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SELF-SIMILARITY IN DETERMINISTIC AND STOCHASTIC DISSIPATIVE SYSTEMS

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Abstract. In this paper, self-similarity is illustrated and compared in deterministic and stochastic dissipative systems. Examples are (1) deterministic self-similarity in reaction-diffusion system and Navier-Stokes equations, where solutions eventually decay to zero due to balance of diffusion (viscosity) and nonlinearity; (2) statistical self-similarity in randomly advected passive scalar model of Kraichnan where solutions undergo turbulent decay due to roughness of advection; (3) self-similarity in blowup of solutions of fourth order nonlinear parabolic equations of the Cahn-Hillard type. Problems for future research are mentioned, especially those where self-similarity is conjectured based on numerical evidence or physical grounds but mathematically open.

1. Introduction. Self-similarity in evolutionary PDEs (partial differential equations) refers to a special spatial-temporal form of solutions in certain asymptotic limit (at large times or blowup times). This allows us to learn about the system as solutions evolve to a simple pattern. Self-similarity is a robust concept that extends under proper conditions to stochastic PDEs when one looks at statistical quantities of solutions. In this paper, we shall present a couple of worked out examples in recent mathematical literature on self-similarity, which is generated from competition of dissipation (dif-

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fusivity) and nonlinearity in the deterministic case or from the roughness of noise (eddy diffusivity) in the stochastic case. Specifically, we shall illustrate the decay self-similar asymptotics of (1) reaction-diffusion systems where rates of decay depend sensitively on powers of nonlinearities; (2) incompressible Navier-Stokes (NS) system in R^2 (R^3) where linear terms eventually dominate over nonlinearities; (3) passive scalar equation á la Kraichnan where Hölder continuous random advection field generates energy decay. Self-similarity in blowup of solutions is shown later in a focusing 4th order nonlinear parabolic equation of Cahn-Hillard (C-H) type where nonlinearity dominates over linear terms and leads to finite time singularity.

Self-similarity is widely used in physics and mathematics. We hope that our examples serve as a representative sample for understanding and future research. A major area not discussed here is self-similarity in nonlinear wave equations, see [28] among others. There is a rich physics literature on scalings in noisy systems arising in fractal surface growth, see [1] for an exposition. A related but much more classical area, where scaling concept is extensively applied, is turbulence, of which the passive scalar equation is a simple prototype, see [21] for a recent review.

The rest of the paper is organized as follows. Section 2 is on self-similar decay asymptotics of reaction-diffusion systems arising in cubic autocatalytic reaction. Section 3 is on self-similar decay of NS solutions in R^2 (R^3). Section 4 is on self-similar decay of equal time two point correlators of passive scalars, its consequences and extensions. Section 5 is on self-similar blowup of a focusing C-H equation. Open problems are mentioned after a presentation of main phenomena and results.

2. Self-Similarity with Anomaly in Reaction-Diffusion Systems. Let us consider the following two component reaction-diffusion system with cubic nonlinearity:

$$u_{1,t} = u_{1,xx} - u_1 u_2^2, \tag{2.1}$$

$$u_{2,t} = d\, u_{2,xx} + u_1 u_2^2, \tag{2.2}$$

where d is a positive constant, the so called Lewis number; $x \in R^1$, and initial data $(u_1, u_2)|_{t=0} = (a_1(x), a_2(x)) \in (L^1(R^1) \cap L^\infty(R^1))^2$, is nonnegative, decaying at infinity, and of arbitrary size. The system arises from modeling isothermal autocatalytic reactions of the type $A + 2B \to 3B$, with rate proportional to $u_1 u_2^2$; u_1 and u_2 are concentrations of the reactant A, and the autocatalyst B. Such cubic autocatalysis occurred for certain liquid-phase reactions, see [17], [25] in the chemistry literature, also [4]. Similar reactions also have been observed in gas phase, system (2.1)-(2.2) is called Gray-Scott models, [14]. System (2.1)-(2.2) can also be viewed as a predator (u_2) - prey (u_1) system.

We are interested in the large time asymptotic behavior of solutions. For initial data of finite total mass $\int_{R^1} (u_1 + u_2)\, dx < +\infty$, hence finite amount of food supply, the preditor u_2 will eventually take over prey u_1 and die out itself. Though this intuition is suggestive in that it suffices to consider small initial data, it does not spell out the rate of decay. An easy case is when the nonlinearity is higher than cubic, $u_1 u_2^{p-1}$, $p > 3$. Under the usual parabolic scaling $x \to Lx$, $t \to L^2 t$, $(u_1, u_2) \to \frac{(u_1, u_2)}{L}$, L large, the system becomes:

$$u_{1,t} = u_{1,xx} - L^{3-p} u_1 u_2^{p-1}, \tag{2.3}$$

$$u_{2,t} = d\, u_{2,xx} + L^{3-p} u_1 u_2^{p-1}. \tag{2.4}$$

The parameter L describes the scaling effect at time L from initial time $t = 1$. If one iterates such a scaling map n times, the nonlinear terms are of order $L^{n(3-p)}$, hence for large time or $n \to \infty$, the linear diffusion part will dominate nonlinear terms, the decay rate is $t^{-1/2}$ given by the fundamental solutions of heat equations. The argument fails if $p = 3$, which we call the critical case.

To see what happens, let us consider sub(super)-solutions. First we observe that by maximum principle the solution of (2.2) is always above the solution $\underline{u}_2$ of the heat equation $\underline{u}_{2,t} = \underline{u}_{2,xx}$, $\underline{u}_2(x,0) = a_2(x)$. Since $\underline{u}_2$ converges to a self similar solution, as $t \to \infty$ we obtain

$$\underline{u}_2 \geq m \frac{1}{\sqrt{d\,t}} f_0^\star + h.o.t \equiv \frac{m}{\sqrt{4\pi d\,t}} e^{-\frac{x^2}{4d\,t}} + h.o.t \tag{2.5}$$

for large enough t and some constant m depending only on the initial data. Ignoring the higher order terms for the moment, we see that by maximum principle, u_1 is bounded from above by $\overline{u}_1$ which solves the equation

$$\overline{u}_{1,t} = \overline{u}_{1,xx} - \frac{m^2}{d\,t}(f_0^\star)^2 \overline{u}_1 \tag{2.6}$$

for $t \gg 1$. The equation (2.6) admits self-similar solutions of the form

$$\overline{u}_1 = t^{-\alpha} f_\alpha^\star\left(\frac{x}{\sqrt{t}}\right). \tag{2.7}$$

Upon substitution, we find that:

$$\left(-\frac{d^2}{d\xi^2} - \frac{\xi}{2}\frac{d}{d\xi} + \frac{m^2}{d}(f_0^\star)^2\right) f_\alpha^\star = \alpha f_\alpha^\star.$$

Let $f_\alpha^\star = e^{-\xi^2/8} g$ to get a self-adjoint eigenvalue problem:

$$Lg \equiv \left[-\frac{d^2}{d\xi^2} + \left(\frac{1}{4} + \frac{\xi^2}{16} + \frac{m^2}{d}(f_0^\star)^2\right)\right] g = \alpha g. \tag{2.8}$$

Without the last potential term in operator L, L reduces to the familiar harmonic oscillator. The exponent α and the function $f_\alpha^\star = f_\alpha^\star(\xi)$, $\xi = \frac{x}{\sqrt{t}}$, are respectively principal eigenvalue and eigenfunction of L in $L^2(R^1)$. The positive potential in L implies that $\alpha > 1/2$, suggesting the *existence of an anomalous exponent*. The following theorem makes it precise.

Let $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, $\mathcal{B}$ is the Banach space of continuous functions on R^1 with the norm

$$\|f\| = \sup_{x \in R^1} |f(x)|(1+|x|)^q, \quad \text{with } q > 1 \text{ fixed.} \tag{2.9}$$

Let $\phi_d = \phi_d(x)$ be the Gaussian:

$$\phi_d(x) = \frac{1}{\sqrt{4\pi d}} \exp\{-\frac{x^2}{4d}\}. \tag{2.10}$$

Given $A > 0$, let ψ_A be the principal eigenfunction (ground state) of the differential operator:

$$\mathcal{L}_A = -\frac{d^2}{dx^2} - \frac{1}{2} x \frac{d}{dx} - \frac{1}{2} + A^2 \phi_d^2(x), \tag{2.11}$$

on $L^2(R^1, d\mu)$, with $d\mu(x) = e^{\frac{x^2}{4}} dx$. The corresponding eigenvalue is denoted by E_A, $E_A > 0$ for $A > 0$. Normalize ψ_A by $\int \psi_A^2(x) d\mu(x) = 1$. The operator $\mathcal{L}_A$ on $L^2(R^1, d\mu)$ has a compact resolvent, a pure point spectrum, a non-degenerate lowest eigenvalue, and conjugate to a perturbation of the Hamiltonian of the harmonic oscillator:

$$e^{\frac{x^2}{8}} \mathcal{L}_A e^{-\frac{x^2}{8}} = H_A \equiv -\frac{d^2}{dx^2} + \frac{x^2}{16} - \frac{1}{4} + A^2 \phi_d^2(x).$$

Theorem 2.1.(Bricmont-Kupianen-Xin, 1996) *Let* $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, $a_i \not\equiv 0, a_i \geq 0, i = 1, 2$. *Let* $A = \int_{R^1} a_1(x) + a_2(x) dx$, *the total mass of the system, conserved in time. Then system* (2.1) – (2.2) *has a unique global classical solution* $(u_1(x,t), u_2(x,t)) \in \mathcal{B} \times \mathcal{B}$ *for* $\forall t \geq 0$. *Moreover, there exists a number* $q(A)$ *such that, if* $q \geq q(A)$ *in* (2.9), *there is positive number* B *depending continuously on* (a_1, a_2) *such that:*

$$\|t^{\frac{1}{2}+E_A} u_1(\sqrt{t}\cdot, t) - B\psi_A(\cdot)\| \underset{t \uparrow \infty}{\longrightarrow} 0, \tag{2.12}$$

$$\|t^{\frac{1}{2}}u_2(\sqrt{t}\cdot,t) - A\phi_d(\cdot)\| \underset{t\uparrow\infty}{\longrightarrow} 0. \tag{2.13}$$

The convergence in (2.12) and (2.13) implies:

$$u_1(x,t) \sim \frac{B\psi_A(\frac{x}{\sqrt{t}})}{t^{\frac{1}{2}+E_A}} + \cdots, \quad u_2(x,t) \sim \frac{A}{t^{\frac{1}{2}}}\phi_d(\frac{x}{\sqrt{t}}) + \cdots.$$

The leading terms are the two parameter self-similar solutions to the reduced system, i.e, (2.1)-(2.2) with $+u_1u_2^2$ dropped from (2.2). The anomalous exponent $E_A > 0$ occurs as a result of the interactions of nonlinearities of opposite signs. It is amusing that the original system (2.1)-(2.2) has no exact self-similar solutions however. The proof proceeds first by establishing global bounds using nonlinear functionals of solutions, then proving decay, and at last unveiling the asymptotic structures by a rescaling procedure called renormalization method, see [5] for details.

The presence of anomalous exponent is rather delicate with respect to power perturbation in nonlinearity. If one changes the reaction nonlinearity from $u_1u_2^2$ to $u_1^{p1}u_2^{p2}$, with $p_i \in (1,2)$ $(i = 1,2)$, $p_1 + p_2 = 3$:

$$u_{1,t} = u_{1,xx} - u_1^{p_1}u_2^{p_2}, \tag{2.14}$$

$$u_{2,t} = d\, u_{2,xx} + u_1^{p_1}u_2^{p_2}, \tag{2.15}$$

then even though the total power of nonlinearity is still critical, the asymptotic behavior is different [16]:

Theorem 2.2. **(Li-Qi, 2002)** *Consider initial data* $(a_1,a_2) \in \mathcal{B}\times\mathcal{B}, a_i \not\equiv 0, a_i \geq 0, i = 1,2$. *Let* $A = \int_{R^1} a_1(x) + a_2(x)dx$, *the total mass of the system, conserved in time. Then system* $(2.14) - (2.15)$ *has a unique global classical solution* $(u_1(x,t), u_2(x,t)) \in \mathcal{B}\times\mathcal{B}$ *for* $\forall t \geq 0$. *Furthermore, there* *exists an explicit positive number*

$$B = B(A, d, p_1, p_2) = \left(\frac{4\pi d^{p_2/2}}{(p_1 - 1)A^{p_2}(p_1 + p_2/d)^{1/2}} \right)^{1/(p_1-1)},$$

so that as $t \to +\infty$*:*

$$\|t^{1/2}(\log t)^{1/(p_1-1)} u_1(\sqrt{t}\cdot, t) - B\phi_1(\cdot)\| \to 0,$$
$$\|t^{1/2} u_2(\sqrt{t}\cdot, t) - A\phi_d(\cdot)\| \to 0. \tag{2.16}$$

We remark that the powers p_1 and p_2 normally relate to the number of particles of the reacting species participating in collision. The models with fractional value of $p_2 > 0$ ($p_1 = 1$) appeared in [22] and references therein as lumped approximation of a chain of single step reactions. We are not yet aware of a paper in the literature where both p_1 and p_2 are fractional. Nevertheless, such a regime is useful for understanding self-similarity in reaction-diffusion systems. The appearance of log is due to u_1 being nonlinear.

By sub(super)-solution argument and a-priori bounds, it is not hard to see that if $p_i \geq 1$, $i = 1, 2$, $p_1 + p_2 < 3$, u_1 decays like $\exp\{-O(t^{(3-p_1-p_2)/2})\}$ for $|x| \leq O(\sqrt{t})$, and behaves like regular diffusion if $|x| \gg O(\sqrt{t})$. There is no power law (scaling) behavior in u_1. Such u_1 also renders nonlinearity insignificant for u_2, which decays diffusively to zero, see [16] for more details.

It is rather interesting to extend the results so far to systems of reaction-diffusion equations with more than two components and inquire whether and how self-similarity with anomalies persists.

3. Self-Similar Decay in Incompressible NS. The two dimensional incompressible NS system is:

$$u_t + (u \cdot \nabla)u = \Delta u - \nabla p, \quad \nabla \cdot u = 0, \tag{3.1}$$

where $u = u(x,t) = (u_1, u_2) \in R^2$ is fluid velocity, $p = p(x,t)$ fluid pressure, $x = (x_1, x_2) \in R^2$. The vorticity $\omega = u_{2,x_1} - u_{1,x_2} \in R^1$ obeys the equation:

$$\omega_t + (u \cdot \nabla)\omega = \Delta\omega, \tag{3.2}$$

$$u = \frac{1}{2\pi}\int_{R^2} \frac{(x-y)^{\perp}}{|x-y|^2}\omega(y)\,dy, \tag{3.3}$$

the Biot-Savart law, where $x^{\perp} = (-x_2, x_1)$.

Our experience of stirring a cup of coffee tells us that viscosity will dissipate all initial fluid energy at long time, in fact it has been proved that solutions from initial data of arbitrary size will eventually decay to zero, see [26] among others. Again one asks: What is the exact long time asymptotics? It suffices to answer this question for small initial data.

Introducing scaling variables:

$$\xi = \frac{x}{\sqrt{1+t}}, \quad \tau = \log(1+t), \tag{3.4}$$

and writing:

$$\omega(x,t) = \frac{1}{1+t} w\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right), \tag{3.5}$$

$$u(x,t) = \frac{1}{1+t} v\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right). \tag{3.6}$$

then $w(\xi,\tau)$ satisfies the equation:

$$w_\tau = \mathcal{L}w - (v \cdot \nabla_\xi)w, \tag{3.7}$$

with:

$$\mathcal{L}w = \Delta_\xi w + \frac{1}{2}(\xi \cdot \nabla_\xi)w + w, \tag{3.8}$$

and:

$$v(\xi,\tau) = \frac{1}{2\pi}\int_{R^2} \frac{(\xi-\eta)^{\perp}}{|\xi-\eta|^2} w(\eta,\tau)\, d\eta. \tag{3.9}$$

Notice that $\mathcal{L}$ here is simply the two dimensional version of the operator $\mathcal{L}_0$ in the last section.

Define a weighted Hilbert space:

$$L^2(m) = \{f \in L^2(R^2) : \|f\|_2 = |(1+|\xi|^2)^{m/2} f|_2 < \infty\},$$

$|\cdot|_2$ being the standard L^2 norm. The spectrum of $\mathcal{L}$ on $L^2(m)$ consists of eigenvalues $-\frac{k}{2}$, $k = 0,1,2,\ldots$, and continuous spectrum $\{\lambda \in C : Re(\lambda) \leq -\frac{m-1}{2}\}$.

Theorem 3.1.(Gallay-Wayne, 2001) *Let* $\mu \in (0,1/2)$. *There exist constants* ϵ, C, *positive, such that if* $\|w_0\|_2 \leq \epsilon$*:*

$$\|w(\cdot,\tau) - AG(\cdot)\|_2 \leq Ce^{-\mu\tau}, \tag{3.10}$$

$A = \int_{R^2} w_0(\xi)\, d\xi$, $G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}$. *The function* G *is a stationary solution of* (3.7)*, the so called Oseen's vortex.*

We refer to [10] for the complete proof using center manifold method. We see that the vorticity decay is self-similar at the rate of linear diffusion $O(1/t)$ in the original variables (x,t) if $A \neq 0$:

$$\omega(x,t) \sim \frac{A}{4\pi(1+t)} \exp\{-|x|^2/4(1+t)\} + \cdots.$$

The next order asymptotics is:

$$\|w(\xi,\tau) - AG(\xi) + \frac{1}{2}(B_1\xi_1 + B_2\xi_2)G(\xi)e^{-\tau/2}\|_3 \leq Ce^{-\mu_1\tau}$$

$\mu_1 \in (1/2,1)$, $B_i = \int_{R^2} \xi_i \omega_0(\xi)\, d\xi$. One can view A as zeroth, and B_1, B_2 as first moments of vorticity ω_0. Similar result holds for decay of solutions to three dimensional NS [11]. Linear diffusion dominates, with no anomalies in decay rate.

4. Self-Similar Decay in Passive Scalar.

4.1. Model and basic properties. The passive scalar equation is a stochastic advection-diffusion equation:

$$\partial_t \theta(\mathbf{r},t) + (\mathbf{v}(\mathbf{r},t) \circ \nabla)\theta(\mathbf{r},t) = \kappa \triangle \theta(\mathbf{r},t), \tag{4.1}$$

where θ is a scalar field (e.g. temperature or concentration); $\kappa > 0$ is the molecular diffusivity; advecting velocity $\mathbf{v}(\mathbf{r},t)$ a given zero mean incompressible Gaussian random field; $\circ$ refers to Stratonovitch product. Kraichnan model [18] hypothesized that the time dependence of velocity field is white noise, and velocity covariance is:

$$\langle v_i(\mathbf{r},t) v_j(\mathbf{r}',t')\rangle = D_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{4.2}$$

where D_{ij} is C^{ζ}, $\zeta \in (0,2)$. The braket means average over velocity ensemble. The vector $\mathbf{r} \in R^d$, $d \geq 2$, and r denotes the length of $\mathbf{r}$. It follows that the random velocity field is spatially Hölder continuous.

Equation (4.1) can be viewed as the vorticity equation of the two dimensional incompressible NS provided velocity field is prescribed (passive).

The above random velocity field is referred to as synthetic turbulent. It was discovered [18] that the equal time two point correlation function (correlator) $\Theta(\mathbf{r},t) = \langle \theta(\mathbf{r},t)\theta(\mathbf{0},t)\rangle$ obeys the following closed equation:

$$\partial_t \Theta(\mathbf{r},t) = [D_{ij}(0) - D_{ij}(\mathbf{r})]\boldsymbol{\nabla}_{\mathbf{r}_i}\boldsymbol{\nabla}_{\mathbf{r}_j}\Theta(\mathbf{r},t) + 2\kappa \triangle \Theta(\mathbf{r},t). \tag{4.3}$$

For recent derivations from different perspectives, including those on closed equations for higher order correlators, see [20], [12], [19], and [8].

The inertial-convective range is the regime: $\kappa \to 0$, $r \ll L$, L is a fixed large (integral) scale. Then the velocity covariance in space has a power-law form

$$D_{ij}(\mathbf{r}) \sim D_{ij}(\mathbf{0}) - D_1 \cdot r^{\zeta}\left[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right]$$

up to correction of order $O\left(\frac{r}{L}\right)^2$ for $0 < \zeta < 2$ and $r \ll L$, under the additional condition that velocity field has statistical isotropy as well as homogeneity (spatially translation invariant). Velocity field is Hölder continuous a.s. with exponent (Hurst) $H = \zeta/2$, so called rough velocity. Equation (4.3) for the two point correlator simplifies to

$$\partial_t \Theta(r,t) = \frac{D_1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d+\zeta-1}\frac{\partial \Theta}{\partial r}(r,t)\right]. \tag{4.4}$$

The scalar energy is : $E_\theta(t) = \frac{1}{2}\langle \theta^2(\mathbf{0},t)\rangle$. In equation (4.4) the limit has been taken of vanishing diffusivity, to examine the effect of **turbulent dissipation**: scalar energy $E_\theta(t) \to 0$ as $t \to \infty$ in the limit $\kappa \to 0$. The variable diffusion in (4.4) is called *eddy diffusivity*.

This corresponds to Onsager's conjecture on the three-dimensional NS energy cascade [23]: the turbulent solution ensemble in the limit of vanishing viscosity of NS should consist of realizations of the inviscid Euler equations which dissipate energy. These must be *weak or distributional solutions*, not classical Euler solutions which conserve energy. This ideal mechanism of dissipation has been called the *dissipative anomaly*. This kind of turbulent mechanism of dissipation is well-illustrated in the Kraichnan model. The operator on the right hand side of (4.4) is homogeneous with degree $-\gamma$ ($\gamma = 2 - \zeta$), using $E_\theta(t) = \frac{1}{2}\Theta(0,t)$:

$$\frac{dE_\theta}{dt}(t) = \frac{D_1}{2r^{d-1}}\frac{\partial}{\partial r}\left[r^{d+\zeta-1}\frac{\partial \Theta}{\partial r}(r,t)\right]\Bigg|_{r=0} = 0, \tag{4.5}$$

if the so called 2nd-order structure function:

$$S_2(r,t) \equiv \langle[\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^2\rangle$$

$$= 2[\Theta(0,t)-\Theta(r,t)] \sim Cr^{\xi}, \tag{4.6}$$

with $\xi > \gamma$. Certain *critical degree of singularity in the solution is required for turbulent dissipation.*

4.2. Family of self-similar solutions. Let us find all possible self-similar decay solutions, in the Kraichnan model, to explicitly illustrate the ideal turbulent dissipation mechanism. Seek self-similar solutions of the form:

$$\Theta(r,t) = \vartheta^2(t)\Phi\left(\frac{r}{L(t)}\right), \quad \Phi(0)=1, \tag{4.7}$$

then one finds:

$$\frac{2\dot{\vartheta}(t)}{D_1\vartheta(t)L^{\zeta-2}(t)} = -\alpha < 0 \tag{4.8}$$

$$\frac{\dot{L}(t)}{D_1 L^{\zeta-1}(t)} = 1 \tag{4.9}$$

$$\rho^{\zeta}\Phi''(\rho) + [(d+\zeta-1)\rho^{\zeta-1}+\rho]\Phi'(\rho) + \alpha\Phi(\rho) = 0, \tag{4.10}$$

which is transformed into Kummer's equation by changing variables $x = -\frac{\rho^{\gamma}}{\gamma}$ ($\gamma = 2-\zeta$):

$$x\frac{\partial^2\Phi}{\partial x^2} + \left[\frac{d}{\gamma}-x\right]\frac{\partial\Phi}{\partial x} - \frac{\alpha}{\gamma}\Phi = 0. \tag{4.11}$$

Of the two linearly independent solutions, one is an entire function satisfying $\Phi(0)=1$, call it $\Phi(\alpha/\gamma, d/\gamma; x)$.

Realizability refers to: solutions of (4.4) must be positive definite functions to be correlation functions of a statistical problem. Physical solutions must be positive definite, and equation (4.4) preserves such positivity. Results on existence of self-similar solutions are:

Theorem 4.1. (Eyink-Xin, 2000) *The self-similar solutions of equation* (4.4) *of the form* (4.7) *exist with* ϑ *and* $L(t)$ *obeying* (4.8) − (4.9) *for any* $\alpha > 0$*, and* Φ *given by:*

$$\Phi(\rho) = \Phi\left(\frac{\alpha}{\gamma}, \frac{d}{\gamma}; -\frac{\rho^\gamma}{\gamma}\right), \tag{4.12}$$

realizable as radially symmetric functions on R^d *for* $\alpha \in (0, d+\gamma]$*, and for no* $\alpha > d+\gamma$*. Moreover, for* $\rho \ll 1$*,*

$$\Phi(\rho) = 1 - \frac{\alpha}{d\gamma}\rho^\gamma + O(\rho^{2\gamma});$$

for $\rho \gg 1$*,* $\alpha \neq d + \gamma l$*,* $l = 0, 1$ *(non-exceptional),*

$$\Phi(\rho) \sim \gamma^{\alpha/\gamma} \frac{\Gamma\left(\frac{d}{\gamma}\right)}{\Gamma\left(-\frac{\alpha-d}{\gamma}\right)} \rho^{-\alpha}.$$

A few properties of solutions are as follows:

• Realizability is verified through integral representation of Kummer functions.

• $L(t) \sim t^{1/\gamma}$, scalar energy is $\vartheta^2(t) \sim O(L^{-\alpha}(t)) = O(t^{-\alpha/\gamma})$.

• Self-similar solutions have Hölder exponent γ at $r = 0$, just the critical singularity to dissipate energy, and have decay in space by power law if α is non-exceptional.

• Explicit solutions for exceptional α are:

$$\Phi_0(\rho) = e^{-\frac{\rho^\gamma}{\gamma}}, \quad \alpha = d, \tag{4.13}$$

$$\Phi_1(\rho) \;=\; \left[1-\frac{\rho^\gamma}{d}\right] e^{-\frac{\rho^\gamma}{\gamma}}, \quad \alpha = d+\gamma. \tag{4.14}$$

Both decay exponentially.

4.3. Self-Similar Solutions as Attractors. One further asks the question: how do self-similar solutions attract time dependent solutions?

To this end, write time dependent solutions as:

$$\Theta(x,t) = \vartheta^2(t)\Gamma(\rho,\tau(t)), \quad x = \gamma^{-1}\rho^\gamma, \quad \rho = r/L(t),$$

$$\tau(t) = \log L^\gamma \sim \log(t-t_0),$$

then:

$$\frac{\partial\Gamma}{\partial\tau}(x,\tau) = x\frac{\partial^2\Gamma}{\partial x^2}(x,\tau) + (c+x)\frac{\partial\Gamma}{\partial x}(x,\tau) + a\Gamma(x,\tau), \tag{4.15}$$

$a = \alpha/\gamma$, $c = d/\gamma$. Self-similar solutions $\Phi = \Phi(a,c;-x)$ are steady states of (4.15).

Writing $\Gamma(x,\tau) = \Phi A(x,\tau)$, A solves:

$$\frac{\partial A}{\partial\tau}(x,\tau) = x\frac{\partial^2 A}{\partial x^2}(x,\tau) + \left(c+x+2x\frac{\Phi'(x)}{\Phi(x)}\right)\frac{\partial A}{\partial x}(x,\tau). \tag{4.16}$$

Our objective is then: show convergence of $A \to 1$ as $\tau \to \infty$.

Observation I: solution of (4.16) has a probabilistic representation:

$$A(x,\tau) = E[A(X_{x,\tau},0)],$$

where $X_{x,\tau}$ is the diffusion process starting at x at $\tau = 0$ and obeying the

Ito equation ($a = c + x + 2x\Phi'(x)/\Phi(x)$, $\sigma^2(x) = x$):

$$dX_\tau = a(X_\tau)d\tau + \sigma(X_\tau)dW_\tau, \tag{4.17}$$

with "invariant density":

$$P(x) \propto x^{c-1}e^x\Phi^2(x), \tag{4.18}$$

which is only normalizable for the exceptional cases when $\alpha = d + \gamma l,\ l = 0, 1, 2, ...$

Observation II: Using $P(x)$, a generalized self-adjoint form of (4.16) is:

$$\frac{\partial A}{\partial \tau}(x,\tau) = \frac{x^{1-c}e^{-x}}{\Phi^2(x)}\frac{\partial}{\partial x}\left[x^c e^x \Phi^2(x)\frac{\partial A}{\partial x}(x,\tau)\right], \tag{4.19}$$

implying the invariants:

$$J(\tau) := \int_0^\infty dx\ x^{c-1}e^x A(x,\tau)\Phi^2(x), \tag{4.20}$$

for all α, however, only finite when α is exceptional. So we have $J_l,\ l = 0, 1, 2, ...$ The J_0 is known as Corrsin invarant, the others are new. Through a careful analysis of sample path behavior of Ito solutions, we have:

Theorem 4.2. (Eyink-Xin, 2000) *The convergence of time dependent solutions to self-similar solutions holds as follows:*

$$\begin{aligned}
&J_0 = J_1 = \infty :\ \Gamma_{(\alpha)}(\tau) \to \Phi_{(\alpha)},\ 0 < \alpha < d\\
&J_0 = 0, J_1 = \infty :\ \Gamma_{(\alpha)}(\tau) \to \Phi_{(\alpha)},\ d < \alpha < d + \gamma\\
&0 < J_0 < \infty, J_1 \text{ arbitrary} :\ \Gamma_{(\alpha)}(\tau) \to \Phi_0,\ \alpha = d\\
&J_0 = 0, 0 < J_1 < \infty :\ \Gamma_{(\alpha)}(\tau) \to \Phi_1,\ \alpha = d + \gamma.
\end{aligned}$$

The first two results apply for initial data with a power-law decay $\sim r^{-\alpha}$ *for*

$\alpha < d+\gamma$, whereas the last two include (among other possibilities) initial power-laws with $\alpha > d+\gamma$. Convergence is uniform on compact sets of $x \in (0,\infty)$.

The above self-similar decay results in the exceptional cases have been numerically supported for non-white NS velocity fields, e.g. two-dimentional turbulence field with inverse enstropy cascade ($\gamma = 2/3$), see [6]. Also in [6], physical arguments in the exceptional cases are given for self-similarity of higher order correlators and one point probability distribution function (PDF) $P(\theta,t) = t^{\nu/(2\gamma)}Q(\theta t^{\nu/(2\gamma)})$ ($\nu = d+\gamma l$, $l = 0,1$; P non-Gaussian). It would be very interesting to give rigorous mathematical proofs of these findings. Useful upper bounds on the higher order equal time correlators are recently obtained in [15].

A few preliminary investigations on the decay of passive scalar were reported earlier in the review article [21], section 4.2. Self-similar decay in Burgers turbulence has been studied in [13]. Scaling properties of active scalars were studied in [7].

A real world example is the decay of turbulence in the wake of airplanes. Trailing vortices in the wake turbulence left by a jumbo jet could raise the safety concern of a subsequent aircraft. Hence enough distance must be allowed between two adjacent flights by airport controlers at takeoff and landing. The decay time of trailing vortices correlates with the turbulent dissipation rate in that stronger turbulence causes faster breakdown of vortices, see Figure 1 of [27]. Due to changing boundary conditions as a result of wind, atmospheric turbulence and stratification, the study on the motion and decay of trailing vortices remains a challenge for both theory and computation, see [27] for a recent review.

5. Self-Similarity in Focusing C-H. A common feature of the systems in the previous sections is that asymptotics of self-similar solutions are related to certain linear second order parabolic equations with spatially dependent coefficients. Either by spectral theory or special functions or probabilistic representation, one has a way to carry out analysis. These tools are hard to come by if the equation is higher order in space. An example is the focusing C-H type equation:

$$u_t = -(u^2)_{xx} - u_{xxxx}, \quad x \in R^1. \tag{5.1}$$

The equation arises from evolution of interfaces between two phases, see [3] for more background discussion and its relationship to Kuramoto-Sivarshinsky type equations. Equation (5.1) has also been utilized for enhancement of speech formants in [24]. A major phenomenon is the focusing of u and formation of a finite time singularity of self-similar type.

To see the focusing property, suppose the initial condition is a smooth function with multiple peaks and valleys, and write (5.1) as:

$$u_t + 2\, u_x u_x = -2\, u u_{xx} - u_{xxxx}.$$

Thus the negative curvature regions ($u_{xx} < 0$) go up, positive curvature regions ($u_{xx} > 0$) go down. Moreover, positive slope regions advects to the right, and negative slope regions advects to the left. Combining these two effects, we see that peaks are focusing. This motion is however highly unstable, and requires high wave number stabilization, hence the role of the fourth order derivative term.

Focusing leads to finite time blowup of solutions, and this is captured by self-similar solutions of the form (t_* a positive constant):

$$u = (t^* - t)^{-1/2} f(x/(t^* - t)^{1/4}), \tag{5.2}$$

which satisfies ($\xi = x/(t^* - t)^{1/4}$, $' = \frac{\partial}{\partial \xi}$):

$$\frac{1}{2}f + \frac{1}{4}\xi f' + f'''' + (f^2)'' = 0. \tag{5.3}$$

The function f is the blowup profile, and is a smooth function. The C-H (5.1) conserves total mass $\int_{R^1} u\,dx$, so we shall consider initial data with spatial decay and finite mass.

Equation (5.3) is invariant under reflection $\xi \to -\xi$, so let us also consider even solution of (5.3), for $\xi \geq 0$. Integrating (5.3) over ξ shows that any spatially decaying solution must change sign and has zero integral (mass). The existence of self-similar solution is now reduced to finding a nontrivial solution to (5.3) with zero total integral on R^1.

Such a solution has been computed in [3]. Moreover, through a refined adaptive time dependent calculation of (5.1), it was also found [3] that the self-similar solution is attractive and unique. The self-similar solution changes sign once along $\xi > 0$. Under this condition, it is proved [24] that the maximum of f is larger in absolute value than its minimum, or f is asymmetric, as was numerically observed in [3].

The interesting open mathematical questions are: (1) prove the existence and/or uniqueness of nontrivial solutions to the variable coefficient, quadratically nonlinear, fourth order ODE (5.3) on the line so that $f(\xi) \to 0$ as $\xi \to \infty$; (2) prove that the resulting self-similar solution (5.2) is dynamically stable under the C-H evolution (5.1).

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