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ON THE RESTRICTED SKOROKHOD SPACES

BY

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Dedicated to S. T. Hu

Abstract. In the theory of stochastic processes, the sample paths are usually continuous (C) or at least of type J, having no discontinuities other than jumps. In some cases of continuous path processes, the continuity modulus of the paths is restricted; for example, the paths of Wiener process are Hölderian (of class $<1/2$). Thus with stronger topology on the paths space, more limit theorems are available. In this paper we discuss two classes of restrictions on J, similar to the Hölderian conditions on C. Conditions of realizability and limit theorems are obtained, and are applied to the special case of Ito processes defined by Ito's stochastic integrals relative to a Lévy process.

0. **Introduction.** In [5] M. Woodroofe introduced a class of "restricted Skorokhod spaces". These are obtained by restricting the behavior of $\bar{\omega}(\cdot;\, X)$, the regularity-modulus of X. Since this modulus $\bar{\omega}$ plays the same role in $\mathcal{J}$ as the continuity-modulus ω plays in the space $\mathcal{C}$, this analogy is the proper setting of the interpretation of these spaces.

We will discuss two classes of restrictions. One is the class of integral restrictions as in Woodroofe [5]. The others are more proper analogues of the Lipschitz-Hölder classes. We will follow the notation of [6].

1. **The space $\mathcal{J}_w$.** Consider the set of all $X \in \mathcal{J}_w$ with $\bar{\omega}(\cdot,\, X)$ satisfying the following condition:

$$\int_0^1 w(\delta)\, \bar{\omega}(\delta;\, X) < \infty\,. \tag{1}$$

Here the weight function w is positive, nonincreasing, continuous,

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and approaching infinity as $\delta \downarrow 0$. On this set $\mathcal{G}_w$ we define the metric d_w by

$$d_w(X;\ Y) = d_H(\Gamma X;\ \Gamma Y) + \int_0^1 |\bar{\omega}(\delta;\ X) - \bar{\omega}(\delta;\ Y)|\, w(\delta)\, d\delta\,.$$

This is obviously stronger than the Prokhorov metric for $\mathcal{G}$.

LEMMA 1. *A subset* $F \subset \mathcal{G}_w$ *is relatively compact iff*

(i) $F(I) = \{X(t) : X \in F,\ t \in I\}$ is relatively compact in E, and

(ii) $\lim_{\delta_1 \downarrow 0} \sup_F \int_0^{\delta_1} w(\delta)\, \bar{\omega}(\delta;\ X) d\delta = 0$.

Proof. *Sufficiency*. By (ii), $\{\bar{\omega}(\cdot;\ X),\ X \in F\}$ is equi-modulated, and F is relatively compact in $\mathcal{G}$ by (i) and (ii). If $(X_m) \subset F$ converge to X_0 in $\mathcal{G}$, then $d\bar{\omega}(\ ;\ X_n) \to d\bar{\omega}(\ ;\ X_0)$ in the weak topology; therefore $\bar{\omega}(\delta;\ X_n) \to \bar{\omega}(\delta;\ X_0)$ except on a countable set, resulting in $\int_0^{\delta} \bar{\omega}(\delta;\ X_0)\, w(\delta)\, d\delta < \infty$ by (ii). Or $X_0 \in \mathcal{G}_w$. Also

$$\int w(\delta)\, |\bar{\omega}(\delta;\ X_n) - \bar{\omega}(\delta;\ X_0)|\, d\delta \to 0 \quad \text{as} \quad n \to \infty\,.$$

Therefore $X_n \to X_0$ in d_w.

Necessity. If F is precompact in $\mathcal{G}_w$, it is relatively compact in $\mathcal{G}$, so that (i) is true. Now for any Cauchy sequence (X_n) in F we have

$$\lim_{m,\, n \to \infty} \int_0^1 d\delta\, |\bar{\omega}(\delta;\ X_m) - \bar{\omega}(\delta,\ X_n)|\, w(\delta) = 0\,.$$

Let $X_n \to X_0$ in $\mathcal{G}$; then the sequence $\bar{\omega}(\cdot, X_n)$ converges to $\bar{\omega}(\cdot, X_0)$ almost everywhere while it is a Cauchy sequence in $\mathcal{L}^1(w(\delta)\, d\delta)$, so that it converges to $\bar{\omega}(\cdot;\ X_0)$ in $\mathcal{L}^1(w(\delta)\, d\delta)$ and therefore $X_n \to X_0$ in $\mathcal{G}_w$. Q.E.D.

We have actually proved the completeness of $\mathcal{G}_w$. The separability of the space is shown by exhibiting the (standard) dense set, the set of all step functions jumping at binary times $(K/2^n)$ and taking values in a countable dense set of E. (We omit the detail.)

2. **The space $\mathcal{G}^w$.** Next we introduce the space $\mathcal{G}^w$ of all functions $X \in \mathcal{G}$ satisfying

$$\lim_{\delta \downarrow 0} \bar{\omega}(\delta;\ X)\, w(\delta) = 0\,, \tag{2}$$

where w is a fixed function with the same properties as before. As the metric we take

$$d^w(X;\ Y) = d_J\,|\overline{\omega}(\cdot;\ X)\,w(\cdot) - \overline{\omega}(\cdot;\ Y)\,w(\cdot)| + d_H(\Gamma X;\ \Gamma Y)\,,$$

where d_J is the Prokhorov distance for $\mathcal{J}$. By the same argument as above we have

LEMMA 2. *A set F is relatively compact in $\mathcal{J}^w$ iff*

(i) *FI is relatively compact in E, and*

(ii) $\lim_{\delta\downarrow 0}\sup_{X\in F}\overline{\omega}(\delta;\ X)\,w(\delta) = 0$.

COROLLARY. *$\mathcal{J}^w$ is a Polish space.*

The particular case of $w(\delta) \equiv \delta^{-(1+\alpha)}$ $(1 \geq \alpha > 0)$ for $\mathcal{J}_w$ is written as D_α by Woodroofe. On the other hand the case of $w(\delta) = \delta^{-\alpha}$ for $\mathcal{J}^w$ will be written as D^α. We see clearly

$$D_\alpha \subset D^\alpha \subset D_{\alpha-} = \bigcap_{\beta<\alpha} D_\beta\,.$$

REMARK. If condition (2) is replaced by

$$\text{(2')} \qquad \lim_{\delta\downarrow 0}\overline{\omega}(\delta;\ X)\,w(\delta) \text{ exists}\,,$$

we get similar results. But if it is replaced by

$$\text{(2'')} \qquad \sup_{\delta}\overline{\omega}(\delta;\ X)\,w(\delta) \cdot < \infty\,,$$

it seems difficult to define a suitable Polish metric, although this set should be the "true" analogue of the Lipschitz-Hölder functions when $w(\delta) = \delta^{-\alpha}$. But this distinction is not of much practical relevance. Indeed the most useful subspaces (with stronger topology) of $\mathcal{J}$ are the "projective limits":

Let $D_{\alpha-} = \bigcap_{\beta<\alpha} D_\beta$. We take the "projective topology" generated by the induced topologies. The assertions above can be obviously modified to this space. For example, we have

LEMMA 3. *A subset $F \subset D_{\beta-}$ is relatively compact iff*

(i) *of Lemma 2 holds, and*

(ii) *of Lemma 2 holds true for all $w(\delta) = \delta^{-1-\alpha}$ with $0 < \alpha < \beta$.*

In particular, $D_{\beta-}$ is a Polish space. $D_{\beta-}$ can also be defined as the projective limit of $(D^\alpha : 0 < \alpha < \beta)$.

3. **Continuity of addition.** We now show that the analogy to the Lipschitz-Hölder class is not very deep. Indeed the space D_α is not even a group with pointwise-addition when $E=\boldsymbol{R}$.

EXAMPLE. f and $g \in D_\alpha$ but $(f-g) \notin D_\alpha$. Let $c_1 = 0 < b_1 < c_2 < b_2 < c_2 < \cdots \to 1$, and let

$$f(x) = l_n\,, \quad \text{for } c_n \leq x < b_n \text{ for some } n \in \boldsymbol{N}\,;$$
$$= 0\,, \quad \text{otherwise.}$$

Let $p_n = b_n - c_n$, $s_n = c_{n+1} - b_n$, $p_0 = 0$. We will assume that $s_{n-1} > s_n > p_n > p_{n+1}$, $l_n \downarrow 0$; therefore $\overline{\omega}(\delta;\, f) = l_n$, if $p_n \leq \delta < p_{n-1}$, for $n \in N$. We see that $f \in \mathcal{G}_w$ iff $\mathcal{L}_w(f) \equiv \sum_1^\infty \left\{ l_n \int_{p_n}^{p_{n-1}} w(z)\, dz \right\} < \infty$.

If $w = w_\alpha = z^{-1-\alpha}$, then the condition is

$$\mathcal{L}_w(f) = \frac{1}{\alpha} \sum_1^\infty l_n \cdot \left(\frac{1}{p_n^\alpha} \frac{1}{p_{n-1}^\alpha} \right) < \infty\,.$$

Now also let $\cdots c_n < a_n < b_n < c_{n+1} < \cdots$,

$$g(x) = l_n\,, \quad c_n \leq x < a_n,$$
$$= 0\,, \quad \text{otherwise,}$$

and $a_n - c_n = q_n$, $t_n = c_{n+1} - a_n$, $q_0 = 1$.

Of course, we have

$$t_{n-1} > t_n > q_n > q_{n+1}\,,$$

and everything calculated above is valid, so that

$$\mathcal{L}_w(g) = \frac{1}{\alpha} \sum (l_n) \left(\frac{1}{q_n^\alpha} - \frac{1}{q_{n-1}^\alpha} \right).$$

Now we set $h = f - g$; thus

$$h(x) = l_n\,, \quad a_n \leq x_n < b_n \text{ for some } n,$$
$$= 0\,, \quad \text{otherwise.}$$

and

$$\mathcal{L}_w(h) \equiv \frac{1}{\alpha} \sum_1^\infty (l_n) \left(\frac{1}{r_n^\alpha} - \frac{1}{r_{n-1}^\alpha} \right),$$

where $r_n \equiv p_n - a_n = p_n - q_n < r_{n-1}$. Thus $\mathcal{L}_w(g) < \infty$, $\mathcal{L}_w(f) < \infty$. And yet $\mathcal{L}_w(f-g) = \infty$, if: we choose the numbers l_n, p_n, q_n, s_n in such a way that:

$$1)\quad l_n > l_{n+1} > \cdots \to 0,$$

$$2)\quad s_{n-1} > s_n > p_n > p_{n+1}, \qquad 3)\quad \sum s_n + \sum p_n = 1,$$

$$4)\quad p_n > q_n > q_{n+1}, \qquad 5)\quad r_n = p_n - q_n > r_{n+1},$$

$$6)\quad \mathscr{L}_w(g) < \infty, \qquad 7)\quad \mathscr{L}_w(f) < \infty,$$

$$8)\quad \mathscr{L}_w(h) = \infty.$$

It is obvious that such a sequence exists. To get rid of (s_n), we assume that $s_n = 2p_n$, so we need only $2') + 3') : p_n > p_{n+1}$, $\sum p_n = \frac{1}{3}$, instead of 2), 3). To ensure 1), let $l_n = n^{-r}$, with $r > 0$. Also we put $p_n \equiv \text{const} \cdot (n+4)^{-\beta}$, with $\beta > 1$, then 2') + 3') are valid; and if $\alpha\beta - r < 0$, then 7) is valid, since $\mathscr{L}_w(f) \sim \text{const} \cdot \sum n^{-r} n^{\alpha\beta-1}$.

To ensure 4) and 5), now let $\beta_1 > \beta$, undetermined, with

$$q_n = p_n - p_n^{\beta_1/\beta} \cdot \text{const} = \text{const} \cdot (n+4)^{-\beta} - \text{const} \cdot (n+4)^{-\beta_1},$$

which is positive if β_1 is large, and 6) is true. Finally let $\alpha\beta_1 - r > 0$, so that 8) is ensured. Q. E. D.

The following is then worth mentioning.

PROPOSITION 1. *If $X \in \mathscr{J}_w$, $Y \in \mathscr{J}_w \cap \mathscr{C}$ and (E, ρ) is a Polish group with left-invariant metric ρ, then $Z \equiv X + Y \in \mathscr{J}_w$ and the "addition" is continuous.*

Proof. With $\bar{\omega}(\delta; Z) \leq \bar{\omega}(\delta; X) + \omega(\delta; Y) \leq \bar{\omega}(\delta; X) + 2\bar{\omega}(\delta, Y)$ we have $Z \in \mathscr{J}_w$. If now $X_n \to X_0$ in $\mathscr{J}_w$, $Y_n \to Y_0$ in $\mathscr{C}_w = \mathscr{J}_w \cap \mathscr{C}$, then $Z_n = X_n + Y_n \to Z_0$ in $\mathscr{J}$. Also the formula shows

$$\lim_{\delta_1 \downarrow 0} \sup_n \int_0^{\delta_1} \bar{\omega}(\delta; Z_n)\, w(\delta)\, d\delta = 0.$$

Now apply the criterion of Lemma 1.

NOTE. Similarly the assertion for $\mathscr{J}^w$ is also valid.

We mention also the following related fact; the easy proof can be omitted.

LEMMA 4. *Let X be a step-function with finitely many jumps, and $Y \in \mathscr{J}_w$. If $D(X) \cap D(Y) = \emptyset$, then $X + Y \in \mathscr{J}_w$.*

4. **Problem of realization.** Let us consider the problem whether a process may be viewed as a random variable valued in D_α or $D_{\alpha-}$.

We adopt the viewpoint of not distinguishing two "equivalent" processes with identical marginal distributions. Standard argument (see, e.g., Woodroofe [5, Th'm 3.1] reduces this to the criterion of equitight distributions. It is clear that the criterion given by the Prokhorov theorem in our space is quite awkward. The criterion of Woodroofe [5, Th'm 3.1 and 3.2], while not a necessary condition, is very convenient. We state a useful corollary of it.

DEFINITION. Let $((X_t)_I, (\mathcal{F}_t), (\Omega, \mathcal{P}))$ be an adapted process. Let

$$\overline{\overline{\omega}}(\delta;\ \varepsilon) \equiv \overline{\overline{\omega}}(\delta,\ \varepsilon,\ X,\ \mathcal{F}) \equiv \sup_{0\leq s\leq t\leq s+\delta} \operatorname{ess\,sup} \mathcal{P}\{\rho(X_t;\ X_s) \geq \varepsilon \mid \mathcal{F}_s\}\,.$$

If $\overline{\overline{\omega}}(\delta;\ \varepsilon) \downarrow 0$ as $\delta \downarrow 0$ (for all $\varepsilon > 0$), then the process is uniformly Markov-continuous (abbreviation: U. M. C.).

This is of course the generalization of the "uniform stochastic continuity" for a Markov transition function. We observe that the combinatorics of Dynkin and Kinney is valid for this kind of process, so that in particular all U. M. C.-processes are $\mathcal{G}$-processes.

LEMMA 5. *Let* $\overline{\overline{\omega}}(\delta;\ \varepsilon) = O(\delta\varepsilon^{-\gamma})$, *where* $\gamma > 1$; *then*

$$\hat{\omega}(\delta;\ \varepsilon) = \mathcal{P}\,\{\overline{\omega}(\delta;\ X) > \varepsilon\} = O(\delta\varepsilon^{-2\alpha})\,.$$

Proof. The well-known estimate of Dynkin-Kinney-Skorokhod says:

If $\omega(\delta \geq \varepsilon) < \alpha$, then

$$\mathcal{P}\Big\{\sup_{a\leq t_1\leq t_2\leq t_3\leq a+\delta} \min\,(\rho(X_{t_1};\ X_{t_2});\ \rho(X_{t_2};\ X_{t_3})) > 4\varepsilon\Big\} < \frac{\alpha^2}{(1-\alpha)^2}$$

(see Skorokhod [4, Lemma 2, §VI. 5]).

Divide the whole interval $I=[0, 1]$ into n subintervals of length $\leq \delta$ and suppose $n \sim 1/\delta$; then

$$\mathcal{P}\,\{\overline{\omega}(\delta;\ X) > 4\varepsilon\} \leq 2\alpha + \frac{\alpha^2}{(1-\alpha)^2}\,\frac{2}{\delta}\,.$$

Therefore, with $\alpha = O(\delta\varepsilon^{-2\alpha})$, $\hat{\omega}(\delta,\ \varepsilon) = O(\delta\varepsilon^{-2\alpha})$.

LEMMA 6. *If* $\hat{\omega}(\delta;\ \varepsilon) = O(\delta\varepsilon^{-2\gamma})$, *then the process belongs to* $D_{(1/2\gamma)-}$.

Proof. $\sum \mathscr{P}\{\overline{\omega}(1/2^n, X) > 2^{-n\alpha}\} \leq K \sum 2^{-n(1-2\alpha r)} < \infty$ if $\alpha < 1/2r$. By the Borel-Cantelli Lemma, almost surely $\sup_n \overline{\omega}(1/2^n, X)\, 2^{n\alpha} < \infty$. Q. E. D.

THEOREM 2. *If* $\omega(\delta, \varepsilon) = O(\delta\varepsilon^{-\alpha})$ *then* $X \in D_{(1/2r)-}$ *almost surely.*

This theorem will now be applied to the Ito processes.

5. **Ito processes.** For convenience we adhere to the notation and conventions of K. Ito [2]; in particular, we will always take the "continuous" or "regular kernels" for the stochastic (indefinite) integrals $\int_0^t\int R(s, \omega)\, dW(s)$ and $\int_0^t\int f(s, u, \omega)\, q(ds\, du)$; they are thus $\mathscr{C}$ and $\mathscr{J}$ processes, respectively.

Assuming f to be the simplest kind of integrand, we then have

$$\left\|\int_I\int_{\mathcal{U}} f\, dq\right\|_{\mathscr{L}^2(\mathscr{P})} = \left[\iint \mathscr{P}\{|f(s, u)|^2\}\, ds\, \lambda(du)\right]^{1/2},$$

where $(\mathcal{U};\ \lambda(du))$ is $(\boldsymbol{R}\backslash 0,\ du/|u^2|)$ as in [2].

We have also

$$\left\|\iint f\, dq\right\|_{\mathscr{L}^1(\mathscr{P})} \leq \left\|\iint |f|\, dp\right\|_{\mathscr{L}^1(\mathscr{P})} + \iint \mathscr{P}\{|f|\}\, ds\, \lambda(du)$$
$$\leq 2\iint \mathscr{P}\{|f|\}\, ds\, \lambda(du)\,.$$

We can apply the Riesz Convexity Theorem, and we have

THEOREM 3. *If* $f(s, u, \omega)$ *is "adapted" and*

$$\iint \mathscr{P}|f(s, u, \omega)|^\alpha\, ds \cdot \lambda(du) < \infty\,,$$

where $1 \leq \alpha \leq 2$, *then the stochastic integral* $\iint f\, dq$ *can be defined in such a way that*

$$\left\|\iint f\, dq\right\|_{\mathscr{L}^\alpha(\mathscr{P})} \leq \left[\iint \{|f|^\alpha\}\, ds\, \lambda(du)\right]^{1/\alpha} \cdot C(\alpha)\,,$$

where $C(\alpha) = 2\exp(2 - 2/\alpha)$.

COROLLARY. *If* $\sup_s \int \mathscr{P}\{|f(s;\ )|^\alpha\, d\lambda = M^\alpha < \infty$ *for all* s, *then*

$$\left\|\iint_{\Delta s} f\, dq\right\|_{\mathscr{L}^\alpha(\mathscr{P})} \leq M(\Delta s)^{1/\alpha}\, C(\alpha)$$

for any interval Δs.

In particular, under the condition in the corollary, we have

LEMMA 7. *If* $X_q(t) = \int_0^t \int f\,dq$, *then* $\overline{\omega}(\delta;\,\varepsilon;\,X) = O(\delta\varepsilon^{-\alpha})$, *and* X *is a* $D_{(1/2\alpha)}$- *process.*

[To prove the last estimation: $\mathcal{P}\left\{\left(\iint_{\Delta s} f\,dq\right) \geq \varepsilon\right\} = \mathcal{P}\{(\ \ )^\alpha \geq \varepsilon^\alpha\} \leq M^\alpha C(\alpha)\,\delta\varepsilon^{-\alpha}$.]

By combining this and Lemma 4, we have

LEMMA 8. *Let* $f(s, u, \omega)$ *be "adapted" and*

$$\int_0^1 \int_{|u|\leq 1} \mathcal{P}\{|f(s, u, \cdot)|^\alpha\}\, ds\, \lambda(du) < \infty\,,$$

where $1 \leq \alpha \leq 2$, *and let* $f(s, \cdot\omega)$ *be almost surely left-continuous in* s *as a function valued in* $\mathcal{L}^\alpha(\mathcal{U}_p)$, $\mathcal{U}_p = \{u : |u| > 1\}$ *with measure* $du/|u|^2$. *Then the process*

$$X(t) = X_q(t) + X_p(t) = \int_0^t \int_{|u|\leq 1} f\,dq + \iint_{|u|>1} f\,dp$$

is a $D_{(1/2\alpha)}$- *process.*

REMARKS. For $\alpha > 2$ the stochastic integral seems not definable. The case of W-integral (based on white noise), which is a "limiting case" of q-integral, is limited to the $\mathcal{L}^2$-norms. We have

LEMMA 9. *If* $\sup_s \mathcal{P}\{(R(s, \omega)|^2\} = M^2 < \infty$, *then* $X_W(t) = \int_0^t R\,dW$ *defines a* $\mathcal{C}_{(1/4)}$- *process.*

By combining Lemmas 8 and 9 we have

THEOREM 4. *The process* $X(t) = X_p(t) + X_q(t) + X_W(t)$, *where* X_p, X_q *are as in Lemma* 8, *with* $\alpha = 2$, *and* X_W *as in Lemma* 9, *is a* $D_{(1/4)}$- *process.*

REMARK. The undesirable factor in Lemma 9 can perhaps be discarded by some smoothness condition on R. (Consider the extreme case $R = 1$ when there is the most precise result of P. Lévy on the Hölder condition of Brownian motion.) On the other hand, for the punctual right-side Hölder condition our rough estimation gives immediately (everything as in Lemma 8)

LEMMA 10. *If* $\gamma < 1/\alpha$, *then* $\overline{\lim}_{t\downarrow 0}\, t^{-\gamma} \rho(X(t);\, X(0) = 0$ *(a.e.).*

This is an extension of a special case of the Blumenthal-Getoor Theorem [1], which is concerned with the "infinitely-divisible" processes.

References

1. R. M. Blumenthal and R. K. Getoor, *Sample functions of stochastic processes with stationary independent increments*, J. Math. Mech. **10** (1961), 493–516.

2. K. Ito, *On stochastic differential equations*, Mem. Amer. Math. Soc. no. 4 (1951).

3. A. V. Skorokhod, *Limit theorems for stochastic processes*, Theor. Probability Appl. **1** (1956), 261–290.

4. ———, *Studies in the Theory of Random Processes*, Addison-Wesley, Reading, Mass., 1965.

5. M. B. Woodroofe, *On the weak convergence of stochastic processes without discontinuities of the second kind*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1968), 18–25.

6. W. Yang, *On the M_1 topology of Skorokhod spaces*, Bull. Inst. Math. Acad. Sinica **2** (1974), 267–279.

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