BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 31, Number 1, March 2003

ON OPERATOR WHOSE NORM IS AN EIGENVALUE

BY

C.-S. LIN (林嘉祥)

Dedicated to Professor Carshow Lin on his retirement

Abstract. We present in this paper various types of characterizations of a bounded linear operator T on a Hilbert space whose norm is an eigenvalue for T, and their consequences are given. We show that many results in Hilbert space operator theory are related to such an operator.

In this article we are interested in the study of a bounded linear operator on a Hilbert space H whose norm is an eigenvalue for the operator. Various types of characterizations of such operator are given, and consequently we see that many results in operator theory are related to such operator. As the matter of fact, our study is motivated by the following two results: (1) A linear compact operator on a locally uniformly convex Banach sapce into itself satisfies the Daugavet equation if and only if its norm is an eigenvalue for the operator [1, Theorem 2.7]; and (2) Every compact operator on a Hilbert space has a norm attaining vector for the operator [4, p.85].

In what follows capital letters mean bounded linear operators on H; T^* and I denote the adjoint operator of T and the identity operator, respectively. We shall recall some definitions first. A unit vector $x \in H$ is

Received by the editors September 4, 2001 and in revised form February 8, 2002.
1991 AMS Subject Classification. 47A10, 47A30, 47A50.
Key words and phrases: Eigenvalue, Daugavet equation, generalized Daugavet equation, normal eigenvalue, numerical range, normaloid operator, norm attaining vector, complete vector.

called a norm attaining vector for T if $\| T \|=\| Tx \|$ [4, p.85]. A compact operator has such useful property, and it is also known that if T is a compact normal operator, then there exists a unit vector $y \in H$ such that $\| T \|=| (Ty,y) |$. A scalar $\alpha \neq 0$ is a normal eigenvalue for T if both $Tx = \alpha x$ and $T^*x = \overline{\alpha}x$ hold associated with the same eigenvector $x \neq 0$. The numerical range of T is the convex set $W(T) = \{(Tx,x) : x \in H, \| x \|= 1\}$, and the numerical radius of T is $w(T) = \sup\{| \lambda |: \lambda \in W(T)\}$ [4, p.108]. T is a normaloid operator if $w(T) =\| T \|$ [4, p.114]. T satisfies the Daugavet equation if $\| I+T \|= 1+ \| T \|$ [2], which is one useful property in solving a variety of problems in approximation theory [3]. The equation $\| I+S+T \|= 1+ \| S \| + \| T \|$ is said to be the generalized Daugavet equation [5]. In our discussion we need the following definition, and the lemma following it is our basic tool in this paper.

Definition. A unit vector $x \in H$ is called a complete vector for T if (Tx,x) is a real number such that $\| T \|= (Tx,x) =\| Tx \|$.

Lemma. *Let $x \in H$ be a unit vector. Then the following are equivalent.*

(1) $\| A \| + \| B \|$ *is an eigenvalue for* $A+B$, *i.e.,* $(A+B)x = (\| A \| + \| B \|)x$;

(2) $1+ \| A \| + \| B \|= \|(I+A+B)x\|$,

(3) $\| A \|$ *and* $\| B \|$ *are eigenvalues for* A *and* B, *respectively, with respect to* x *i.e.,* $Ax = \|A\|x$ *and* $Bx = \|B\|x$;

(4) $\| A \|$ *and* $\| B \|$ *are in the numerical ranges of* A *and* B, *respectively, with respect to* x, *i.e.,* $\| A \|= (Ax,x)$ *and* $\| B \|= (Bx,x)$;

(5) x *is a complete vector for* A *and* B, *respectively, i.e.,* $\| A \|= (Ax,x) = \| Ax \|$ and $\| B \|= (Bx,x) =\| Bx \|$;

(6) $\| A \| + \| B \|$ *is in the numerical range of* $A+B$, *i.e.,* $\| A \| + \| B \|= ((A+B)x,x)$;

(7) $Ax =\| A \| x =\| Ax \| x$ *and* $Bx =\| B \| x =\| Bx \| x$.

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Since

$$
\begin{aligned}
&\|Ax-\| A \| x\|^2+\|\|Ax+x\|Bx-\| B \| (Ax+x) \|^2 \\
= \;& \{\| Ax \|^2 -2 \| A \| \operatorname{Re}(Ax,x)+ \| Ax \|^2\} \\
& +\{\| Ax+x\|^2 \| Bx \|^2 -2 \| B \|\| Ax+x\|[\operatorname{Re}(Bx,x)+\operatorname{Re}(Ax,Bx)] \\
& + \| Ax+x\|^2 \| B \|^2\} \\
\leq \;& 2 \| A \| [\| A \| -\operatorname{Re}(Ax,x)] \\
& +2 \| B \|\| Ax+x\|[\| B \|\| Ax+x\| - \operatorname{Re}(Bx,x) - \operatorname{Re}(Ax,Bx)] \\
\leq \;& 2(1+ \| A \| + \| B \|)^2[\| A \| -\operatorname{Re}(Ax,x)] \\
& +2(1+ \| A \| + \| B \|)^2[\| B \| (1+ \| A \|) - \operatorname{Re}(Bx,x) - \operatorname{Re}(Ax,Bx)] \\
= \;& 2(1+ \| A \| + \| B \|)^2[\| A \| + \| B \| + \| A \|\| B \| \\
& -\operatorname{Re}(Ax,x) - \operatorname{Re}(Bx,x) - \operatorname{Re}(Ax,Bx)] \\
\leq \;& (1+ \| A \| + \| B \|)^2\{2(\| A \| + \| B \|) + (\| A \| + \| B \|)^2 \\
& - \| Ax \|^2 - \| Bx \|^2 -2[\operatorname{Re}(Ax,x) + \operatorname{Re}(Bx,x) + \operatorname{Re}(Ax,Bx)]\} \\
= \;& (1+ \| A \| + \| B \|)^2\{(1+ \| A \| + \| B \|)^2 - \|(I+A+B)x\|^2\} \\
= \;& 0
\end{aligned}
$$

by (2). Hence $Ax-\| A \| x=0$ and $\|Ax+x\|Bx-\| B \| (Ax+x)=0$. Equalities in (3) thus follows.

Implications (3)⇒(1), (3)⇒(5)⇒(4)⇒(6) and (3)⇔(7) are trivial. It remains to prove (6)⇒(3). Since $\|Ax-\| A \| x\|^2 \leq 2 \| A \| [\| A \| - \operatorname{Re}\,(Ax,x)]$ by above, $\| B \| \, \|Ax-\| A \| x\|^2+ \| A \| \, \|Bx-\| B \| x\|^2 \leq 2 \| A \|\| B \| [\| A \| + \| B \| - \operatorname{Re}\,((A+B)x,x)] = 0$ by (6), and (3) follows.

Corollary 1. *Let $x \in H$ be a unit vector. Then any statement in Lemma implies the following.*

(i) *x is a norm attaining vector for $A+B$, i.e., $\| A+B \|=\| (A+B)x \|$.*

(ii) $\| A+B \|=\| A \| + \| B \|$.

(iii) *x is a norm attaining vector for $I+A+B$, i.e.,* $\| I+A+B \|= \| (I+A+B)x \|$.

(iv) *A and B satisfy the generalized Daugavet equation, i.e.,* $\| I+A+B \|= 1+ \| A \| + \| B \|$.

Proof. (i) and (ii) are due to (1) in Lemma. (2) in Lemma implies (iii) and (iv).

Remark that a further discussion of the equality (ii) in Corollary 1 can be found in [6] which is related to the unilateral shift on H. Also, many results related to the generalized Daugavet equality (iv) in Corollary 1 can be found in [5]. Notice that by (3) in Lemma we may say that Lemma is characterizations of two norms being eigenvalues for two operators. Correspondingly Corollary 2 below is characterizations of a norm attaining vector for two operators. We shall omit the proof since it is nothing but replacements of A by A^*A, and B by B^*B in Lemma.

Corollary 2. *Let $x \in H$ be a unit vector. Then the following are equivalent.*

(1) $\| A \|^2 + \| B \|^2$ *is an eigenvalue for A^*A+B^*B, i.e.,* $(A^*A+B^*B)x = (\| A \|^2 + \| B \|^2)x$;

(2) $1+ \| A \|^2 + \| B \|^2= \|(I+A^*A+B^*B)x\|$;

(3) $\| A \|^2$ *and* $\| B \|^2$ *are eigenvalues for A^*A and B^*B, respectively, with respect to x, i.e.,* $A^*Ax =\| A \|^2 x$ *and* $B^*Bx =\| B \|^2 x$;

(4) *x is a norm attaining vector for A and B, i.e.,* $\| A \|=\| Ax \|$ *and* $\| B \|=\| Bx \|$;

(5) *x is a complete vector for A^*A and B^*B, respectively, i.e.,* $\| A \|^2= \| A^*Ax \|=\| Ax \|^2$ *and* $\| B \|^2=\| B^*Bx \|=\| Bx \|^2$;

(6) $\| A \|^2 + \| B \|^2$ *is in the numerical range of A^*A+B^*B, i.e.,* $\|A\|^2 + \|B\|^2 = ((A^*A+B^*B)x,x)$;

(7) $A^*Ax = \|A^*Ax\|x = \|A\|^2x$ *and* $B^*Bx = \|B^*Bx\|x = \|B\|^2x$.

Remark that (5) *in Lemma implies* (4) *in Corollary* 2. *Hence we may conclude that any statement in Lemma implies any statement in Corollary* 2.

Corollary 3. *Let* $x \in H$ *be a unit vector. Then any statement in Corollary* 2 *implies the following.*

(i) *x is a norm attaining vector for* $A^*A + B^*B$, *i.e.,* $\| A^*A + B^*B \| = \| (A^*A + B^*B)x \|$.
(ii) $\| A^*A + B^*B \| = \| A \|^2 + \| B \|^2$.
(iii) *x is a norm attaining vector for* $I + A^*A + B^*B$, *i.e.,* $\| I + A^*A + B^*B \| = \| (I + A^*A + B^*B)x \|$.
(iv) A^*A+B^*B *satisfies the generalized Daugavet equation, i.e.,* $\| I+A^*A+B^*B \| = 1+ \| A \|^2 + \| B \|^2$.
(v) *x is a norm attaining vector for* A^*A *and* B^*B, *respectively, i.e.,* $\| A \|^2 = \| A^*Ax \|$ *and* $\| B \|^2 = \| B^*Bx \|$.

Proof. (1) in Corollary 2 implies (i) and (ii); while (2) implies (iii) and (iv); and (7) implies (v).

The next result is characterizations of the norm being an eigenvalue for an operator.

Theorem 1. *Let* $x \in H$ *be a unit vector. Then the following are equivalent.*

(1) $\| T \|$ *is an eigenvalue for T i.e.,* $Tx = \| T \| x$;
(2) $1+ \| T \| = \| (I + T)x \|$;
(3) $\| T \|$ *is an eigenvalue for T and* $Tx = \| Tx \| x$, *i.e.,* $Tx = \| T \| x$ *and* $Tx = \| Tx \| x$*;*
(4) $\| T \|$ *is in the numerical range of T, i.e.,* $\| T \| = (Tx, x)$;
(5) *x is a complete vector for T, i.e.,* $\| T \| = (Tx, x) = \| Tx \|$;

(6) $2 \parallel T \parallel$ *is an eigenvalue for* $T + T^*$, *i.e.,* $(T + T^*)x = 2 \parallel T \parallel x$;

(7) $\parallel T \parallel$ *and* $\parallel T \parallel^2$ *are eigenvalues for* T *and* T^*T, *respectively, with respect to* x, *i.e.,* $Tx = \parallel T \parallel x$ *and* $T^*Tx = \parallel T \parallel^2 x$;

(8) $(1+ \parallel T \parallel) \parallel T \parallel$ *is an eigenvalue for* $(I + T^*)T$, *i.e.,* $(I + T^*)Tx = (1+ \parallel T \parallel) \parallel T \parallel x$;

(9) $\parallel T \parallel$ *is a normal eigenvalue for* T, *i.e.,* $Tx = \parallel T \parallel x = T^*x$;

(10) x *is a complete vector for* T *and* T^*, *i.e.,* $\parallel T \parallel = (Tx, x) = \parallel Tx \parallel = \parallel T^*x \parallel$;

(11) x *is a complete vector for* T *and* T^*T, *i.e.,* $\parallel T \parallel = (Tx, x) = \parallel Tx \parallel$ *and* $\parallel T \parallel^2 = \parallel Tx \parallel^2 = \parallel T^*Tx \parallel$;

(12) $1+ \parallel T \parallel + \parallel T \parallel^2 = \parallel (I + T + T^*T)x \parallel$.

Proof. Let $A = T$ and $B = O$ in Lemma. Then (1), (2), (3), (4) and (5) are all equivalent. The proof of each other case is as follows.

(4)⇔(6). Let $A = T$ and $B = T^*$ in (1) and (4) of Lemma.
(5)⇔(7). Let $A = T$ and $B = T^*T$ in (3) and (4) of Lemma.
(5)⇔(8). Let $A = T$ and $B = T^*T$ in (1) and (4) of Lemma.
(6)⇔(9). Let $A = T$ and $B = T^*$ in (1) and (3) of Lemma.
(6)⇔(10). Let $A = T$ and $B = T^*$ in (1) and (6) of Lemma.
(5)⇔(11). Let $A = T$ and $B = T^*T$ in (4) and (5) of Lemma.
(5)⇔(12). Let $A = T$ and $B = T^*T$ in (2) and (4) of Lemma.

The next result is characterizations of a vector being a norm attaining vector for an operator.

Theorem 2. *Let* $x \in H$ *be a unit vector. Then the following are equivalent.*

(1) x *is a norm attaining vector for* T, *i.e.,* $\parallel T \parallel = \parallel Tx \parallel$;

(2) $\parallel T \parallel^2$ *is an eigenvalue for* T^*T, *i.e.,* $T^*Tx = \parallel T \parallel^2 x$;

(3) $1+ \parallel T \parallel^2 = \parallel (I + T^*T)x \parallel$;

(4) x *is a complete vector for* T^*T, *i.e.,* $\parallel T \parallel^2 = \parallel T^*Tx \parallel = \parallel Tx \parallel^2$;

(5) $\| T \|^2$ *is an eigenvalue for* T^*T, *and* $T^*Tx = \| T^*Tx \| x$, *i.e.*, $T^*Tx = \| T \|^2 x$ *and* $T^*Tx = \| T^*Tx \| x$*;*

(6) $\| T \|^2$ *is in the numerical range of* T^*T, *i.e.*, $\| T \|^2 = (T^*Tx, x)$*;*

(7) $1 + \| T \|^2 + \| T \|^4 = \| (I + T^*T + (T^*T)^2)x \|$;

(8) $\| T \|^2$ *and* $\| T \|^4$ *are eigenvalues for* T^*T *and* $(T^*T)^2$, *respectively, with respect to x, i.e.*, $T^*Tx = \| T \|^2 x$ *and* $(T^*T)^2x = \| T \|^4 x$*;*

(9) $(1 + \| T \|^2) \| T \|^2$ *is an eigenvalue for* $(I + T^*T)T^*T$, *i.e.*, $(I + T^*T)T^*Tx = (1 + \| T \|^2) \| T \|^2 x$*;*

(10) x *is a complete vector for* T^*T *and* $(T^*T)^2$, *i.e.*, $\| T \|^2 = \| Tx \|^2 = \| T^*Tx \|$ *and* $\| T \|^4 = \| T^*Tx \|^2 = \| (T^*T)^2x \|$.

Proof. Let $A = T^*T$ and $B = O$ in Lemma. Then (1), (2), (3), (4), (5) and (6) are all equivalent. Next,

(2)⇔(7). Let $A = T^*T$ and $B = (T^*T)^2$ in (2) and (3) of Lemma.
(4)⇔(8). Let $A = T^*T$ and $B = (T^*T)^2$ in (3) and (4) of Lemma.
(4)⇔(9). Let $A = T^*T$ and $B = (T^*T)^2$ in (1) and (4) of Lemma.
(4)⇔(10). Let $A = T^*T$ and $B = (T^*T)^2$ in (4) and (5) of Lemma.

The next result shows that if $\| T \|$ is an eigenvalue for T with respect to x, then x is a norm attaining vector for T. Related results are also given.

Corollary 4. *Let* $x \in H$ *be a unit vector. Then any statement in Theorem* 1 *implies the following.*

(i) *Any statement in Theorem* 2.

(ii) T *satisfies the Daugavet equation, i.e.*, $\| I + T \| = 1 + \| T \|$.

(iii) T *and* T^* *satisfy the generalized Daugavet equation, i.e.*, $\| I + T + T^* \| = 1 + 2 \| T \|$.

(iv) T *and* T^*T *satisfy the generalized Daugavet equation, i.e.*, $\| I + T + T^*T \| = 1 + \| T \| + \| T \|^2$.

(v) T *is a normaloid operator, i.e.*, $w(T) = \| T \|$.

(vi) x *is a norm attaining vector for* $I + T$, *i.e.*, $\| I + T \| = \| (I + T)x \|$.

(vii) *x is a norm attaining vector for $I+T+T^*$, i.e.,* $\| I+T+T^* \| = \| (I+T+T^*)x \|$.

(viii) *x is a norm attaining vector for $I+T+T^*T$, i.e.,* $\| I+T+T^*T \| = \| (I+T+T^*T)x \|$.

Proof. (i) follows by (1) and (5) in Theorem 1. (ii) by (2) in Theorem 1. (iii) by (6) in Theorem 1. (iv) by (12) in Theorem 1. (vi) by (2) in Theorem 1. (vii) by (6) in Theorem 1. (viii) by (12) in Theorem 1. It remains to show (v). Since $\| T \| \in W(T)$ by (4) in Theorem 1, and it is true that $w(T) \leq \| T \|$ in general [4, p.114] so that $w(T) = \| T \|$.

Finally, consequences of Theorem 2 are indicated in the next result.

Corollary 5. *Let $x \in H$ be a unit vector. Then any statement in Theorem* 2 *implies the following.*

(i) *x is a norm attaining vector for $I+T^*T$, i.e.,* $\| I+T^*T \| = \| (I+T^*T)x \|$.

(ii) *T^*T satisfies the Daugavet equation, i.e.,* $\| I+T^*T \| = 1+ \| T \|^2$.

(iii) *x is a norm attaining vector for $I+T^*T+(T^*T)^2$, i.e.,* $\| I+T^*T+(T^*T)^2 \| = \| (I+T^*T+(T^*T)^2)x \|$.

(iv) *T^*T and $(T^*T)^2$ satisfies the generalized Daugavet equation, i.e.,* $\| I+T^*T+(T^*T)^2 \| = 1+ \| T \|^2 + \| T \|^4$.

Proof. (i) and (ii) follow from (3) in Theorem 2. (iii) and (iv) are obtained from (7) in Theorem 2.

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Department of Mathematics, Bishop's University, Lennoxville, Quebec, J1M 1Z7, Canada.

E-mial: plin@ubishops.ca