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A LINEAR OPERATOR AND ITS APPLICATIONS ON MEROMORPHIC p-VALENT FUNCTIONS

BY

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Abstract. Let $\sum_p$ be the class of functions of the form $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}$ which are analytic and p-valent in $D = \{z : 0 < |z| < 1\}$. A new linear operator $L_p(a, c)$ is introduced. The object of this paper is to investigate some properties of the operator $L_p(a, c)$.

1. Introduction. Let $\sum_p$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in N = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic and p-valent in $D = \{z : 0 < |z| < 1\}$.

For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p,j} z^{k-p}, \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p,1} a_{k-p,2} z^{k-p}. \tag{1.3}$$

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Now we define the function $\phi_p(a,c;z)$ by

(1.4) $$\phi_p(a,c;z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \qquad (c \neq 0, -1, -2, \ldots),$$

where $(x)_k$ is the Pochhammer symbol. We note that

$$\phi_p(a,c;z) = z^{-p} \ {}_2F_1(1,a;c;z)$$

where

$$ {}_2F_1(1,a;c;z) = \sum_{k=0}^{\infty} \frac{(1)_k (a)_k}{(c)_k} \frac{z^k}{k!}$$

Corresponding to the function $\phi_p(a,c;z)$, we define a new linear operator $L_p(a,c)$ on $\sum_p$ by the Hadamard product for $f(z) \in \sum_p$

(1.5) $$L_p(a,c)f(z) = \phi_p(a,c;z) * f(z).$$

In [2], Yang investigated the following operator D^{n+p-1} for $f(z) \in \sum_p$

(1.6) $$\begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p(1-z)^{n+p}} * f(z) \\ &= \frac{(z^{n+2p-1}f(z))^{(n+p-1)}}{(n+p-1)!z^p}, \end{aligned}$$

where n is any integer greater than $-p$.

For a function $f(z) \in \sum_p$, we define the integral operator $J_{v,p}$ by

(1.7) $$J_{v,p}f(z) = \frac{v}{z^{v+p}} \int_0^z t^{v+p-1} f(t)dt \qquad (v > 0).$$

There are many papers [1,2,3] in which the operator $J_{v,p}$ was investigated.

Putting $a = n+p$ and $c = 1$ in (1.5), we have $L_p(n+p,1)f(z) = D^{n+p-1}f(z)$. Setting $a = v$ and $c = v+1$ in (1.5), we obtain $L_p(v,v+1)f(z) = J_{v,p}f(z)$.

Let $f(z)$ and $F(z)$ be analytic in the open unit disc $E = \{z : |z| < 1\}$. The function $f(z)$ is subordination to $F(z)$, written $f(z) \prec F(z)$, if $F(z)$ is univalent, $f(0) = F(0)$ and $f(E) \subset F(E)$.

In this paper, we shall investigate some properties of the linear operator $L_p(a, c)$ defined on meromorphic p-valent functions

2. Preliminaries. In order to give our theorems, we need following lemmas.

Lemma 1. *If* $f(z) \in \sum_p$, *then*

$$z(L_p(a,c)f(z))' = aL_p(a+1,c)f(z) - (a+p)L_p(a,c)f(z), \tag{2.1}$$

where $c \neq 0, -1, -2, \ldots$.

Proof. Note that

$$L_p(a,c)f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k-p} z^{k-p}$$

and

$$L_p(a+1,c)f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a+1)_k}{(c)_k} a_{k-p} z^{k-p}.$$

These give that

$$\begin{aligned}
&aL_p(a+1,c)f(z) - (a+p)L_p(a,c)f(z) \\
&= [az^{-p} + \sum_{k=1}^{\infty} (a+k) \frac{(a)_k}{(c)_k} a_{k-p} z^{k-p}] - [(a+p)z^{-p} + \sum_{k=1}^{\infty} (a+p) \frac{(a)_k}{(c)_k} a_{k-p} z^{k-p}] \\
&= -pz^{-p} + \sum_{k=1}^{\infty} (k-p) \frac{(a)_k}{(c)_k} a_{k-p} z^{k-p} \\
&= z(L_p(a,c)f(z))'.
\end{aligned}$$

This proves (2.1).

Lemma 2.(see [4]). *Let* Ω *be a set in the complex plane* C *and let* b *be a complex number satisfying* $\mathrm{Re}\, b > 0$. *Suppose that the function* $\psi : C^2 \times E \to C$ *satisfies the condition*

$$\psi(ix, y; z) \notin \Omega \tag{2.2}$$

for all real $x, y \leq -|b - ix|^2/(2\mathrm{Re}\, b)$ *and all* $z \in E$. *If the function* $p(z)$ *defined by* $p(z) = b + a_1 z + a_2 z^2 + \cdots$ *is analytic in* E *and if* $\psi(p(z), zp'(z); z) \in \Omega$, *then* $\mathrm{Re}\, p(z) > 0$ *in* E.

3. Main Results. In this secion, we shall derive some properties of the linear operator $L_p(a, c)$.

Theorem 1. *Let* $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \in \sum_p$. *Let* $\beta \geq 1, a > 0$ *and* $0 \leq \alpha < 1$. *Then*

$$\mathrm{Re}\left\{\frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)}\right\} < 1 + \frac{1-\alpha}{a} \qquad (z \in E) \tag{3.1}$$

implies

$$\mathrm{Re}\{(z^p L_p(a, c)f(z))^{-1/2\beta(1-\alpha)}\} > 2^{-1/\beta} \qquad (z \in E). \tag{3.2}$$

The result is sharp.

Proof. From (2.1) and (3.1), we have

$$\mathrm{Re}\left\{-\frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)}\right\} > p + \alpha - 1 \qquad (z \in E).$$

That is,

$$-\frac{1}{2(1-\alpha)}\left[\frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)} + p\right] \prec \frac{z}{1-z}. \tag{3.3}$$

Let

$$p(z) = [z^p L_p(a, c)f(z)]^{-1/2(1-\alpha)},$$

then (3.3) may be written as

$$z[\log p(z)]' \prec z\left[\log \frac{1}{1-z}\right]'. \tag{3.4}$$

Using a well-known result [9] to (3.4), we find that

$$p(z) = [z^p L_p(a,c)f(z)]^{-1/2(1-\alpha)} \prec \frac{1}{1-z},$$

that is, that

$$[z^p L_p(a,c)f(z)]^{-1/2\beta(1-\alpha)} = \Big(\frac{1}{1-w(z)}\Big)^{1/\beta}, \tag{3.5}$$

where $w(z)$ is analytic in E, $w(0)=0$ and $|w(z)|<1$ for $z \in E$.

According to $\mathrm{Re}(t^{1/\beta}) \geq (\mathrm{Re}\ t)^{1/\beta}$ for $\mathrm{Re}\ t > 0$ and $\beta \geq 1$, (3.5) yields

$$\begin{aligned} \mathrm{Re}\left\{[z^p L_p(a,c)f(z)]^{-1/2\beta(1-\alpha)}\right\} &\geq \left[\mathrm{Re}\,\frac{1}{1-w(z)}\right]^{1/\beta} \\ &> 2^{-1/\beta} \qquad (z \in E). \end{aligned}$$

Further, we see that the result is sharp for the function

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{c(c+1)\cdots(c+k-1)(2\alpha-2)(2\alpha-1)\cdots(2\alpha+k-3)}{a(a+1)\cdots(a+k-1)} z^{k-p}.$$

This completes the proof.

Theorem 2. *Let* $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \in \sum_p$ *and let* $a > 0$, $\alpha > 1$ *and* $0 \leq \lambda < a+1$. *Then*

$$\mathrm{Re}\left\{(1-\lambda)\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} + \lambda \frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}\right\} < \alpha \qquad (z \in E) \tag{3.6}$$

implies

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right\}< \beta \qquad (z \in E), \tag{3.7}$$

where $\beta \in [\alpha, +\infty)$ *is the positive root of the equation*

$$2(a+1-\lambda)x^2 + [3\lambda - 2(a+1)\alpha]x - \lambda = 0. \tag{3.8}$$

Proof. Let

$$p(z) = \frac{1}{\beta-1}\left[\beta - \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right], \tag{3.9}$$

then $p(z)$ is analytic in E and $p(0) = 1$. Differentiating (3.9) and using Lemma 1 we deduce that

$$\begin{aligned}
&(1-\lambda)\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} + \lambda\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}\\
&= \beta - \frac{\lambda(\beta-1)}{a+1} - \frac{(\beta-1)(a+1-\lambda)}{a+1}p(z) - \frac{\lambda(\beta-1)}{a+1}\frac{zp'(z)}{\beta-(\beta-1)p(z)}\\
&= \psi(p(z), zp'(z))
\end{aligned}$$

where

$$\psi(r,s) = \beta - \frac{\lambda(\beta-1)}{a+1} - \frac{(\beta-1)(a+1-\lambda)}{a+1}r - \frac{\lambda(\beta-1)}{a+1}\frac{s}{\beta-(\beta-1)r}. \tag{3.10}$$

Using (3.6) and (3.10), we have

$$\{\psi(p(z), zp'(z)) : z \in E\} \subset \Omega = \{w \in C : \operatorname{Re} w < \alpha\}.$$

Now for all real $x, y \le -\frac{1+x^2}{2}$, we obtain

$$\begin{aligned}
\operatorname{Re}\{\psi(ix,y)\} &= \beta - \frac{\lambda(\beta-1)}{a+1} - \frac{\lambda(\beta-1)y}{a+1}\frac{\beta}{\beta^2+(\beta-1)^2x^2}\\
&\ge \beta - \frac{\lambda(\beta-1)}{a+1} + \frac{\lambda(\beta-1)\beta}{2(a+1)}\frac{1+x^2}{\beta^2+(\beta-1)^2x^2}
\end{aligned}$$

$$\begin{aligned} &\geq \beta - \frac{\lambda(\beta-1)}{a+1} + \frac{\lambda(\beta-1)}{2\beta(a+1)} \\ &= \beta + \frac{\lambda(\beta-1)(1-2\beta)}{2(a+1)\beta} = \alpha, \end{aligned}$$

where β is the positive root of the equation (3.8).

Note that $0 \leq \lambda < a+1$ and $f(\alpha) = -\lambda(2\alpha-1)(\alpha-1) \leq 0$, then we have $\beta \in [\alpha, +\infty)$. Hence for each $z \in E$, $\psi(ix, y) \notin \Omega$. By Lemma 2, we get $\operatorname{Re} p(z) > 0$. This proves (3.7).

Theorem 3. *Let λ, α, a be real number with $\lambda \geq 0, \alpha > 1$ and $a > 0$. Let $g(z) \in \sum_p$ satisfy*

$$\operatorname{Re}\left\{\frac{L_p(a,c)g(z)}{L_p(a+1,c)g(z)}\right\} > \delta \qquad (0 \leq \delta < 1). \tag{3.11}$$

If $f(z) \in \sum_p$ satisfies

$$\operatorname{Re}\left\{(1-\lambda)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \lambda\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} < \alpha \qquad (z \in E), \tag{3.12}$$

then we have

$$\operatorname{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} < \frac{2\alpha a + \lambda\delta}{2a + \lambda\delta} \qquad (z \in E). \tag{3.13}$$

Proof. Let $\beta = \dfrac{2a\alpha + \lambda\delta}{2a + \lambda\delta}$ and consider the function

$$u(z) = \frac{1}{\beta - 1}\left[\beta - \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right].$$

The function $u(z)$ is analytic in E and $u(0) = 1$. Set $B(z) = L_p(a,c)g(z)/L_p(a+1,c)g(z)$, then $\operatorname{Re} B(z) > \delta$. Differentiating $u(z)$ and using Lemma 1, we have

$$(1-\lambda)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}+\lambda\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}$$
$$=\beta-(\beta-1)u(z)-\frac{\lambda(\beta-1)}{a}B(z)zu'(z).$$

Let,

$$\psi(r,s)=\beta-(\beta-1)r-\frac{\lambda(\beta-1)}{a}B(z)s$$

then from (3.12), we deduce that

$$\{\psi(u(z),zu'(z)):z\in E\}\subset\Omega=\{w\in C:\operatorname{Re}w<\alpha\}.$$

Now for all real $x, y\le -(1+x^2)/2$ we have

$$\begin{aligned}\operatorname{Re}\{\psi(ix,y)\} &= \beta-\frac{\lambda(\beta-1)y}{a}\operatorname{Re}B(z)\\ &\ge \beta+\frac{\lambda(\beta-1)\delta}{2a}(1+x^2)\\ &\ge \beta+\frac{\lambda(\beta-1)\delta}{2a}=\alpha\end{aligned}$$

Hence for each $z\in E$, $\psi(ix,y)\notin\Omega$. Thus by Lemma 2, $\operatorname{Re}u(z)>0$ in E. The proof of the theorem is complete.

From the proof of Theorem 3, we can easily have the following corollaries.

Corollary 1. *Let λ,α,a be real number such that $\lambda\ge 1,\alpha>1$ and $a>0$. Let $g(z)\in\sum_p$ satisfy the condition* (3.11). *If $f(z)\in\sum_p$ satisfies* (3.12)*, then we have*

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\}<\frac{\alpha(2a+\delta)+\delta(\lambda-1)}{2a+\lambda\delta}\qquad(z\in E).$$

Taking $a=n+p$ and $c=1$ in Theorem 3, we have

Corollary 2. *Let λ,α be real number with $\lambda\ge 0,\alpha>1$. Let $g(z)\in\sum_p$*

satisfy

$$\mathrm{Re}\left\{\frac{D^{n+p-1}g(z)}{D^{n+p}g(z)}\right\} > \delta \qquad (0 \le \delta < 1).$$

If $f(z) \in \sum_p$ *satisfies*

$$\mathrm{Re}\left\{(1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} < \alpha \qquad (z \in E),$$

then we have

$$\mathrm{Re}\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\} < \frac{2\alpha(n+p)+\lambda\delta}{2(n+p)+\lambda\delta} \qquad (z \in E).$$

Putting $a = v$ and $c = v + 1$ in Theorem 3, we obtain

Corollary 3. *Let* λ, α, v *be real number with* $\lambda \ge 0, \alpha > 1$ *and* $v > 0$. *Let* $g(z) \in \sum_p$ *satisfy*

$$\mathrm{Re}\left\{\frac{J_{v,p}g(z)}{g(z)}\right\} > \delta \qquad (0 \le \delta < 1).$$

If $f(z) \in \sum_p$ *satisfies*

$$\mathrm{Re}\left\{(1-\lambda)\frac{J_{v,p}f(z)}{J_{v,p}g(z)} + \lambda\frac{f(z)}{g(z)}\right\} < \alpha \qquad (z \in E),$$

then we have

$$Re\left\{\frac{J_{v,p}f(z)}{J_{v,p}g(z)}\right\} < \frac{2\alpha v+\lambda\delta}{2v+\lambda\delta} \qquad (z \in E).$$

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