BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 30, Number 4, December 2002

VARIATIONAL METHODS FOR THE TYPICALLY REAL FUNCTIONS AND APPLICATIONS

BY

PAVEL G. TODOROV

Abstract. We derive new variational methods and formulas for the functions stated in the title and apply them for finding the sharp estimates for the coefficients of the inverse functions of the examined functions. Two conjectures for these coefficients are stated.

1. Introduction. Rogosinski [1] introduced the class TR of the typically real functions $f(z)$ analytic in the unit disk $\mid z \mid< 1$ and satisfying the conditions $f(0) = f'(0) - 1 = 0$ and $(\mathrm{Im} f(z))(\mathrm{Im} z) > 0$ for all nonreal z in $\mid z \mid< 1$. For the class TR, Robertson [2] found the following Stieltjes integral representation

$$(1) \qquad w = f(z) = \int_{-1}^{1} \frac{z d\mu(t)}{1 - 2tz + z^2} = \sum_{n=1}^{\infty} a_n z^n \in TR, \quad \mid z \mid< 1,$$

where $\mu(t)$ is a probability measure on $[-1, 1]$, and

$$(2) \quad a_n = \int_{-1}^{1} \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} d\mu(t), n = 1, 2, \dots, a_1 = 1, 0 \leq \arccos t \leq \pi.$$

Received by the editor December 7, 2000.

AMS 2000 Subject Classification: Primary 30C45, 30C50, 30C70, 30C75. Secondary 12D10, 26C10, 26C15, 30B10, 30C10, 30C15.

Key words and phrases: Variational methods and results for the typically real functions, applications to the maxima and minima of the coefficients of the inverse functions of these functions, two conjectures.

Goluzin [3, p. 202, Theorem 1] rediscovered the representations (1)-(2) but Goluzin [3, p. 212, Corollary 5] was the first to prove that in the inside of the curve

$$C = \{z : [(|\, z+i\,|=\sqrt{2}) \cap (\mathrm{Im}z \geq 0)] \cup [(|\, z-i\,|=\sqrt{2}) \cap (\mathrm{Im}z \leq 0)]\} \tag{3}$$

all functions (1) are univalent. With the help of other methods Brannan and Kirwan [4, pp. 154-155, Theorem 3] and Goodman [5, pp. 100-101, Theorem 3] proved that the Goluzin domain of univalence is maximal for the class TR. Brannan and Kirwan [4, p. 154, Theorem 2] also proved that the image of the disk $|\, z \,|< 1$ under every function $f(z) \in TR$ in the w-plane covers the disk $|\, w \,|< 1/4$. Goodman [6, p. 236, Theorem 2] found the Koebe domain for the class TR, i.e. the largest possible domain that is covered under the image of the disk $|\, z \,|< 1$ by every function $f(z) \in TR$ in the w-plane, where the Brannan-Kirwan disk $|\, w \,|< 1/4$ is maximal with a centre $w = 0$ in the Goodman Koebe domain. Since any function $w = f(z)$ in TR is univalent in the inside of the curve C determined by (3) it follows that its inverse function $z = g(w)$ exists in a certain neighborhood of the origin $w = 0$ where it has the Taylor series

$$z = g(w) = \sum_{n=1}^{\infty} b_n w^n, \qquad b_1 = 1, \tag{4}$$

where the coefficients b_n are determined by the coefficients a_n in (2) with the help of Theorem 3 below.

Let S be the well known class of analytic and univalent functions

$$w = F(z) = \sum_{n=1}^{\infty} A_n z^n, \qquad |\, z \,|< 1, \qquad A_1 = 1. \tag{5}$$

The Koebe domain for the class S is the disk $|\, w \,|< 1/4$ (see, for example, Goodman [7, Vol. I, pp. 62-63, Theorem 1]). Let

$$z = G(w) = \sum_{n=1}^{\infty} B_n w^n, \qquad |\, w \,| < \frac{1}{4}, \qquad B_1 = 1, \tag{6}$$

denote the inverse function of any functioin (5) for $|\, z \,| < 1$ where the coefficients B_n are determined by the coefficients A_n with the help of Theorem 3 below. According to the Loewner result (see, for example, Goodman [7, Vol. II, p. 205, Theorem 24]) the coefficients B_n in (6) satisfy the sharp inequalities

$$|\, B_n \,| \leq \frac{1}{n+1} \binom{2n}{n}, \qquad n = 2, 3, \ldots, \tag{7}$$

where the equalities for all $n = 2, 3, \ldots$ are attained only for the inverse functions of the rotations of the Koebe function

$$K(z) = \frac{z}{(1-z)^2} \in S. \tag{8}$$

Since the Koebe function (8) belongs to the class TR as well, so the Loewner inequalities (7) also hold for the subclass STR of univalent typically real functions (1) in $|\, z \,| < 1$.

In this paper we will derive variational methods which yield more precise information in comparison with the Kirwan result [8, pp. 942-943, Theorem 1.3] for the extremal functions of a given bounded real-valued continuous functional in the class TR. As an application of these methods we will find the minima and maxima of the coefficients b_2, b_3, b_4 and state two conjectures for the extrema of all coefficients $b_n, n = 2, 3, 4, \ldots$, in the full class TR.

2. Variational formulas for the class TR. The following variational methods and results represented by Theorems 1 and 2 are new.

Theorem 1. *Let* ε *with* $-1 < \varepsilon < 1$, $\varepsilon \neq 0$, *be an arbitrary number and let the function* $f(z)$ *belong to the class* TR. *Then the varied function*

$$f_*(z) = \int_{-1}^{1} \frac{z d\mu(t)}{1 - 2\frac{t-\varepsilon}{1-t\varepsilon} z + z^2}, \qquad |\, z \,| < 1, \tag{9}$$

also belongs to the class TR *and it has the following asymptotic representation*

$$f_*(z) = f(z) - 2\varepsilon z^2 \int_{-1}^{1} \frac{1-t^2}{(1-2tz+z^2)^2} d\mu(t) + O(\varepsilon^2), \quad |\, z \,| < 1, \tag{10}$$

where $O(\varepsilon^2)$ *denotes a magnitude, the ratio of which to* ε^2 *is uniformly bounded for* z *lying in an arbitrary closed set of the disk* $|\, z \,| < 1$.

Proof. The linear fractional function

$$\tau = \frac{t-\varepsilon}{1-t\varepsilon}, \quad -1 \le t \le 1, \quad -1 < \varepsilon < 1, \quad \varepsilon \neq 0, \tag{11}$$

for fixed ε, increases with t from -1 to 1. This property of (11) permits us to substitute $(t-\varepsilon)/(1-t\varepsilon)$ for t in (1) to obtain (9). The function (9) belongs to the class TR with the probability measure

$$\nu(\tau) := \mu\Big(\frac{\tau+\varepsilon}{1+\tau\varepsilon}\Big), \quad -1 \le \tau \le 1.$$

The difference between (9) and (1) is

$$\begin{aligned} f_*(z) - f(z) &= -2\varepsilon z^2 \int_{-1}^{1} \frac{1-t^2}{(1-2tz+z^2)^2} \cdot \frac{d\mu(t)}{1 - \varepsilon \frac{t-2z+tz^2}{1-2tz+z^2}} \\ &= -2\varepsilon z^2 \int_{-1}^{1} \frac{1-t^2}{(1-2tz+z^2)^2} \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \Big(\frac{t-2z+tz^2}{1-2tz+z^2}\Big)^{\nu} d\mu(t) \end{aligned} \tag{12}$$

for $|\, z \,| < 1$ and sufficiently small $|\, \varepsilon \,|$. Thus from (12) we obtain (10), which completes the proof of Theorem 1.

Theorem 2. *For a given point z of the disk $|z|<1$ and a given analytic function $\phi(u_0, u_1, \ldots, u_n; z)$, $n\geq 0$, on the set $\bigcup_{TR}\{f(z), f'(z), \ldots, f^{(n)}(z); z\}$, the minimum (the maximum) of the functional*

$$\text{Re}\,\phi(f(z), f'(z), \ldots, f^{(n)}(z); z) \tag{13}$$

in the class TR is attained only either in the subclass $(TR)_1 \subset TR$ of functions

$$f(z) = \frac{cz}{(1+z)^2} + \frac{(1-c)z}{(1-z)^2} \in (TR)_1, \qquad 0 \leq c \leq 1, \tag{14}$$

or in the subclass $(TR)_2 \subset TR$ of functions

$$f(z) \;=\; \sum_{k=1}^{p} \frac{c_k z}{1 - 2t_k z + z^2} \in (TR)_2, \tag{15}$$

where

$$1\leq p\leq n+2, \qquad -1\leq t_1\leq t_2\leq\cdots\leq t_p\leq 1, \quad 0\leq c_k\leq 1, \quad \sum_{k=1}^{p} c_k = 1, \tag{16}$$

where $t_1, t_2, \ldots, t_p$ are among the numbers -1 and 1 and the roots in the interval $-1 \leq t \leq 1$ of the equation

$$P(t) \equiv \text{Re}\left[\sum_{s=0}^{n} \frac{\partial\phi[f(z)]}{\partial u_s} \frac{\partial^s}{\partial z^s}\Big(\frac{z}{1-2tz+z^2}\Big)^2\right] = 0 \tag{17}$$

in t for which we assume that at the extremum of (13) *it is not an identity in t in the interval $-1 \leq t \leq 1$ where $\phi[f(z)] \equiv \phi(f(z), f'(z), \ldots, f^{(n)}(z); z)$.*

Proof. Let z with $|\, z \,|< 1$ be fixed. The extremal functions $f(z)\in TR$ exist since the functional (13) is continuous and bounded on TR and the class TR is normal and compact in $|\, z \,|< 1$.

(i) If we set

$$u_s = f^{(s)}(z), \quad u_s^* = f_*^{(s)}(z) \quad (0 \leq s \leq n), \tag{18}$$

then the increments by the asymptotic formula (10) are

$$
\begin{aligned}
(19) \qquad du_s &= u_s^* - u_s \\
&= -2\varepsilon \int_{-1}^{1} (1-t^2) \frac{\partial^s}{\partial z^s} \Big(\frac{z}{1-2tz+z^2} \Big)^2 d\mu(t) + O(\varepsilon^2)
\end{aligned}
$$

for $0 \le s \le n$. Further, we introduce the abridged notatioins

$$
(20) \qquad \phi \equiv \phi(u_0, u_1, \dots, u_n; z), \quad \phi^* \equiv \phi(u_0^*, u_1^*, \dots, u_n^*; z),
$$

where u_s and $u_s^*(0 \le s \le n)$ are given by (18). Then for sufficiently small $|\,\varepsilon\,|$, we have the Taylor series

$$
(21) \qquad \phi^* = \phi + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left(\sum_{s=0}^{n} \frac{\partial}{\partial u_s} du_s \right)^{\nu} \phi
$$

for the functions (20). From (21) and (19) we obtain

$$
(22) \quad \phi^* = \phi - 2\varepsilon \sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s} \int_{-1}^{1} (1-t^2) \frac{\partial^s}{\partial z^s} \Big(\frac{z}{1-2tz+z^2} \Big)^2 d\mu(t) + O(\varepsilon^2).
$$

It follows from (22) that

$$
(23) \quad \operatorname{Re} \phi^* = \operatorname{Re} \phi - 2\varepsilon \int_{-1}^{1} (1-t^2) \operatorname{Re} \left[\sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s} \frac{\partial^s}{\partial z^s} \Big(\frac{z}{1-2tz+z^2} \Big)^2 \right] d\mu(t) + O(\varepsilon^2).
$$

The extremality of the function $f(z)$ in the class TR at the fixed point z and the arbitrariness of ε imply that the coefficient of ε in (23) vanishes, i.e. that

$$
(24) \qquad \int_{-1}^{1} (1-t^2) \operatorname{Re} \left[\sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s} \frac{\partial^s}{\partial z^s} \Big(\frac{z}{1-2tz+z^2} \Big)^2 \right] d\mu(t) = 0.
$$

If the equation (17) does not vanish identically, i.e. if the conditions for the equation (17) hold, then the equation of the extremality (24) is fulfilled if and only if the measure $\mu(t)$ is a step functioin with points of discontinuity at -1 and 1 and the roots of the equation (17) in t in the closed interval

$[-1,1]$ and the corresponding jumps with a sum equal to unit. In fact, this is evident if $\mu(t)$ is a corresponding step function. Conversely, it follows from the Goluzin variational formula applied to the class TR (see, for example, in our paper [9, p. 93, formula (19)]) that $\mu(t)$ is a constant between any two adjacent roots of the equation (17) for the extremal function $f(z)$ (see the comments for our formulas (27)-(28) in [9, pp. 94-95]). Hence, the extremal functions $f(z)$ belong to the subclasses $(TR)_1 \subset TR$ and $(TR)_2 \subset TR$ of functions (14) and (15)-(16), respectively, where the upper bound of the natural number p is determined as follows:

(ii) The s-th derivative in (17) is equal to

$$\frac{\partial^s}{\partial z^s}\Big(\frac{z}{1-2tz+z^2}\Big)^2 = z^2 D^s(1-2tz+z^2)^{-2} + 2szD^{s-1}(1-2tz+z^2)^{-2} + s(s-1)D^{s-2}(1-2tz+z^2)^{-2} \tag{25}$$

for $0 \le s \le n$, where $D = d/dz$ is the symbol for the differentiation and $D^0 = 1$ and $D^m = 0$ for $m = -1, -2$ by convention. Further, we use our method in [10, pp. 226-227]. We set

$$\varphi(u) = u^{-2}, \quad u = 1-2tz+z^2; \quad \varphi(v) = v^{-2}, \quad v = 1-2t\xi+\xi^2, \tag{26}$$

where

$$v - u = (\xi - z)(\xi + z - 2t). \tag{27}$$

Then from (26)-(27) we obtain

$$\begin{aligned} &\varphi(v) \\ = \;& \varphi(u) + \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(u)}{k!}(v-u)^k \\ = \;& \varphi(u) + \sum_{k=1}^{\infty} \frac{(-1)^k(k+1)}{(1-2tz+z^2)^{k+2}}(\xi - z)^k(\xi + z - 2t)^k \\ = \;& \varphi(u) + \sum_{k=1}^{\infty} \frac{(-1)^k(k+1)}{(1-2tz+z^2)^{k+2}}(\xi-z)^k \sum_{s=0}^{k} \binom{k}{s}(\xi-z)^s(2z-2t)^{k-s} \end{aligned} \tag{28}$$

$$= \varphi(u) + \sum_{s=1}^{\infty}(\xi - z)^s$$
$$\cdot \sum_{k=0}^{s-1}(-1)^{s-k}(s-k+1)\binom{s-k}{k}\frac{(2z-2t)^{s-2k}}{(1-2tz+z^2)^{s-k+2}}.$$

It follows from (28) that

$$\text{(29)}\quad D^s_{\xi=z}\varphi(v) = D^s(1-2tz+z^2)^{-2} = s!\sum_{k=0}^{[\frac{s}{2}]}(-1)^{s-k}(s-k+1)\binom{s-k}{k}\frac{(2z-2t)^{s-2k}}{(1-2tz+z^2)^{s-k+2}}$$

for $s = 0, 1, 2, \ldots$, where $[\frac{s}{2}]$ denotes the greatest integer less than or equal to $s/2$. Now with the help of the formula (29) we obtain the s-th derivative in (25), where the empty sums for D^{-1} and D^{-2} are zeros by convention. It follows from (29) and (25) that the equation (17) is algebraic of degree no more than $2n+2$ in t.

(iii) Let the real number ε be with a sufficiently small $\mid \varepsilon \mid$. If the extremal function $f(z) \in (TR)_2$ and in (15)-(16) we substitute $c_k + \varepsilon$ and $c_{k+1} - \varepsilon$ for c_k and c_{k+1}, respectively, then the varied function

$$\text{(30)}\qquad f_{**}(z) = f(z) + \varepsilon\Big(\frac{z}{1-2t_kz+z^2} - \frac{z}{1-2t_{k+1}z+z^2}\Big)$$

also belongs to the subclass $(TR)_2$. If we analogously set

$$\text{(31)}\qquad u_s = f^{(s)}(z), \quad u_s^{**} = f^s_{**}(z) \quad (0 \le s \le n),$$

then the increments by formula (30) are

$$\text{(32)}\quad du_s = u_s^{**} - u_s = \varepsilon\Big(\frac{\partial^s}{\partial z^s}\frac{z}{1-2t_kz+z^2} - \frac{\partial^s}{\partial z^s}\frac{z}{1-2t_{k+1}z+z^2}\Big) \quad (0 \le s \le n).$$

For brevity, we again denote

$$\phi \equiv \phi(u_0, u_1, \ldots, u_n; z), \quad \phi^{**} \equiv \phi(u_0^{**}, u_1^{**}, \ldots, u_n^{**}; z), \tag{33}$$

where u_s and $u_s^{**}(0 \leq s \leq n)$ are given by (31). Then, the corresponding Taylor series (21) for the functions (33) and (32) analogously yield

$$\operatorname{Re}\phi^{**} = \operatorname{Re}\phi + \varepsilon \operatorname{Re}\sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s}\Big(\frac{\partial^s}{\partial z^s}\frac{z}{1-2t_k z+z^2} - \frac{\partial^s}{\partial z^s}\frac{z}{1-2t_{k+1}z+z^2}\Big) + O(\varepsilon^2). \tag{34}$$

In addition, it follows from the conditions for the equation (17) that we have the inequality

$$\frac{\partial \phi[f(z)]}{\partial u_s} \equiv \frac{\partial}{\partial u_s}\phi(f(z), f'(z), \ldots, f^{(n)}(z); z) \neq 0 \tag{35}$$

at least for one $s \in \{0, 1, \ldots, n\}$. Then, the extremality of $f(z) \in (TR)_2$ at the fixed point z and the arbitrariness of ε in (34), having in mind the inequality (35), imply the condition

$$\operatorname{Re}\sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s}\Big(\frac{\partial^s}{\partial z^s}\frac{z}{1-2t_k z+z^2} - \frac{\partial^s}{\partial z^s}\frac{z}{1-2t_{k+1}z+z^2}\Big) = 0. \tag{36}$$

The condition (36) shows that the function

$$Q(t) \equiv \operatorname{Re}\sum_{s=0}^{n} \frac{\partial \phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\frac{z}{1-2tz+z^2}, \quad -1 \leq t \leq 1, \tag{37}$$

has equal values at any two adjacent points of discontinuity t_k and t_{k+1} of the measure $\mu(t)$ for the subclass $(TR)_2$, i.e. $Q(t)$ has equal values at all the points of discontinuity of the measure $\mu(t)$ for the subclass $(TR)_2$. Hence, the derivative $Q'(t)$ vanishes at least at one point inside the intervals between any two adjacent points of discontinuity of $\mu(t)$ in $-1 \leq t \leq 1$. But from

(37) and (17) we conclude that

$$Q'(t) = 2\,\mathrm{Re}\sum_{s=0}^{n} \frac{\partial\phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\Big(\frac{z}{1-2tz+z^2}\Big)^2 = 2P(t). \tag{38}$$

The equation (17) has no more than $2n+2$ roots in t. Taking into account the endpoints -1 and 1, the step measure $\mu(t)$ has no more than $2n+4$ points of discontinuity in the interval $-1 \le t \le 1$. It follows from (38) that if the points of discontinuity of $\mu(t)$ in $-1 \le t \le 1$ are more than $n+2$, then the equation (17) will have more than $2n+2$ roots in $-1 \le t \le 1$, which is impossible. Hence, the number p satisfies the inequalities in (16). If the extremal function $f(z) \in (TR)_1$ from (14), the corresponding assertions are established in the same way.

This completes the proof of Theorem 2.

3. Application to the coefficient problem of the inverse functions in the class TR. For this aim we need the following

Theorem 3. *In terms of the coefficients a_n in (2), the coefficients b_n in (4) have the following simplest explicit form*

$$b_n = \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\binom{n+k-1}{k}D_{n-1,k}(a_2,a_3,\dots,a_{n-k+1}), \quad n\ge 2, \tag{39}$$

where

$$D_{n-1,k}(a_2,a_3,\dots,a_{n-k+1}) \equiv \sum \frac{k!(a_2)^{\nu_1}(a_3)^{\nu_2}\dots(a_{n-k+1})^{\nu_{n-k}}}{\nu_1!\nu_2!\dots\nu_{n-k}!} \tag{40}$$

for $1\le k\le n-1$, $n\ge 2$, are the ordinary Bell polynomials in $a_2, a_3, \dots, a_{n-k+1}$, where the sum is taken over all nonnegative integers $\nu_1, \nu_2, \dots, \nu_{n-k}$ satisfying

$$\begin{aligned} &\nu_1+\nu_2+\cdots+\nu_{n-k} = k, \\ &\nu_1+2\nu_2+\cdots+(n-k)\nu_{n-k} = n-1, \quad 1\le k\le n-1, \quad n\ge 2. \end{aligned} \tag{41}$$

Proof. This is done by means of the method in [9, pp. 91-93, Theorem 1], which is applicable to each analytic function $F(z)$ in $|z|<1$ normalized by the requirements $F(0)=F'(0)-1=0$ (see in [9] a recurrence relation for the polynomials (40) and tables for the polynomials (40) and the coefficients (39)).

Theorem 4. *The minimum (the maximum) of the coefficients* b_n, $n\geq 2$, *from* (39) *in the class* TR *is attained only either in the subclass* $(TR)_1 \subset TR$ *of functions* (14) *or in the subclass* $(TR)_2 \subset TR$ *of functions* (15) – (16) *with:*

(i) $1\leq p\leq m$ if $n=2m$, $m=1,2,\ldots,$

(ii) $1\leq p\leq m+1$ if $n=2m+1$, $m=1,2,\ldots,$

where in (16) *the points* $t_1,t_2,\ldots,t_p$ *are among the numbers* -1 *and* 1 *and the roots in the interval* $-1\leq t\leq 1$ *of the algebraic equation*

$$P(t)\equiv \frac{1}{2}\sum_{s=2}^{n}\frac{\partial b_n}{\partial a_s}U'_{s-1}(t)=0, \quad n\geq 2, \tag{42}$$

of degree $n-2$ *in* t *(for* $n=2$ *this equation is impossible – see below Corollary* 4.1*), and the function*

$$Q(t)\equiv \sum_{s=2}^{n}\frac{\partial b_n}{\partial a_s}U_{s-1}(t), \quad Q'(t)=2P(t), \quad n\geq 2, \tag{43}$$

has equal values at any two adjacent points of the sequence $t_1,t_2,\ldots,t_p$, *where* $U_{s-1}(t)$ *are the Chebyshev polynomials of the second kind*

$$U_{s-1}(t)=\sum_{k=0}^{[\frac{s-1}{2}]}(-1)^k\binom{s-1-k}{k}(2t)^{s-1-2k}, \quad 2\leq s\leq n, \quad n\geq 2, \tag{44}$$

where $[\frac{s-1}{2}]$ *denotes the greatest integer less than or equal to* $\frac{s-1}{2}$.

Proof. We apply Theorem 2 for $z = 0$ and the function

$$(45)\qquad \begin{aligned} b_n &\equiv \phi(u_0, u_1, \ldots, u_n; 0)\\ &\equiv \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\binom{n+k-1}{k}D_{n-1,k}\Big(\frac{u_2}{2!}, \frac{u_3}{3!}, \ldots, \frac{u_{n-k+1}}{(n-k+1)!}\Big)\end{aligned}$$

on the set $\underset{TR}{\cup}\{f(0), f'(0), \ldots, f^{(n)}(0); 0\}$ where $n \geq 2$, having in mind (39)-(41). For the function (45), the equation (17) and the function (37) for the condition (36) are reduced to (42) and (43)-(44), respectively, where

$$(46)\qquad \frac{\partial b_n}{\partial a_s} = s!\frac{\partial \phi}{\partial u_s}, \quad a_s = \frac{f^{(s)}(0)}{s!} \equiv \frac{u_s}{s!}, \quad 2 \leq s \leq n.$$

In fact, for $z = 0$, the s-th derivative in (17) is determined immediately by the equations (25) and (29). According to the method in section (ii) of the proof of Theorem 2, the s-th derivative in (37) is determined by the equations

$$(47)\qquad \frac{\partial^s}{\partial z^s}\frac{z}{1-2tz+z^2} = zD^s(1-2tz+z^2)^{-1} + sD^{s-1}(1-2tz+z^2)^{-1}$$

and (in(26)-(28) we choose $\varphi(u) = u^{-1}$ and $\varphi(v) = v^{-1}$; see also [10, pp. 229-230, formula (13)]

$$(48)\quad D^s(1-2tz+z^2)^{-1} = s!\sum_{k=0}^{[\frac{s}{2}]}(-1)^{s-k}\binom{s-k}{k}\frac{(2z-2t)^{s-2k}}{(1-2tz+z^2)^{s-k+1}}$$

for $s = 0, 1, 2, \ldots$, where $D = d/dz$, $D^0 = 1$, $D^{-1} = 0$ and the empty sum for D^{-1} is zero by convention. Now for $z = 0$, the equations (37) and (46)-(48) yield (43)-(44). It is clear from (46) and (39)-(41) that for the function (45) the equation (42) is not an identity in t in the interval $-1 \leq t \leq 1$, since, for example, $\partial b_n/\partial a_n = -1 \neq 0$, $n \geq 2$, and it is the coefficient of $(n-1)(2t)^{n-2}$ according to (44). Hence, the equation (42) is algebraic of degree $n-2$. Further:

(i) For $n = 2m$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (1) for the extremum of (45) has no more than $2m$ points of discontinuity among the roots of the equation (42) and the points -1 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than 1, the function (43) has equal values. If $\mu(t)$ has more than $m(m > 1)$ points of discontinuity in $-1 \leq t \leq 1$, then the equation (42) will have more than $2m - 2$ roots in $-1 \leq t \leq 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $-1 \leq t \leq 1$ are not more than $m(m \geq 1)$. Therefore, the interval of the integer p in (16) is contracted to $1 \leq p \leq m$.

(ii) For $n = 2m + 1$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (1) for the extremum of (45) has no more than $2m + 1$ points of discontinuity among the roots of the equation (42) and the points -1 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than 1, the function (43) has equal values. If $\mu(t)$ has more than $m + 1$ points of discontinuity in $-1 \leq t \leq 1$, then the equation (42) will have more than $2m - 1$ roots in $-1 \leq t \leq 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $-1 \leq t \leq 1$ are not more than $m + 1$. Therefore, the interval of the integer p in (16) is contracted to $1 \leq p \leq m + 1$.

This completes the proof of Theorem 4.

Corollary 4.1. *The coefficient* b_2 *from* (39) *satisfies the sharp inequalities*

$$-2 \leq b_2 \leq 2, \tag{49}$$

where the equalities hold only for the following extremal functions: on the left-hand side of (49)*, for the function*

$$g(w) = \frac{1 + 2w - \sqrt{1 + 4w}}{2w} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \binom{2n}{n} w^n, \tag{50}$$

inverse of the function

$$f(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty} nz^n \in (TR)_1, \tag{51}$$

and on the right-hand side of (49), *for the function*

$$g(w)=\frac{1-2w-\sqrt{1-4w}}{2w}=\sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}w^n, \tag{52}$$

inverse of the function

$$f(z)=\frac{z}{(1+z)^2}=\sum_{n=1}^{\infty}(-1)^{n-1}nz^n \in (TR)_1, \tag{53}$$

where the series in (50) *and* (52) *converge for* $\mid w \mid < \frac{1}{4}$.

Proof. For $n=2$, Theorem 4, (i) yields $p=1$. For $n=2$ from (39)-(41), (42) and (44) we obtain

$$b_2=-a_2, \quad \frac{\partial b_2}{\partial a_2}=-1, \tag{54}$$

and

$$P(t)=-1\neq 0, \tag{55}$$

respectively. It follows from (55) and (15)-(17) that the point of discontinuity of $\mu(t)$ can be either $t_1=-1$ or $t_1=1$ with the corresponding jumps $c_1=1$ and $c_1=1$. Therefore, we obtain the two extremal functions (51) and (53) of the form (14), the inverse of which (50) and (52) supply the equalities in (49), respectively.

Remark. The inequalities (49) and the extremal functions (51) (or (50)) and (53) (or (52)) follow from the first equation in (54) and (2) for $n=2$ as well.

Corollary 4.2. *The coefficient* b_3 *from* (39) *satisfies the sharp inequalities*

$$-3 \le b_3 \le 5, \tag{56}$$

where the equalities hold only for the following extremal functions: on the left-hand side of (56), *for the function*

$$\begin{aligned} g(w) &= \frac{1+\sqrt{1+16w^2}-\sqrt{2(1+\sqrt{1+16w^2})}}{4w} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left[2^{2n+1}\binom{2n}{n} - \frac{1}{2}\binom{4n+2}{2n+1}\right] w^{2n+1} \end{aligned} \tag{57}$$

(*the series converges for* $\mid w \mid < \frac{1}{4}$), *inverse of the function*

$$f(z) = \frac{z(1+z^2)}{(1-z^2)^2} = \sum_{n=0}^{\infty}(2n+1)z^{2n+1} \in (TR)_1, \tag{58}$$

and on the right-hand side of (56), *for the functions* (50) *and* (52), *inverse of the functions* (51) *and* (53), *respectively.*

Proof. For $n = 3$, Theorem 4, (ii) yields $p = 1, 2$. For $n = 3$ from (39)-(41), (42)-(44) and (46) we obtain

$$b_3 = -a_3 + 2a_2^2, \quad a_2 = \frac{f''(0)}{2}, \quad a_3 = \frac{f'''(0)}{6}, \tag{59}$$

$$\frac{1}{4}P(t) = a_2 - t = 0 \tag{60}$$

and

$$Q(t) = 8a_2t - 4t^2 + 1, \quad Q'(t) = 2P(t), \tag{61}$$

respectively. If $p = 1$, then (15)-(16) are reduced to

$$f(z) = \frac{z}{1-2tz+z^2} = \sum_{n=1}^{\infty} U_{n-1}(t)z^n \in (TR)_2, \quad \mid z \mid < 1, \tag{62}$$

where t can be either the root $t = a_2$ from (60) or any of the points -1 and 1. The series in (62) follows from (46)-(48) and (44). Therefore $a_2 = U_1(t) = 2t$ according to (44), and hence $t = 0$. If $t = 0$ in (62), it follows that the coefficients $a_2 = 0$ and $a_3 = -1$, and the first equation in (59) yields $b_3 = 1$. If $t = -1$ in (62), then we obtain the function (53) for which $a_2 = -2$ and $a_3 = 3$, and the first equation in (59) yields $b_3 = 5$. If $t = 1$ in (62), then we obtain the function (51) for which $a_2 = 2$ and $a_3 = 3$, and the first equation in (59) yields $b_3 = 5$. From the comparison of these three cases it follows that for the corresponding inverse functions (52) and (50) the bound 5 in (56) is attained.

If $p=2$, then (15) has two terms, corresponding to the condition $Q(t_1)= Q(t_2)$, where $Q(t)$ is determined by (61) and t_1 and t_2 are among the numbers -1, $t = a_2$ and 1. According to this condition and the Rolle theorem, the equation (60) has an odd number of roots between t_1 and t_2. This is possible only if $t_1 = -1$ and $t_2 = 1$. Hence, the extremal function $f(z)$ is

$$f(z) = \frac{cz}{(1+z)^2} + \frac{(1-c)z}{(1-z)^2} \in (TR)_2, \quad 0 < c < 1. \tag{63}$$

Further, the condition $Q(-1) = Q(1)$, where $Q(t)$ is given by (61), yields $a_2 = 0$. On the other hand from (63), (53) and (51) we obtain $a_2 = 2(1-2c)$, and hence $c = \frac{1}{2}$. For $c = \frac{1}{2}$ from (63), we obtain the extremal function (58) and its inverse function (57) for which the bound -3 in (56) is attained.

Corollary 4.3. *The coefficient* b_4 *from* (39) *satisfies the sharp inequalities*

$$-14 \leq b_4 \leq 14, \tag{64}$$

where the equalities hold only for the following extremal functions: on the left-hand side of (64), *for the function* (50), *inverse of the function* (51), *and* *on the right-hand side of* (64)*, for the function* (52)*, inverse of the function* (53)*.*

Proof. For $n = 4$, Theorem 4, (i) yields $p = 1, 2$. For $n = 4$ from (39)-(41), (42)-(44) and (46) we obtain

$$\begin{aligned} &(65) \quad & b_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ &(66) \quad & P(t) &= 5a_3 - 15a_2^2 + 20a_2t - 12t^2 + 2 = 0, \\ &(67) \quad & Q(t) &= (5a_3 - 15a_2^2)2t + 5a_2(4t^2 - 1) - 8t^3 + 4t, \quad Q'(t) = 2P(t), \end{aligned}$$

where $a_{2,3,4}$ are the coefficients of the extremal function $f(z) \in (TR)_2$, i.e.

$$a_2 = \frac{f''(0)}{2}, \quad a_3 = \frac{f'''(0)}{6}, \quad a_4 = \frac{f^{(4)}(0)}{24}, \tag{68}$$

respectively. If $p = 1$, then (15)-(16) are reduced to (62), where t can be either any root of (66) in $-1 \le t \le 1$ or any of the points -1 and 1. With the help of (62), (44) and (65) we obtain

$$b_4 = -8t^3 - 6t, \quad -1 \le t \le 1. \tag{69}$$

The maximum 14 and the minimum -14 of (69) are obtained for $t = -1$ and $t = 1$, respectively. For these values of t in (62) we obtain the functions (53) (or (52)) and (51) (or (50)) for which the corresponding boundaries in (64) are attained.

If $p = 2$, then (15), having in mind (16), can have the following forms

$$\begin{aligned} &(70) \quad f(z) = \frac{cz}{(1+z)^2} + \frac{(1-c)z}{1-2tz+z^2} \in (TR)_2, \quad 0 < c < 1, \quad -1 < t < 1, \\ &(71) \quad f(z) = \frac{cz}{1-2tz+z^2} + \frac{(1-c)z}{(1-z)^2} \in (TR)_2, \quad 0 < c < 1, \quad -1 < t < 1, \end{aligned}$$

and (63), where t (in general different for each function) is a real root of (66) and the other root of (66) has to lie in the open intervals $(-1, t)$, $(t, 1)$ and

$(-1,1)$ in accordance to (67) and the conditions

$$Q(-1)=Q(t),\quad Q(t)=Q(1),\quad Q(-1)=Q(1), \tag{72}$$

respectively.

(a) From the first equation of (72) and (67), we obtain the corresponding equation for (70), namely

$$5a_3-15a_2^2+10a_2(t-1)-2(2t^2-2t+1)=0. \tag{73}$$

It follows from (66) and (73) that

$$a_2=\frac{2(2t-1)}{5},\quad a_3=\frac{2(14t^2-14t+1)}{25}. \tag{74}$$

On the other hand from (70) with the help of (62) and (44), we get the coefficients (68) as follows

$$\begin{aligned} a_2 &= 2t-2c(t+1), \\ a_3 &= 4t^2-1-4c(t^2-1), \\ a_4 &= 8t^3-4t-4c(2t^3-t+1). \end{aligned} \tag{75}$$

The equations (74) and (75) yield the equations

$$c=\frac{3t+1}{5(t+1)},\quad 12t^2+68t-7=0,\quad t=\frac{-17+\sqrt{310}}{6}=0.101136\ldots \tag{76}$$

(the other root of the second equation is not in the open interval $(-1,1)$). From (76), (74) and (75), we obtain

$$\begin{aligned} &c=\frac{145-4\sqrt{310}}{315},\quad a_2=-\frac{2(20-\sqrt{310})}{15},\\ &a_3=-\frac{56\sqrt{310}-985}{45},\quad a_4=-\frac{2(919\sqrt{310}-16265)}{135}. \end{aligned} \tag{77}$$

By the values of $a_{2,3}$ from (77), the equation (66) becomes

$$108t^2 + 24(20 - \sqrt{310})t - (40\sqrt{310} - 701) = 0. \tag{78}$$

But the equation (78) is not fulfilled for the value of t in (76), i.e. the extremal function $f(z)$ is not of the form (70).

(b) If in the function (70) we replace z, t and c by $-z, -t$ and $1 - c$, respectively, then we will obtain the function (71), multiplied by -1. Hence the function (71) is not extremal as well.

(c) From the third equation of (72) and (67), we obtain the corresponding equation for (63), namely

$$5a_3 - 15a_2^2 - 2 = 0. \tag{79}$$

The coefficients $a_{2,3,4}$ of (63) can be obtained from (75) for $t = 1$, i.e.

$$a_2 = 2 - 4c, \quad a_3 = 3, \quad a_4 = 4 - 8c. \tag{80}$$

From (79)-(80), we get the values

$$c_{1,2} = \frac{30 \pm \sqrt{195}}{60}. \tag{81}$$

The equations (80) and (81) yield

$$a_2 = \mp\frac{\sqrt{195}}{15}, \quad a_3 = 3, \quad a_4 = \mp\frac{2\sqrt{195}}{15}, \tag{82}$$

respectively. By the values of $a_{2,3}$ from (82), the equation (66) becomes

$$9t^2 \pm \sqrt{195}t - 3 = 0. \tag{83}$$

Really, each equation of (83) has one root in $-1 \leq t \leq 1$, respectively.

Finally, from (82) and (65), we find

$$b_4 = \mp \frac{182\sqrt{195}}{225} = \mp 11.295518\ldots, \tag{84}$$

respectively. Now the comparison of (69) and (84) leads us to (64), which completes the proof of Corollary 4.3.

For the following coefficients $b_5, b_6, \ldots$, we can proceed in the same way.

Conjecture 1. *In the full class TR of typically real functions* (1) *each coefficient $b_n, n = 2, 3, \ldots,$ from* (39) *attains its minimum (maximum) only for the rational functions of the form* (14).

Conjecture 2. *In the full class TR of typically real functions* (1) *the maxima $M_n, n = 2, 3, \ldots,$ of the corresponding coefficients b_n, $n = 2, 3, \ldots,$ from* (39) *are*

$$M_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 2, 3, \ldots, \tag{85}$$

which are attained only for the functions (14) *with $c = 1$ for even $n = 2, 4, \ldots,$ and with $c = 1$ and $c = 0$ for odd $n = 3, 5, \ldots,$ respectively* (*the functions* (50)-(53)*, respectively*).

In the subclass STR of typicall real functions (1) univalent in $|z| < 1$ Conjecture 2 is true according to the Loewner result (7)-(8).

In the full class TR of typically real functions (1) Conjectures 1 and 2 for $n = 2, 3, 4$ are proved in Corollaries 4.1, 4.2, 4.3 above.

4. Properties of the coefficients of the inverse function of the function (62). The function (62) for $t = \mp 1$ belongs to the special subclass $(TR)_1$ of functions (14) and for $-1 \leq t \leq 1$ belongs to the general subclass $(TR)_2$ of functions (15)-(16). For $-1 \leq t \leq 1$, the inverse function of the function (62) is

$$(86)\qquad g(w) = \frac{2tw+1-\sqrt{4(t^2-1)w^2+4tw+1}}{2w} \equiv \frac{2w}{2tw+1+\sqrt{4(t^2-1)w^2+4tw+1}}, \quad \sqrt{1}=1.$$

The first expression in (86) leads us to an inconvenient form of its Taylor series. Therefore we will use the expression (62) as follows:

Theorem 5. *The Taylor series of the function* (86) *is*

$$(87)\qquad g(w) = \sum_{n=1}^{\infty} \frac{1}{n} C_{n-1}^{-n}(t) w^n \equiv \sum_{n=1}^{\infty} b_n(t) w^n, \quad |\, w \,| < \frac{1}{4},$$

where the $C_{n-1}^{-n}(t)$ *are the Gegenbauer polynomials*

$$(88)\qquad C_{n-1}^{-n}(t) = (-1)^{n-1} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1-k}{k} \binom{n}{k+1} (2t)^{n-1-2k},$$

where $[\frac{n-1}{2}]$ *denotes the greatest integer less than or equal to* $\frac{n-1}{2}$.

Proof. According to the Lagrange formula for the n-th derivative of inverse functions (see, for example, Carathédory [11, p.231, formula (228.4)]), applied to the function (62), we have

$$(89)\qquad b_n(t) = \frac{1}{n!} D_{z=0}^{n-1} \Big(\frac{z}{f(z)}\Big)^n = \frac{1}{n!} D_{z=0}^{n-1} (1-2tz+z^2)^n, \qquad n \geq 1, \quad -1 \leq t \leq 1.$$

In addition, we have the Taylor series

$$(90)\qquad (1-2tz+z^2)^n = \sum_{\nu=0}^{2n} C_\nu^{-n}(t) z^\nu, \quad n \geq 1,$$

which generates the Gegenbauer polynomials

$$(91)\quad C_\nu^{-n}(t)=(-1)^\nu\sum_{k=0}^{[\frac{\nu}{2}]}\binom{\nu-k}{k}\binom{n}{\nu-k}(2t)^{\nu-2k},\quad 0\le\nu\le 2n,$$

where $[\nu/2]$ denotes the greatest integer less than or equal to $\nu/2$ (the formula (91) can be obtained if we choose $\varphi(u)=u^n$ and $\varphi(v)=v^n$ in (26)-(28); see details in [10, pp. 227-228, formula (7)] where the notation $C_\nu^{(n)}(t)$ is used instead of the notation $C_\nu^{-n}(t)$). Now it follows from (89)-(91) that (87)-(88) hold, which completes the proof of Theorem 5.

Corollary 5.1. *For $-1\le t\le 1$, we have the sharp estimates*

$$(92)\quad -\frac{1}{2m+1}\binom{4m}{2m}\le b_{2m}(t)\le\frac{1}{2m+1}\binom{4m}{2m},\quad m=1,2,\dots,$$

where the equalities on the left-hand side and on the right-hand side hold only for $t=1$ and $t=-1$, respectively, and

$$(93)\qquad 0\le b_{2m+1}(t)\le\frac{1}{2m+2}\binom{4m+2}{2m+1},\quad m=1,2\dots,$$

where the equalities on the left-hand side and on the right-hand side hold only for $t=0$ and $t=\pm1$, respectively.

Proof. From (87) we have

$$(94)\qquad b_n(t)=\frac{1}{n}C_{n-1}^{-n}(t),\quad n\ge1,\quad -1\le t\le1,\quad b_1(t)=1.$$

It follows from (88) that $C_{n-1}^{-n}(t)$ decreases for $-1\le t\le 1$ if n is even, and it decreases for $-1\le t\le 0$ and increases for $0\le t\le 1$ if n is odd. From (90) we obtain

$$(95)\ C_{n-1}^{-n}(-1)=\binom{2n}{n-1},\quad C_{n-1}^{-n}(1)=(-1)^{n-1}\binom{2n}{n-1},\quad n\ge1.$$

Therefore from (94)-(95) the inequalities (92)-(93) follow which are in accordance to the conjecture (85).

Corollary 5.2. *The coefficients $b_n(t)$ in (87) satisfy the recurrence relations*

$$b_n'(t) = -2(n-1)b_{n-1}(t), \quad n \geq 2, \quad b_1(t) = 1, \tag{96}$$

and

$$b_{2m}(t) = -2(2m-1)\int_0^t b_{2m-1}(\tau)d\tau, \quad m \geq 1, \quad b_1(t) = 1, \tag{97}$$

and

$$b_{2m+1}(t) = -4m\int_0^t b_{2m}(\tau)d\tau + \frac{1}{m+1}\binom{2m}{m}, \quad m \geq 1. \tag{98}$$

Proof. If we differentiate (90) with respect to t, we will obtain

$$\frac{d}{dt}C_{n-1}^{-n}(t) = -2nC_{n-2}^{-n+1}(t), \quad n \geq 2. \tag{99}$$

Thus (99) and (94) lead us to (96). From (96) we obtain (97)-(98), having in mind (90) for $t = 0$, which are also convenient for calculation of b_n, $n = 2, 3, \ldots$.

For example, from (87)-(88) or (97)-(98) we obtain the coefficients

$$b_2(t) = -2t, \quad b_3(t) = 4t^2 + 1,$$
$$b_4(t) = -8t^3 - 6t, \quad b_5(t) = 16t^4 + 24t^2 + 2.$$

5. General properties of the coefficients of the inverse functions in the class TR. We need the following

Theorem 6. *If the function $f(z) \in TR$ with the probability measure $\mu(t)$ on $[-1, 1]$ in (1), then the function $\alpha(z) \equiv -f(-z) \in TR$ with the probability measure $\sigma(t) \equiv \mu(1) - \mu(-t)$ on $[-1, 1]$ in (1).*

Proof. This follows from (1) if we replace z and t by $-z$ and $-t$, respectively.

Theorem 7. *If the inverse function $z = g(w)$ of the function $w = f(z) \in TR$ has the Taylor expansion* (4)*, then the inverse function $z = \beta(w)$ of the function $w = \alpha(z) = -f(-z) \in TR$ has the Taylor expansion*

$$z = \beta(w) = \sum_{n=1}^{\infty} (-1)^{n-1} b_n w^n, \quad b_1 = 1. \tag{100}$$

First proof. This follows from (1), (4) and (39)-(41).

Second proof. It follows from the Lagrange formula in (89) that the derivatives

$$\begin{aligned} D^n_{w=0}\beta(w) &= D^{n-1}_{z=0}\Big(\frac{z}{\alpha(z)}\Big)^n = (-1)^{n-1} D^{n-1}_{z=0}\Big(\frac{z}{f(z)}\Big)^n \\ &= (-1)^{n-1} D^n_{w=0} g(w) \end{aligned} \tag{101}$$

for $n = 1, 2, \ldots$. Now (100) follows from (101) and (4).

Theorem 8. *Let*

$$m_n = \min_{TR} b_n, \quad M_n = \max_{TR} b_n, \quad n = 2, 3, \ldots, \tag{102}$$

where b_n, $n \geq 2$, are those in (4)*.*

Then

$$m_{2n} = -M_{2n}, \quad n = 1, 2, \ldots. \tag{103}$$

Proof. If $f(z) \in TR$, it follows from (102) and (4) that the even coefficients

$$b_{2n} \leq M_{2n}, \quad n = 1, 2, \ldots. \tag{104}$$

Since $\alpha(z)$ in Theorems 6 and 7 belongs to TR, it follows from (102) and (100) that the even coefficients

$$-b_{2n} \le M_{2n}, \quad n = 1, 2, \ldots. \tag{105}$$

Thus (104) and (105) yield

$$-M_{2n} \le b_{2n} \le M_{2n}, \quad n = 1, 2, \ldots. \tag{106}$$

Therefore it follows from (106) that (103) holds.

Corollary 8.1. *If Conjecture* 2 *for* (85) *is true, then the minima* m_{2n}, $n = 1, 2, \ldots$, *of the corresponding even coefficients* b_{2n}, $n = 1, 2, \ldots$, *from* (39) *in the full class* TR *of typically real functions* (1) *are*

$$m_{2n} = -\frac{1}{2n+1}\binom{4n}{2n}, \quad n = 1, 2, \ldots,$$

which are attained only for the functions (14) *with* $c = 0$ *(the functions* (50)-(51)*).*

References

1. W. Rogosinski, *Über positive harmonische Entwiklungen und typisch-reele Potenzreihen*, Math. Z., **35**(1932), 93-121.

2. M. S. Robertson, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc., **41**(1935), 565-572.

3. G. M. Goluzin, *On typically real functions* (*Russian*), Mat. Sbornik N. S., **27**(69) (1950), No. 2, 201-218.

4. D. A. Brannan and W. E. Kirwan, *A covering theorem for typically real functions*, Glasgow Math. J., **10**(1969), 153-155.

5. A. W. Goodman, *The critical points of a typically-real function*, Proc. Amer. Math. Soc., **38**(1)(1973), 95-102.

6. A. W. Goodman, *The domain covered by a typically-real function*, Proc. Amer. Math. Soc., **64**(2)(1977), 233-237.

7. A. W. Goodman, *Univalent Functions*, Vol. I and II, Mariner Publishing Company, Inc., Tampa, Florida, U.S.A., 1983.

8. W. E. Kirwan, *Extremal problems for the typically real functions*, Amer. J. Math., **88**(4)(1966), 942-954.

9. P. G. Todorov, *Sharp estimates for the coefficients of the Nevanlinna univalent functions of the classes* N_1 *and* N_2, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Band **68**(1998), 91-102.

10. P. G. Todorov, *The nth derivative of the function* $\varphi(1-2tx+t^2)$ *and its application to one unified obtaining of the classical Gegenbauer, Legendre, Chebyshev and Hermite polynomials* (Bulgarian, summaries in Russian and English), Travaux scientifiques - Mathématiques, Université de Plovdiv "Paissii Hilendarski", **19**(1981), fasc. 1, 225-231.

11. C. Carathéodory, *Theory of Functions of a Complex Variable*, Volume One, Second English Edition, Chelsea Publishing Company, New York, 1978.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bontchev Str., Block 8, 1113 Sofia, Bulgaria.

E-mail: pgtodorov@abv.bg