem 1.2, or Theorem 5.3 in [11], can be used to extend some classical estimates and interior regularity results for Poisson's equation.

The space $C^{0,\alpha}(\overline{\Omega})$ of (uniformly) Hölder continuous functions with exponent α , $\alpha \in (0, 1]$, see [1] or [9], is formed by all the bounded and uniformly continuous functions defined on Ω and such that the following quantity, called the α -Hölder coefficient of the function, is finite

$$[v]_{C^{0,\alpha}(\Omega)} = \sup_{x,y \in \Omega} \frac{|v(x) - v(y)|}{|x - y|^\alpha}.$$

With the norm $\|v\|_{C^{0,\alpha}(\overline{\Omega})} = \|v\|_{L^\infty(\Omega)} + [v]_{C^{0,\alpha}(\Omega)}$, $C^{0,\alpha}(\overline{\Omega})$ is a Banach space and it might be thought of as a space of functions with derivatives of fractionary order. Using induction, for $k \in \mathbb{N}$ one can say that $u \in C^{k,\alpha}(\overline{\Omega})$ if and only if $u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \in C^{k-1,\alpha}(\overline{\Omega})$. If $0 < \alpha < \beta \leq 1$ and $k \in \mathbb{N}$, the following inclusions hold $C^{k,\beta}(\overline{\Omega}) \subset C^{k,\alpha}(\overline{\Omega}) \subset C^k(\overline{\Omega})$.

If $p > N$ the Sobolev imbedding theorem asserts that $W^{1,p}$ is continuously injected into $C^{0,1-N/p}$, hence in this case, our result deals only with Lusin approximation by functions in $C^{0,\alpha}$ with $\alpha \in (1 - N/p, 1)$. In the limiting case $\alpha = 1$ we can not expect to improve Theorem 1.1 since then the approximant is chosen from the space of uniformly Lipschitz continuous functions which coincides with $W^{1,\infty}(\Omega)$.

Notation. We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω . $B(x, r)$ denotes the ball in \mathbb{R}^N centered at x and with radius r . ω_N is the volume of the N -dimensional unit ball and γ_N will be the volume of the intersection of two N -dimensional unit balls whose centers are at distance one. It will always be assumed that $p \in [1, \infty)$ with p' being the conjugate exponent of p , this is $p' = \frac{p}{p-1}$ if $p > 1$ and $p' = \infty$ if $p = 1$. α will be a positive real number such that $1 - N/p < \alpha < 1$.

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2. Preliminary results. One important ingredient in the proof of Theorem 1.1 is the Hardy-Littlewood maximal function of $|\text{grad } u|$, defined for a function $u \in W^{1,p}(\mathbb{R}^N)$ as

$$M(|\text{grad } u|)(x) = \sup_{r>0} \frac{1}{\omega_N r^N} \int_{B(0,r)} |\text{grad } u(x+y)| \, dy,$$

and which is finite almost everywhere in \mathbb{R}^N . It is known that M satisfies a weak-type inequality, this is that there exists a constant A , depending only on N and p , such that

$$\mu(\{x : M(|\text{grad } u|)(x) > \lambda\}) \leq A (\|\text{grad } u\|_{L^p}/\lambda)^p \quad \text{for all } \lambda > 0,$$

see Theorem 1 on page 5 of [17]. This bound is instrumental in obtaining property (ii) of Theorem 1.1.

Eventhough $M(|\text{grad } u|)$ is well adapted to prove Theorem 1.1, it is not the right object to prove Theorem 1.2 and then we define $M'_\alpha(u)$, a weighted maximal function of $\text{grad } u$ and which will be shown to satisfy a weak-type inequality. Let be

$$g_r(y) = \frac{1}{\omega_N r^N} |y|^{1-\alpha} \chi_{B(0,r)}(y),$$

where $\chi_{B(0,r)}$ is the characteristic function associated to $B(0,r)$. For $u \in W^{1,p}(\mathbb{R}^N)$ we define

$$M'_{\alpha,r}(u)(x) = \int_{\mathbb{R}^N} g_r(y) |\text{grad } u(x+y)| dy,$$

which is finite due to Hölder's inequality, to finally define

$$M'_\alpha(u)(x) = \sup_{r>0} M'_{\alpha,r}(u)(x).$$

For $\lambda > 0$ let the set $H_\lambda(u)$ be defined as

$$H_\lambda(u) = \{ x \in \mathbb{R}^N : M'_\alpha(u)(x) \leq \lambda \}.$$

Lemma 1. *There exists a constant C_1 such that for all $\lambda > 0$ and $u \in \mathcal{D}(\mathbb{R}^N)$*

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_1 \lambda \quad \text{for all } x, y \in H_\lambda(u).$$

Proof. For $x \in H_\lambda(u)$ we define

$$S_{K,r}(x) = \left\{ y \in B(x,r) : \frac{|u(x) - u(y)|}{|x - y|^\alpha} \geq K\lambda \right\}.$$

Then

$$K\lambda \frac{\mu(S_{K,r}(x))}{\mu(B(x,r))} \leq \frac{1}{\omega_N r^N} \int_{B(x,r)} \frac{|u(x) - u(y)|}{|x - y|^\alpha} dy$$

$$\begin{aligned}
&= \frac{1}{\omega_N r^N} \int_{B(0,r)} \frac{|u(x+y) - u(x)|}{|y|^\alpha} dy \\
&= \frac{1}{\omega_N r^N} \int_{B(0,r)} \frac{1}{|y|^\alpha} \left| \int_0^1 \langle \text{grad } u(x+ty), y \rangle dt \right| dy \\
&\leq \frac{1}{\omega_N r^N} \int_{B(0,r)} |y|^{1-\alpha} \int_0^1 |\text{grad } u(x+ty)| dt dy.
\end{aligned}$$

Now we interchange the order of integration by Fubini's Theorem and after that we make the following change of variables for the interior integral: $z = ty$, where now t is just a parameter. The previous line then becomes equal to

$$\begin{aligned}
&\int_0^1 \frac{1}{\omega_N r^N} \int_{B(0,tr)} t^{\alpha-1} |z|^{1-\alpha} |\text{grad } u(x+z)| t^{-N} dz dt \\
&= \int_0^1 t^{\alpha-1} \frac{1}{\omega_N (tr)^N} \int_{B(0,tr)} |z|^{1-\alpha} |\text{grad } u(x+z)| dz dt \\
&\leq \frac{1}{\alpha} M'_\alpha(u)(x) \leq \frac{\lambda}{\alpha}.
\end{aligned}$$

Therefore for any $x \in H_\lambda(u)$

$$\mu(S_{K,r}(x)) \leq \frac{1}{K\alpha} \omega_N r^N \quad \text{for all } K, r > 0.$$

Fix $x, y \in H_\lambda(u)$ and choose K large enough so that

$$(B(x,r) \cap B(y,r)) \setminus (S_{K,r}(x) \cup S_{K,r}(y)) \neq \emptyset,$$

where $r = |x - y|$. We can do this since from $\gamma_N = \mu(B(x,r) \cap B(y,r))/r^N$, we have

$$\mu \{ (B(x,r) \cap B(y,r)) \setminus (S_{K,r}(x) \cup S_{K,r}(y)) \} \geq \gamma_N r^N - \frac{2}{K\alpha} \omega_N r^N.$$

Hence it suffices to take

$$K > \frac{2\omega_N}{\alpha\gamma_N}.$$

Therefore there exists $z \in (B(x, r) \cap B(y, r)) \setminus (S_{K,r}(x) \cup S_{K,r}(y))$ such that

$$\frac{|u(x) - u(z)|}{|x - z|^\alpha} \leq K\lambda \quad \text{and} \quad \frac{|u(y) - u(z)|}{|y - z|^\alpha} \leq K\lambda,$$

and so

$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(y) - u(z)| \leq 2K\lambda r^\alpha = 2K\lambda|x - y|^\alpha$$

and hence

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 2K\lambda,$$

which then gives the $C^{0,\alpha}$ property by choosing $C_1 = 2K$, which is greater than 8 since $\omega_N \geq 2\gamma_N$.

The key point of the proof is an adaptation of the proof of Theorem 3 in [12].

Lemma 2. *Let $u \in W^{1,p}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ such that $M(|\text{grad } u|)(x) < \infty$. Then $M'_\alpha(u)(x) < \infty$.*

Proof. If $r \leq 1$ then

$$M'_{\alpha,r}(u)(x) \leq r^{1-\alpha} M(|\text{grad } u|)(x) \leq M(|\text{grad } u|)(x).$$

Now for $r > 1$ we have from Hölder's inequality

$$M'_{\alpha,r}(u)(x) \leq \|g_r\|_{L^{p'}} \|\text{grad } u\|_{L^p} \leq C \|\text{grad } u\|_{L^p},$$

since $\|g_r\|_{L^{p'}} = Cr^{1-\alpha+N(1/p'-1)} = Cr^{1-\alpha-N/p} \leq C$ because $\alpha > 1 - N/p$.

Lemma 3. *Let be $u \in W^{1,p}(\mathbb{R}^N)$ and let $\{u_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^N)$ that converges to u in $W^{1,p}$. For $x \in \mathbb{R}^N$ such that $M(|\text{grad } u_n|)(x) \rightarrow$*

$M(|\text{grad } u|)(x)$, with $M(|\text{grad } u|)(x) < \infty$, we have that when $n \rightarrow \infty$

$$M'_\alpha(u_n)(x) \rightarrow M'_\alpha(u)(x).$$

Proof. For $r > 0$ we have that

$$\begin{aligned} |M'_{\alpha,r}(u_n)(x) - M'_{\alpha,r}(u)(x)| &= \left| \int_{\mathbb{R}^N} g_r(y) (|\text{grad } u_n(x+y)| - |\text{grad } u(x+y)|) dy \right| \\ &\leq \int_{\mathbb{R}^N} g_r(y) |\text{grad } (u_n - u)(x+y)| dy, \end{aligned}$$

but now we apply Hölder's inequality to this integral to get that it is less or equal than

$$\|g_r\|_{L^{p'}} \|\text{grad } (u_n - u)\|_{L^p}.$$

Therefore for a fixed r : $M'_{\alpha,r}(u_n)(x) \rightarrow M'_{\alpha,r}(u)(x)$ as $n \rightarrow \infty$.

Now, from Lemma 2 we have that $M'_\alpha(u)(x)$, $M'_\alpha(u_n)(x) < \infty$, and then for $\varepsilon > 0$ there exists $r > 0$ such that for all n

$$M'_\alpha(u)(x) - M'_\alpha(u_n)(x) \leq M'_{\alpha,r}(u)(x) - M'_{\alpha,r}(u_n)(x) + \varepsilon,$$

but $M'_{\alpha,r}(u_n)(x) \rightarrow M'_{\alpha,r}(u)(x)$ when $n \rightarrow \infty$, therefore

$$\liminf_{n \rightarrow \infty} M'_\alpha(u_n)(x) \geq M'_\alpha(u)(x).$$

Now we want to show that

$$\limsup_{n \rightarrow \infty} M'_\alpha(u_n)(x) \leq M'_\alpha(u)(x).$$

If this were false, there would exist $\varepsilon > 0$, $N \in \mathbb{N}$ and a subsequence, that we do not relabel, such that if $n \geq N$ then

$$M'_\alpha(u)(x) + \varepsilon \leq M'_\alpha(u_n)(x).$$

But then, using Lemma 2 we can select a sequence of radii r_n such that

$$M'_\alpha(u_n)(x) \leq M'_{\alpha,r_n}(u_n)(x) + \frac{1}{n}.$$

Now since

$$M'_{\alpha,r_n}(u_n)(x) \leq r_n^{1-\alpha} M(|\text{grad } u_n|)(x),$$

we have that

$$\liminf_{n \rightarrow \infty} M'_{\alpha,r_n}(u_n)(x) \leq \liminf_{n \rightarrow \infty} r_n^{1-\alpha} M(|\text{grad } u_n|)(x).$$

But $M(|\text{grad } u_n|)(x) \rightarrow M(|\text{grad } u|)(x)$ when $n \rightarrow \infty$, and then if $\liminf_{n \rightarrow \infty} r_n = 0$ we would get that

$$\liminf_{n \rightarrow \infty} M'_{\alpha,r_n}(u_n)(x) = 0,$$

which is impossible.

Let then be $\delta > 0$ and $N_0 \in \mathbb{N}$ such that $r_n > \delta$ if $n > N_0$, then since

$$\begin{aligned} M'_\alpha(u_n)(x) - M'_\alpha(u)(x) &\leq M'_{\alpha,r_n}(u_n)(x) - M'_{\alpha,r_n}(u)(x) + \frac{1}{n} \\ &\leq \|g_{r_n}\|_{L^{p'}} \|\text{grad } (u_n - u)\|_{L^p} + \frac{1}{n} \\ &\leq C\delta^{1-N/p-\alpha} \|\text{grad } (u_n - u)\|_{L^p} + \frac{1}{n}, \end{aligned}$$

and this quantity goes to zero when n grows to ∞ . Then

$$\limsup_{n \rightarrow \infty} M'_\alpha(u_n)(x) \leq M'_\alpha(u)(x).$$

Lemma 4. *For any $u \in W^{1,p}(\mathbb{R}^N)$ there exists a set E , with zero Lebesgue measure, such that for every $\lambda > 0$*

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_1 \lambda \quad \text{for all } x, y \in H_\lambda(u) \setminus E.$$

Proof. Let $u_n \in \mathcal{D}(\mathbb{R}^N)$ be such that $u_n \rightarrow u$ in the $W^{1,p}$ norm. Then, possibly after extracting subsequences, we can also assume that $u_n \rightarrow u$ and $M(|\text{grad } u_n|) \rightarrow M(|\text{grad } u|)$ pointwise for almost every $x \in \mathbb{R}^N$. Furthermore since $M(|\text{grad } u|)$ is finite almost everywhere, using Lemma 3 we also have almost everywhere convergence of $M'_\alpha(u_n)$ to $M'_\alpha(u)$.

Let E be a subset of \mathbb{R}^N , with zero Lebesgue measure, such that for $x \in \mathbb{R}^N \setminus E$:

$$u_n(x) \rightarrow u(x) \quad \text{and} \quad M'_\alpha(u_n)(x) \rightarrow M'_\alpha(u)(x).$$

Then for $x, y \in H_\lambda(u) \setminus E$ and any $\delta > 0$ there exists an N_λ such that if $n > N_\lambda$ then $x, y \in H_{\lambda+\delta}(u_n)$, therefore we can apply Lemma 1 to u_n , to get

$$\frac{|u_n(x) - u_n(y)|}{|x - y|^\alpha} \leq C_1(\lambda + \delta)$$

and now by letting $n \rightarrow \infty$ we obtain

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_1(\lambda + \delta).$$

But we can just let $\delta \rightarrow 0$ to get that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_1\lambda.$$

Proposition 2.1. *Let be $u \in W^{1,p}(\mathbb{R}^N)$. Then $M'_\alpha(u)$ is continuous when restricted to the complement of the negligible set*

$$E_0 = \{x \in \mathbb{R}^N : M(|\text{grad } u|)(x) = \infty\}.$$

Proof. Let be $x \in \mathbb{R}^N \setminus E_0$.

We show first that $M'_\alpha(u)$ is upper semicontinuous at x . Let $\{x_n\} \subset \mathbb{R}^N \setminus E_0$ be a sequence converging to x and such that the limit of $M'_\alpha(u)(x_n)$ exists

and it is greater than $M'_\alpha(u)(x)$. Then there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$n \geq N \implies M'_\alpha(u)(x_n) \geq M'_\alpha(u)(x) + \varepsilon.$$

Then, because $M'_\alpha(u)(x), M'_\alpha(u_n)(x) < \infty$, for $n \geq N$ there exist $r_n > 0$ such that

$$n \geq N \implies M'_{\alpha, r_n}(u)(x_n) - M'_{\alpha, r_n}(u)(x) \geq \varepsilon/2.$$

But

$$\begin{aligned} & M'_{\alpha, r_n}(u)(x_n) - M'_{\alpha, r_n}(u)(x) \\ &= \frac{1}{w_N r_n^N} \int_{B(0, r_n)} |y|^{1-\alpha} (|\text{grad } u(x_n + y)| - |\text{grad } u(x + y)|) dy \\ &\leq \frac{1}{w_N r_n^N} \| |y|^{1-\alpha} \|_{L^{p'}(B(0, r_n))} \| \text{grad } u(x_n + \cdot) - \text{grad } u(x + \cdot) \|_{L^p(B(0, r_n))}, \\ &\leq C r_n^{1-N/p-\alpha} \| \text{grad } u(x_n + \cdot) - \text{grad } u(x + \cdot) \|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

But now we can repeat the argument in the proof of Lemma 3 above, to get that the r_n stay away from zero and then from the continuity of the translation in L^p , see [1], we get that the last quantity goes to zero when $n \rightarrow \infty$.

The lower semicontinuity is obtained in the same way.

Lemma 5. *There exists a constant C_2 , depending only on p , N and α , such that for all $u \in W^{1,p}(\mathbb{R}^N)$ and all $\lambda > 0$ we have that*

$$\mu(\{x : M'_\alpha(u)(x) > \lambda\}) \leq C_2 (\|\text{grad } u\|_{L^p}/\lambda)^{p+\delta} \quad \text{with } \delta = \frac{p^2(1-\alpha)}{N-p(1-\alpha)}.$$

Proof. Let us fix $\lambda > 0$ and define

$$E_\lambda = \{x \in \mathbb{R}^N \setminus E_0 : M'_\alpha(u)(x) > \lambda\},$$

which, from Proposition 2.1, is a measurable set.

For every $x \in E_\lambda$ there exists $r(x) > 0$ such that

$$(1) \quad \frac{1}{\omega_N r(x)^N} \int_{B(x, r(x))} |y - x|^{1-\alpha} |\text{grad } u(y)| \, dy \geq \lambda.$$

From Hölder's inequality we have for $p > 1$ that

$$\begin{aligned} & \int_{B(x, r(x))} |y - x|^{1-\alpha} |\text{grad } u(y)| \, dy \\ & \leq \left(\int_{B(x, r(x))} |\text{grad } u(y)|^p \, dy \right)^{1/p} \left(\int_{B(0, r(x))} |y|^{p'(1-\alpha)} \, dy \right)^{1/p'}, \end{aligned}$$

but

$$\left(\int_{B(0, r(x))} |y|^{p'(1-\alpha)} \, dy \right)^{1/p'} = \left(\frac{N\omega_N}{N + p'(1-\alpha)} \right)^{1/p'} r(x)^{1-\alpha+N/p'},$$

then calling

$$B = \left(\frac{N}{N + p'(1-\alpha)} \right)^{1/p'},$$

we have from (1) that

$$(2) \quad \lambda \leq B \omega_N^{-1/p} r(x)^{1-\alpha-N/p} \left(\int_{B(x, r(x))} |\text{grad } u(y)|^p \, dy \right)^{1/p},$$

and therefore

$$\lambda \leq B \omega_N^{-1/p} r(x)^{1-\alpha-N/p} \|\text{grad } u\|_{L^p}$$

from where we deduce that necessarily

$$r(x) \leq r_{max} = \left(B \omega_N^{-1/p} \|\text{grad } u\|_{L^p} / \lambda \right)^{p/(N-p(1-\alpha))}.$$

Now from (2) we get that

$$\lambda \left(\omega_N r(x)^N \right)^{1/p} \leq B r(x)^{1-\alpha} \left(\int_{B(x, r(x))} |\text{grad } u(y)|^p \, dy \right)^{1/p},$$

and so

$$(3) \quad \lambda^p \omega_N r(x)^N \leq B^p r_{max}^{p(1-\alpha)} \int_{B(x,r(x))} |\text{grad } u(y)|^p dy.$$

Clearly

$$E_\lambda \subset \bigcup_{x \in E_\lambda} B(x, r(x))$$

and therefore, by the Vitali covering lemma (see [17]), we can select a disjoint subfamily of this family of bounded balls, with centers in $V \subset E_\lambda$, such that

$$\sum_{x \in V} \mu(B(x, r(x))) \geq C_v \mu(E_\lambda),$$

where C_v only depends on N and the best constant is $C_v = 3^{-N}$.

Now we add all the inequalities (3) for the balls in V , to get that

$$\lambda^p 3^{-N} \mu(E_\lambda) \leq B^p r_{max}^{p(1-\alpha)} \|\text{grad } u\|_{L^p}^p,$$

hence

$$\mu(E_\lambda) \leq 3^N B^p r_{max}^{p(1-\alpha)} \frac{\|\text{grad } u\|_{L^p}^p}{\lambda^p},$$

but since $\delta = \frac{p^2(1-\alpha)}{N-p(1-\alpha)}$, from the definition of r_{max} we obtain that

$$\mu(E_\lambda) \leq C_2 (\|\text{grad } u\|_{L^p} / \lambda)^{p+\delta},$$

with

$$C_2 = 3^N \omega_N^{-\delta/p} B^{p+\delta}.$$

To finish the proof, we study the case $p = 1$, but we just need to notice that

$$\int_{B(x,r(x))} |y-x|^{1-\alpha} |\text{grad } u(y)| dy \leq r(x)^{1-\alpha} \int_{B(x,r(x))} |\text{grad } u(y)| dy,$$

and then after similar arguments we get that $\delta = \frac{1-\alpha}{N-(1-\alpha)}$ and

$$\mu(E_\lambda) \leq 3^N \omega_N^{-\delta} (\|\text{grad } u\|_{L^1}/\lambda)^{1+\delta},$$

so this case is contained in the general situation with $B = 1$.

Now, using an argument similar to the Lipschitz extension results: Lemma 3.2 in [7], known as Mac Shane lemma, or Theorem 1 of chapter 3 in [8], we prove an extension lemma for $C^{0,\alpha}(\bar{\Omega})$. After we have obtained this result, we were not surprised to learn that much more general results have been known for a long time, as Theorem 1 in [15] or even [5]. Anyways since we only need the real-valued case, we present it here with no claim of originality.

Lemma 6. *Let $F, G \subset \mathbb{R}^N$ with $F \subset G$ and let $f \in C^{0,\alpha}(\bar{F})$. There exists a function $g \in C^{0,\alpha}(\bar{G})$ such that g extends f to G and*

$$\|g\|_{L^\infty(G)} = \|f\|_{L^\infty(F)}, \quad [g]_{C^{0,\alpha}(G)} = [f]_{C^{0,\alpha}(F)}.$$

Proof. Let $K = [f]_{C^{0,\alpha}(F)}$, which is finite. Then for $x \in G$ we just define

$$g(x) = \max \left\{ -\|f\|_{L^\infty(F)}, \sup_{y \in F} \{ f(y) - K|x - y|^\alpha \} \right\}.$$

Then it is easy to verify that all the properties mentioned in the lemma hold for g .

3. Proof of the main result. First we need to extend u to all of \mathbb{R}^N and to have that

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \leq C_3 \|u\|_{W^{1,p}(\Omega)}.$$

It is here where the regularity of Ω plays a role and the precise conditions are discussed in remark 1 below. From now on we work with the extension of u , so that we can use Lemmas 4 and 5.

For $\lambda = \frac{K}{2C_1}$ we define now

$$H'_\lambda(u) = \{x : M(u)(x) + M'_\alpha(u)(x) \leq \lambda\}.$$

If $x \in H'_\lambda(u)$, because $C_1 > 1$ we have that $|u(x)| \leq M(u)(x) \leq \lambda < K/2$ and then $u \in L^\infty(H'_\lambda(u))$.

Also, since $H'_\lambda(u) \subset H_\lambda(u)$, by Lemma 4 we get

$$[u]_{C^{0,\alpha}(H'_\lambda(u) \setminus E)} \leq C_1 \lambda = K/2,$$

with $\mu(E) = 0$.

By Lemma 6 we can extend u from $H'_\lambda(u) \setminus E$ to all of Ω by $v \in C^{0,\alpha}(\overline{\Omega})$ with

$$\|v\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(H'_\lambda(u) \setminus E)} + [u]_{C^{0,\alpha}(H'_\lambda(u) \setminus E)} \leq K.$$

Now we have

$$\begin{aligned} \{x : u(x) \neq v(x)\} &\subset (\Omega \setminus H'_\lambda(u)) \cup E \\ &\subset \{x : M(u)(x) > \lambda/2\} \cup \{x : M'_\alpha(u)(x) > \lambda/2\} \cup E \end{aligned}$$

and so

$$(4) \quad \mu(\{x : u(x) \neq v(x)\}) \leq \mu(\{x : M(u)(x) > \lambda/2\}) + \mu(\{x : M'_\alpha(u)(x) > \lambda/2\}).$$

Now, since $u \in W^{1,p}$, we have that if $p < N$ the Sobolev imbedding theorem gives the continuous injection $W^{1,p} \hookrightarrow L^{p^*}$ with $p^* = \frac{Np}{N-p}$, the so-called Sobolev critical exponent, and then by the classical weak type inequality for the Hardy-Littlewood maximal function, Theorem 1 in [17], we have that

$$\mu(\{x : M(u)(x) > \lambda/2\}) \leq C (\|u\|_{L^{p^*}}/\lambda)^{p^*} \leq C (\|u\|_{W^{1,p}}/\lambda)^{p^*}$$

and with the second term we just use Lemma 5 to get that

$$\mu(\{x : M'_\alpha(u)(x) > \lambda/2\}) \leq C_2 2^{p+\delta} (\|\text{grad } u\|_{L^p}/\lambda)^{p+\delta}.$$

But $p^* \geq p + \delta$ then for large values of λ we get that

$$\mu(\{x : u(x) \neq v(x)\}) \leq C (\|u\|_{W^{1,p}}/\lambda)^{p+\delta} = C (\|u\|_{W^{1,p}}/K)^{p+\delta}.$$

If $p = N$ then the Sobolev imbedding theorem guarantees that $W^{1,p} \hookrightarrow L^q$ with any $q \geq p$, then we can repeat the previous analysis.

Finally if $p > N$ then, again by the Sobolev imbedding theorem, we get that $u \in L^\infty(\Omega)$, hence for large values of λ the first term in (4) will be zero and therefore we also get the result in this case.

Remark 1. *The degree of regularity we need for the boundary of Ω is the one needed to have an extension theorem from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^N)$ so that*

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \leq C_3 \|u\|_{W^{1,p}(\Omega)},$$

see Theorems 4.26, 4.28 or 4.32 in [1]. This will be either that Ω is a half-space or that the boundary of Ω is bounded and having the uniform C^1 -regularity property or, that the boundary of Ω satisfies a special modification of the uniform cone property, see Theorem 4.32 in [1], the Calderón extension theorem.

4. Relation to the Sobolev Imbedding Theorem. In the case $p > N$ the Sobolev imbedding theorem gives that $W^{1,p}$ is continuously injected into $C^{0,1-N/p}$ and we got a rate of decay of the measure of the sets where the approximants differ from the function being approximated, that explodes to infinity as α approaches $1 - N/p$.

If $p = N$, $W^{1,p}$ is not injected in any space of Hölder continuous functions, but it is injected in all the L^q spaces. In this case we can let α approach zero and then the rate of decay explodes to infinity.

In the case when $p < N$, $W^{1,p}$ is injected into L^{p^*} and we obtain a rate of decay that grows from p , when the approximants are chosen in $W^{1,\infty} = C^{0,1}$, up to p^* when α approaches zero, because δ will then approach pp^*/N . Through a Lusin type approximation in L^q , we can continue to improve the rate of decay which will also explode to infinity when q approaches p^* .

The rate of decay becoming infinite simply reflects the fact that the function to approximate is already in the space of the approximants.

Let $p \in [1, N)$ and $u \in W^{1,p}$ for which we want to find an approximating function in L^q , with $q > p^*$. For $\lambda > 1$ let be

$$E_\lambda = \{x \in \Omega : Mu(x) > \lambda\},$$

then we define the following function:

$$v(x) = \begin{cases} u(x) & \text{if } x \notin E_\lambda \\ 0 & \text{if } x \in E_\lambda, \end{cases}$$

then

$$\|v\|_{L^q}^q = \int_{\Omega \setminus E_\lambda} |u(x)|^q dx = \int_{\Omega \setminus E_\lambda} |u(x)|^{p^*} |u(x)|^{q-p^*} dx \leq C \lambda^{q-p^*} \|u\|_{W^{1,p}}^{p^*},$$

then if we want to have $\|v\|_{L^q} \leq K$ we just need to require that

$$\lambda = \left(K^q / C \|u\|_{W^{1,p}}^{p^*} \right)^{\frac{1}{q-p^*}}.$$

But then

$$\mu(\{x : u(x) \neq v(x)\}) \leq \mu(E_\lambda) \leq C (\|u\|_{W^{1,p}} / \lambda)^{p^*} = C (\|u\|_{W^{1,p}} / K)^{qp^*/q-p^*}.$$

Therefore if $q \rightarrow p^*$ from above, the exponent of K grows to infinity, recovering again the situation for the Sobolev imbedding theorem. On the other hand if $q \rightarrow \infty$ the exponent will approach p^* , which coincides with the limit of the exponent in Theorem 1.2 when $\alpha \rightarrow 0$.

5. Application. The important role played by Hölder continuous functions in the theory of linear elliptic partial differential equations, see [9] chapters 4 and 6, lead us to look for applications of our result in that direction.

One manner in which our result can be used is to obtain partial interior regularity results for classical solutions to $\Delta u = f$ in Ω with $f \in W^{1,p}(\Omega)$. There is however a prerequisite, namely that our truncation-extension procedure should not change the function f on an open set.

For $f \in W^{1,p}(\Omega)$, the sets where the proof of theorem 1.2 does not change f could have been formed in principle by purely isolated points. This is not the case if, after we have extended f to all of \mathbb{R}^N , we have that the following holds:

- (a) $M(|\text{grad } f|)(x) < \infty$ for all $x \in \Omega$.
- (b) There exists $\{f_n\} \subset \mathcal{D}(\mathbb{R}^N)$, with $f_n \rightarrow f$ in $W^{1,p}(\mathbb{R}^N)$ and

$$f_n \rightarrow f \quad \text{and} \quad M(|\text{grad } f_n|) \rightarrow M(|\text{grad } f|) \quad \text{pointwise in } \Omega.$$

Since, from Proposition 2.1, $M'_\alpha(f)$ will then be continuous and the negligible set E in Lemma 4 will not intersect Ω .

In the applications below the function f will, additionally, be continuous and then its maximal function will also be continuous.

Under these conditions there exists $\lambda_0 \geq 0$ such that for $\lambda > \lambda_0$, $\Omega \cap H'_\lambda(f)$ contains an open set, namely

$$\{x \in \Omega : M(f)(x) + M'_\alpha(f)(x) < \lambda\},$$

λ_0 is chosen as the infimum of $M(f)(x) + M'_\alpha(f)(x)$. The complement in Ω of this set has also its measure bounded by a constant times $1/\lambda^{p+\delta}$.

In chapter 4 of [9] several interior regularity results are presented for Poisson's equation with f being assumed to be locally Hölder continuous,

denoted $f \in C^{0,\alpha}(\Omega)$. More generally $C^{k,\alpha}(\Omega)$ is defined as the space containing the functions whose partial derivatives up to order k have finite α -Hölder coefficient over compact subsets of Ω . To present the results we need to use three other norms for Hölder continuous functions. For $x, y \in \Omega$, $d_x = \text{dist}(x, \partial\Omega)$ and $d_{x,y} = \min(d_x, d_y)$. Then, following [9], we denote

$$[u]_{C^k(\Omega)}^* = \sup_{x \in \Omega, |\beta|=k} d_x^k |D^\beta u(x)|,$$

$$\|u\|_{C^k(\Omega)}^* = \sum_{j=0}^k [u]_{C^j(\Omega)}^*,$$

$$[u]_{C^{k,\alpha}(\Omega)}^* = \sup_{x,y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}$$

to finally define the norm

$$\|u\|_{C^{k,\alpha}(\Omega)}^* = \|u\|_{C^k(\Omega)}^* + [u]_{C^{k,\alpha}(\Omega)}^*.$$

Also we can define

$$\|u\|_{C^{0,\alpha}(\Omega)}^{(k)} = \sup_{x \in \Omega} d_x^k |u(x)| + \sup_{x,y \in \Omega} d_{x,y}^{k+\alpha} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

If additionally Ω is bounded, with $d = \text{diam}(\Omega)$, we can define

$$\begin{aligned} \|u\|'_{C^k(\Omega)} &= \sum_{j=0}^k d^j \|D^j u\|_{C(\Omega)}, \\ \|u\|'_{C^{k,\alpha}(\Omega)} &= \|u\|'_{C^k(\bar{\Omega})} + d^{k+\alpha} [D^k u]_{C^{0,\alpha}(\Omega)}. \end{aligned}$$

First we use the following result due to Korn and corresponding to theorem 4.6 in [9].

Proposition 5.1. *Let Ω' be an open subset of \mathbb{R}^N , $f \in C^{0,\alpha}(\Omega')$ and $u \in C^2(\Omega')$ be a solution to $\Delta u = f$ in Ω' . Then $u \in C^{2,\alpha}(\Omega')$ and for*

$B_1 = B(x_0, R)$, $B_2 = B(x_0, 2R) \subset\subset \Omega'$ we have

$$\|u\|'_{C^{2,\alpha}(B_1)} \leq C(\|u\|_{C(B_2)} + R^2\|f\|'_{C^{0,\alpha}(B_2)}),$$

where $C = C(N, \alpha)$.

We can prove a result on a similar spirit but for $f \in W^{1,p}(\Omega)$.

Proposition 5.2. Let $f \in W^{1,p}(\Omega)$ satisfying (a) and (b) and let $u \in C^2(\Omega)$ be a solution to $\Delta u = f$ in Ω . Then for every $\varepsilon > 0$ there exists $E_\varepsilon \subset \Omega$, with $\mu(E_\varepsilon) < \varepsilon$, such that $u \in C^{2,\alpha}(\Omega \setminus E_\varepsilon)$ and for $B_1 = B(x_0, R)$, $B_2 = B(x_0, 2R) \subset\subset \Omega \setminus E_\varepsilon$ we have

$$\|u\|'_{C^{2,\alpha}(B_1)} \leq C(\|u\|_{C(B_2)} + R^2\|f\|'_{C^{0,\alpha}(B_2)}),$$

where $C = C(N, \alpha, \varepsilon)$.

Proof. Let $\varepsilon > 0$ be given and choose λ to be a positive real number large enough, so that

$$E_\varepsilon = \{x \in \Omega : M(f)(x) + M'_\alpha(f)(x) \geq \lambda\},$$

is such that $\mu(E_\varepsilon) < \varepsilon$.

From $M(f)$ and $M'_\alpha(f)$ being continuous we have that $\Omega \setminus E_\varepsilon$ is an open set in Ω . Now by theorem 1.2 over $\Omega \setminus E_\varepsilon$ we can replace f by its Hölder continuous approximant f_λ , which coincides with f and then over $\Omega \setminus E_\varepsilon$ we still have that $\Delta u = f_\lambda$, so we can just apply Proposition 5.1 to get the result.

We can similarly prove a partial version of a global estimate, Theorem 4.8 in [9] and also due to Korn.

Proposition 5.3. Let $f \in W^{1,p}(\Omega)$ satisfying (a) and (b) with $u \in C^2(\Omega)$ a solution to $\Delta u = f$ in Ω . Then for every $\varepsilon > 0$ there exists $E_\varepsilon \subset \Omega$,

with $\mu(E_\varepsilon) < \varepsilon$, such that $u \in C^{2,\alpha}(\Omega \setminus E_\varepsilon)$ and

$$\|u\|_{C^{2,\alpha}(\Omega \setminus E_\varepsilon)}^* \leq C(\|u\|_{C(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega \setminus E_\varepsilon)}^{(2)}),$$

where $C = C(N, \alpha, \varepsilon)$.

Finally we establish a compactness result for solutions to Poisson's equation similar to Corollary 4.7 in [9].

Proposition 5.4. *Let $f \in W^{1,p}(\Omega)$ satisfying (a) and (b) and let $\{u_n\} \subset C^2(\Omega)$ be a bounded sequence of solutions to $\Delta u = f$ in Ω . Then there exists a measurable set $E \subset \Omega$, with $\mu(E) = 0$, and a subsequence $\{u_{n_k}\}$ that converges uniformly over any compact subset of $\Omega \setminus E$ to a solution of $\Delta u = f$ in $\Omega \setminus E$.*

Proof. Let $\{E_{1/n}\}$ be a sequence of nested closed sets chosen as in the proof of Proposition 5.2. Then $E = \bigcap_{n \in \mathbb{N}} E_{1/n}$ will be a closed set with measure zero. Since over $\Omega \setminus E_1$ f is Hölder continuous, by Corollary 4.7 in [9] we can extract a subsequence of $\{u_n\}$ that will converge uniformly on compact subsets of $\Omega \setminus E_1$, we continue this process until we get a subsequence that will converge uniformly to a solution on compact subsets of $\Omega \setminus E$.

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