BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 3, Number 1, June 1975

# POWER-CENTRAL TRACES AND SKEW TRACES

BY

PJEK-HWEE LEE([1])

**Abstract.** A well-known theorem due to Kaplansky asserts that if $R$ is a semisimple ring in which every element is power-central, that is, $x^{n(x)} \in Z(R)$ ($n(x)$ a natural number, $Z(R)$ the center of $R$) for each $x$ in $R$, then $R$ is commutative. For rings with involution, Chacron [2] showed that a semisimple ring with power-central symmetric elements must be a subdirect product of fields, four-dimensional division algebras and $2\times 2$ matrix algebras over fields. In this paper, we prove a parallel theorem on semisimple rings with power-central skew traces and extend Chacron's theorem by conditioning merely the traces.

In what follows, $R$ will always denote a ring with involution $*$. The *center* of $R$ will be denoted by $Z(R)$, or simply $Z$ in case of no ambiguity. $S(R)$, or simply $S$, will stand for the set of all *symmetric elements* in $R$, that is, $S(R) = \{x \in R \mid x^* = x\}$; and $K(R)$ will be $\{x \in R \mid x^* = -x\}$, the set of all *skew elements* in $R$. By the set of *traces* in $R$ we mean the set $T = \{x + x^* \mid x \in R\}$, and the set of *skew traces* will be $K_0 = \{x - x^* \mid x \in R\}$. Besides, we shall use $Z^+$ for $Z \cap S$, the subring of central symmetric elements.

## 1. Division rings.

We begin the paper with

THEOREM 1. *Let $R$ be a division ring with power-central skew elements. Then $k^2 \in Z$ for all $k \in K$ and $R$ is either commutative or 4-dimensional over $Z$.*

**Proof.** If $\operatorname{char} R = 2$, then $K = S$ and every symmetric element is power-central. By Chacron's theorem [2, Theorem 2], $xx^* \in Z$ for all $x \in R$ and $\dim_Z R \leq 4$. Assume that $\operatorname{char} R \neq 2$. In view of Baxter's theorem [4, Theorem 2.3], it suffices to show that $k^2 \in Z$ for

Received by the editors March 15, 1975.

([1]) The author was recently informed that many of the results in this paper were obtained independently by M. Chacron and I. N. Herstein.

all $k \in K$. If $Z \not\subseteq S$, then there exists $\alpha \in Z \cap K$, $\alpha \neq 0$. For each $s \in S$, we have $\alpha s \in K$, so that both $\alpha$ and $\alpha s$ are power-central. Thus $\alpha^{n(s)} \in Z$ and $(\alpha s)^{n(s)} \in Z$ for some natural number $n(s)$. Consequently, $s^{n(s)} \in Z$, that is, each symmetric element is power-central. By Chacron's theorem [2, Theorem 1], $S \subseteq Z$ and hence $k^2 \in Z$ for all $k \in K$. Now to the case $Z \subseteq S$. For each $k \in K$, consider the fields $Z \subseteq Z(k^2) \subseteq Z(k)$. Let $a \in Z(k^2) \subseteq S$. Since $ak \in K$, we have $k^{n(a)} \in Z$ and $(ak)^{n(a)} \in Z$ for some natural number $n(a)$. Thus $a^{n(a)} \in Z$ for each $a \in Z(k^2)$. By a lemma due to Kaplansky [7], there are three possibilities: (a) $Z(k^2) = Z$, (b) $Z(k^2)$ is purely inseparable over $Z$, or (c) $Z(k^2)$ is algebraic over a finite field. In case char $R = 0$, only (a) can be true, and then $k^2 \in Z$ for all $k \in K$. If (c) is true for some $k \in K$, then $Z$ is algebraic over a finite field. Thus each skew element is periodic, namely, $k^{n(k)} = k$ for some $n(k) > 1$. By Herstein-Montgomery's theorem [5, Theorem 3], $R$ must be a field. Therefore, we may assume that char $R = p$ for some odd prime $p$ and that for each $k \in K$ we have $k^{2p^{n(k)}} \in Z$ for some $n(k) \geq 0$. Assume, by way of contradiction, that there exists $k_0 \in K$ such that $k_0^{2p} \in Z$ but $a = k_0^2 \notin Z$. Define a mapping $\delta$ on $R$ by $x\delta = xa - ax$. Since $a^p \in Z$, $\delta^p = 0$; while $\delta \neq 0$ because $a \notin Z$. So $b\delta \neq 0$ for some $b \in R$. Let $n > 1$ be the integer such that $b\delta^n = 0$ but $b\delta^{n-1} \neq 0$. Put $c = b\delta^{n-1}$; then $c = d\delta = da - ad$ for some $d \in R$. Write $c = ae$; then $ae = da - ad$ so $a = dae^{-1} - ade^{-1} = (de^{-1})a - a(de^{-1})$. In other words, $a = af - fa$ for some $f \in R$. Recall that $a = k_0^2 \in S$, so that $a = a^* = f^*a - af^*$ and hence $a = ah - ha$ where $h = \frac{1}{2}(f - f^*) \in K$. By assumption, $h^{2p^m} \in Z$ for some $m \geq 0$. So $h^{2p^m} = ah^{2p^m}a^{-1} = (aha^{-1})^{2p^m} = (1 + h)^{2p^m} = (1 + h^{p^m})^2 = 1 + 2h^{p^m} + h^{2p^m}$ and hence $h^{p^m} = -\frac{1}{2} \in Z \cap K$, a contradiction. This completes the proof that $k^2 \in Z$ for all $k \in K$ and that $\dim_Z R \leq 4$.

Next we weaken the hypothesis on power-central skew elements and assume merely that every skew trace is power-central. In the situation when char $R \neq 2$, each skew element is a skew trace. So we need only consider the case char $R = 2$. Note that $K_0 = T$ and $K = S$ in this case.

THEOREM 2. *Let $R$ be a division ring of characteristic 2 with* *power-central traces. Then any two traces in $R$ commute and $R$ is of dimension at most 4 over $Z$.*

**Proof.** Assume first that $T \cap Z \neq 0$. Let $\alpha \in Z$ such that $\alpha = a + a^* \neq 0$ for some $a \in R$. Then $1 = (\alpha^{-1} a) + (\alpha^{-1} a)^* \in T$ and hence $xx^* \in xTx^* \subseteq T$ for all $x \in R$. Thus for each $s \in S$, $s^2 = ss^*$ is power-central and hence each symmetric element is power-central. By Chacron's theorem [2, Theorem 2], $T \subseteq Z$ and $\dim_Z R \leq 4$. Now suppose that $T \cap Z = 0$. It follows that $Z \subseteq S$. Let $t \in T$. Consider the fields $Z \subseteq Z(t^2) \subseteq Z(t)$. For each $x \in Z(t^2)$, $xt \in T$ so both $t$ and $xt$ are power-central. Thus for some natural number $n(x)$, $t^{n(x)} \in Z$ and $(xt)^{n(x)} \in Z$, so $x^{n(x)} \in Z$. Again by Kaplansky's lemma, we have the following three cases: (a) $Z(t^2) = Z$, (b) $Z(t^2)$ is purely inseparable over $Z$, or (c) $Z(t^2)$ is algebraic over the field of two elements. In case (c), every trace is periodic and then $R$ is commutative by a theorem of Herstein [10]. There remains the case that every trace is purely inseparable over $Z$. Assume by way of contradiction that there exist two traces $a$ and $b$ such that $ab \neq ba$. Since $a$ is purely inseparable over $Z$, there exists an integer $m \geq 1$ such that $a^{2^m} b = ba^{2^m}$ but $a^{2^{m-1}} b \neq ba^{2^{m-1}}$. Similarly, let $n$ be the least natural number such that $b^{2^n}$ commutes with $a^{2^{m-1}}$, that is, $a^{2^m} b^{2^n} = b^{2^m} a^{2^n}$ but $a^{2^m} b^{2^{n-1}} \neq b^{2^{n-1}} a^{2^m}$. Set $c = a^{2^{m-1}}$ and $d = b^{2^{n-1}}$. Then $c^2 d = dc^2$ and $cd^2 = d^2 c$ while $cd \neq dc$. Thus $\beta = cd + dc \neq 0$ but commutes with both $c$ and $d$, and hence $1 = (\beta^{-1} c) d + d(\beta^{-1} c) \in T$ contrary to the assumption $Z \cap T = 0$. Hence any two traces commute and it follows that $\dim_Z R \leq 4$ [9, Theorem 1].

Combining Chacron's theorem together with Theorem 2, we easily get the following

THEOREM 3. *Let $R$ be a division ring with power-central traces. Then any two traces in $R$ commute and* $\dim_Z R \leq 4$.

On the other hand, combining Theorems 1 and 2 we have

THEOREM 4. *Let $R$ be a division ring with power-central skew traces. Then* $\dim_Z R \leq 4$. *Furthermore, we have $k^2 \in Z$ for all $k \in K_0$ if* $\operatorname{char} R \neq 2$, *and $k_1 k_2 = k_2 k_1$ for all $k_1, k_2 \in K_0$ if* $\operatorname{char} R = 2$.

Note that the condition $k^2 \in Z$ for all $k \in K$ does not imply the commutativity of the skew elements. For example, in the division ring of real quaternions with the usual conjugation $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mapsto \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$ as involution, each skew element is square-central but the skew elements do not commute.

2. ***-Simple rings.** By a **-ideal* $I$ of $R$, we mean a (two-sided) ideal of $R$ with $I^* = I$. A ring $R$ is called **-simple* if $R^2 \neq 0$ and $R$ has no *-ideal other than 0 and $R$. One can easily verify that a ring is *-simple if and only if it is either a simple ring or a direct sum of a simple ring and its opposite with the exchange involution. Now we extend our Theorems 3 and 4 to *-simple rings.

THEOREM 5. *Let $R$ be a nonradical *-simple ring with power-central traces. Then $R$ is a field, a 4-dimensional division algebra, the direct sum of two isomorphic fields, or the $2 \times 2$ matrix algebra over a field. Also, any two traces of $R$ commute with each other.*

**Proof.** Assume first that $R = A \oplus A^*$ for some simple ring $A$. For any $x \in A$, we have $(x, x^*) \in T(R)$, so that $(x, x^*)^{n(x)} \in Z(R)$ for some natural number $n(x)$. Thus $A$ consists of power-central elements and hence is a field by Kaplansky's theorem. Accordingly $R$ is the direct sum of two isomorphic fields. Now assume that $R$ is simple. Since $Z$ is either 0 or a field, every trace in $R$ is then either nilpotent or invertible. By a theorem of Herstein and Montgomery [6, Theorem 7], $R$ is a division ring or the $2 \times 2$ matrix ring over a field. In case $R$ is a division ring, we have $\dim_Z R \leq 4$ and $t_1 t_2 = t_2 t_1$ for all $t_1, t_2 \in T$ by Theorem 3. Now to the case $R = F_2$ for some field $F$. If $F_2$ has the symplectic involution, that is, $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^* = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$, then the traces are just the scalar matrices and the commutativity follows trivially. If the involution is given by $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^* = \begin{bmatrix} \bar{\alpha} & \pi^{-1}\bar{\gamma} \\ \pi\bar{\beta} & \bar{\delta} \end{bmatrix}$, where the overbar denotes an automorphism on $F$ of period 2 and $\pi$ is a fixed element in $F$ with $\bar{\pi} = \pi$, then for each $\alpha \in F$, we have $\begin{bmatrix} \alpha+\bar{\alpha} & 0 \\ 0 & 0 \end{bmatrix} \in T$, so that $\begin{bmatrix} (\alpha+\bar{\alpha})^{n(\alpha)} & 0 \\ 0 & 0 \end{bmatrix} \in Z$ and hence $\alpha + \bar{\alpha} = 0$. That is, $\bar{\alpha} = -\alpha$ for all $\alpha \in F$, and it follows that $\operatorname{char} F = 2$. Thus the traces are the matrices of the form $\begin{bmatrix} 0 & \beta \\ \pi\beta & 0 \end{bmatrix}$ and hence any two traces commute.

Using another theorem of Herstein and Montgomery [6, Theorem 8] and proceeding as above, we easily get

THEOREM 6. *Let $R$ be a nonradical *-simple ring with power-central skew traces. Then $R$ is a field, a 4-dimensional simple algebra, or the direct sum of two isomorphic fields. Also, $k^2 \in Z$ for all $k \in K_0$ if* $\operatorname{char} R \neq 2$, *and* $k_1 k_2 = k_2 k_1$ *for all* $k_1, k_2 \in K_0$ *if* $\operatorname{char} R = 2$.

3. ***-Primitive rings.** In [1] the notion of **-primitive ring* was introduced as a ring admitting a **-faithful*, irreducible module (i. e. $Mr = Mr^* = 0$ implies $r = 0$). It is easy to see that a ring is *-primitive if and only if it is either a primitive ring or a subdirect sum of a primitive ring and its opposite with the involution induced by exchanging coordinates. In view of Kaplansky's theorem on polynomial identities [3, Theorem 6.3.1], a *-primitive ring satisfying a polynomial identity must be *-simple.

Let $R$ be a *-primitive ring with power-central traces or skew traces. If $Z = 0$, then every trace or skew trace is nilpotent. The theorems of Herstein and Montgomery describe the structure of such a ring and we see that $Z \neq 0$ in any case. The contradiction shows that $Z \neq 0$. As a matter of fact, $Z^+ \neq 0$. For otherwise, we have $\alpha + \alpha^* = 0$ and $\alpha\alpha^* = 0$, so that $\alpha^2 = 0$ for all $\alpha \in Z$, which is impossible in a semiprime ring with a nonzero center.

Now that $Z^+ \neq 0$, we may consider the ring $Q = \{a/\alpha \mid a \in R,\ \alpha \in Z^+ \backslash 0\}$, because each nonzero element in $Z^+$ is regular in $R$. We can equip $Q$ with an involution by defining $(a/\alpha)^* = a^*/\alpha$. Then each trace or skew trace in $Q$ is power-central. Besides, we claim that $Q$ is a *-simple ring. Let $I$ be a nonzero *-ideal of $Q$. Then $I \cap R$ is a nonzero *-ideal of the *-primitive ring $R$ and hence is itself a *-primitive ring [8, Lemma 14]. From the previous paragraph, we know that $Z^+(I \cap R) \neq 0$. Since $Z^+(I \cap R) \subseteq Z^+(R)$ [8, Lemma 6], $I$ contains an invertible element in $Q$, so $I = Q$. That is, $Q$ must be *-simple. By Theorems 5 and 6, $Q$ satisfies a polynomial identity and so does $R$. Thus $R$ must be also *-simple. Therefore, we have

THEOREM 7. *Let $R$ be a *-primitive ring with power-central traces. Then $R$ is a field, a 4-dimensional division algebra, the direct* *sum of two isomorphic fields, or the* $2 \times 2$ *matrix ring over a field. Also, any two traces commute with each other.*

THEOREM 8. *Let R be a *-primitive ring with power-central skew traces. Then R is a field, a 4-dimensional simple algebra, or the direct sum of two isomorphic fields. Moreover,* $k^2 \in Z$ *for all* $k \in K_0$ *if* $\operatorname{char} R \neq 2$, *and* $k_1 k_2 = k_2 k_1$ *for all* $k_1, k_2 \in K_0$ *if* $\operatorname{char} R = 2$.

Since a semisimple ring is a subdirect product of *-primitive rings, we get

THEOREM 9. *Let R be a semisimple ring with power-central traces. Then R is a subdirect product of fields, 4-dimensional division algebras and* $2 \times 2$ *matrix algebras over fields, and any two traces in R commute with each other.*

THEOREM 10. *Let R be a semisimple ring with power-central skew-traces. Then R is a subdirect product of fields and 4-dimensional simple algebras.*

## REFERENCES

1. W. E. Baxter and W. S. Martindale III, *Rings with involution and polynomial identities*, Canad. J. Math. **20** (1968), 465–473.
2. M. Chacron, *A generalization of a theorem of Kaplansky and rings with involution*, Michigan Math. J. **20** (1973), 45–53.
3. I. N. Herstein, *Noncommutative rings*, Carus Monographs, No. 15, Math. Assoc. Amer., 1968.
4. ———, *Topics in ring theory*, University of Chicago Press, Chicago, Ill., 1969.
5. I. N. Herstein and S. Montgomery, *A note on division rings with involutions.* Michigan Math. J. **18** (1971), 75–79.
6. ———, *Invertible and regular elements in rings with involution*, J. Algebra **25** (1973), 390–400.
7. I. Kaplansky, *A theorem on division rings*, Canad. J. Math. **3** (1951), 290–292.
8. P. H. Lee, *On subrings of rings with involution*, Pacific J. Math. (to appear).
9. S. Montgomery, *Lie structure of simple rings of characteristic 2*, J. Algebra **15** (1970), 387–407.
10. ———, *A generalization of a theorem of Jacobson.* II, Pacific J. Math. **44** (1973), 233–240.

NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN