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LAWS OF LARGE NUMBERS FOR THE MAXIMAL ORDER STATISTICS FROM A TRIANGULAR ARRAY

BY

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Abstract. Let $\{X_{nj}, 1 \leq j \leq n, n \geq 1\}$ be independent and identically distributed random variables with common density $f(x) = px^{-p-1}I(x \geq 1)$, where $p > 0$. Let X_{nk} be the k^{th} largest order statistic from $\{X_{n1}, \ldots, X_{nn}\}$. We consider various laws of large numbers of the form

$$\frac{\sum_{n=k}^{N} a_n X_{nk}}{b_N} \to L$$

for some finite nonzero constant L. It turns out that Strong Laws exist when $pk \geq 1$, while not even a Weak Law exists when $pk < 1$.

1. Introduction. Let $\{X_{nj}, 1 \leq j \leq n, n \geq 1\}$ be independent and identically distributed random variables with density $f(x) = px^{-p-1}I(x \geq 1)$, where $p > 0$. Let X_{nk} be the k^{th} largest order statistic from each row of this triangular array. Therefore the density of X_{nk} is

$$f_{nk}(x) = \frac{p \cdot n!}{(n-k)!(k-1)!}(1 - x^{-p})^{n-k} x^{-pk-1} I(x \geq 1).$$

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In this paper we examine when Strong Laws and Weak Laws of the form

$$\frac{\sum_{n=k}^{N} a_n X_{nk}}{b_N} \to L \tag{1}$$

exist for constants a_n, b_N and L. Our goal is not only to find nonzero limits, L, but to show when these laws of large numbers do and do not exist.

2. Preliminaries. We need a couple of lemmas before we proceed. From page 29 of Riordan [5] we have

Lemma 1.

$$\sum_{j=0}^{n} \frac{\binom{n}{j}(-1)^j}{j+m} = \frac{n!}{m(m+1)\cdots(m+n)} = \frac{\Gamma(m)n!}{\Gamma(n+m+1)}$$

and from page 257 of Abramowitz and Stegun [1] we have

Lemma 2.

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b} \qquad \text{as } n \to \infty.$$

As usual we set $\lg x = \max\{1, \log x\}$ and we use C to denote a generic constant which need not be the same in each appearance. Also, we let $L(x)$ denote any slowly varying function. For more literature on slowly varying function see Seneta [6].

3. pk greater than one. It is important to note that even though pk exceeds one, we may have $p \leq 1$. Thus our original random variables may not possess a first moment. Hence we can use our second and third largest order statistics to achieve our limit results.

Theorem 1. *If $pk > 1$ and $\alpha > -1 - 1/p$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=k}^N n^\alpha L(n) X_{nk}}{N^{\alpha+1+1/p} L(N)} = \frac{\Gamma(k-1/p)}{(\alpha+1+1/p)(k-1)!} \qquad \textit{almost surely}$$

for all slowly varying functions $L(n)$.

Proof. In this case we set $a_n = n^\alpha L(n)$, $b_n = n^{\alpha+1+1/p} L(n)$ and $c_n = b_n/a_n = n^{1+1/p}$ and we partition the series $b_N^{-1} \sum_{n=k}^N a_n X_{nk}$ into the following three terms:

$$\begin{aligned} b_N^{-1} \sum_{n=k}^N a_n X_{nk} &= b_N^{-1} \sum_{n=k}^N a_n [X_{nk} I(X_{nk} \le c_n) - E X_{nk} I(X_{nk} \le c_n)] \\ &\quad + b_N^{-1} \sum_{n=k}^N a_n X_{nk} I(X_{nk} > c_n) \\ &\quad + b_N^{-1} \sum_{n=k}^N a_n E X_{nk} I(X_{nk} \le c_n). \end{aligned}$$

The second term vanishes almost surely since $pk > 1$ and

$$\begin{aligned} \sum_{n=k}^\infty P\{X_{nk} > c_n\} &= \sum_{n=k}^\infty \frac{p \cdot n!}{(n-k)!(k-1)!} \int_{c_n}^\infty (1-x^{-p})^{n-k} x^{-pk-1} dx \\ &< C \sum_{n=k}^\infty \frac{n!}{(n-k)!} \int_{c_n}^\infty x^{-pk-1} dx < C \sum_{n=k}^N \frac{n^k}{c_n^{pk}} \\ &= C \sum_{n=k}^\infty \left(\frac{n}{c_n^p}\right)^k = C \sum_{n=k}^\infty \left(\frac{n}{n^{1+p}}\right)^k = C \sum_{n=k}^\infty \frac{1}{n^{pk}} < \infty. \end{aligned}$$

As for the first term we use the usual Khintchine-Kolmogorov Convergence Theorem and Kronecker's lemma argument, see Chow and Teicher [3]. We examine three different possibilities. The first one is $1 < pk < 2$. In this case

$$\sum_{n=k}^\infty c_n^{-2} E X_{nk}^2 I(X_{nk} \le c_n)$$

$$
\begin{aligned}
&= \sum_{n=k}^{\infty} \frac{p \cdot n!}{c_n^2 (n-k)!(k-1)!} \int_1^{c_n} (1-x^{-p})^{n-k} x^{-pk+1} dx \\
&< C \sum_{n=k}^{\infty} \frac{n^k}{c_n^2} \int_1^{c_n} x^{-pk+1} dx < C \sum_{n=k}^{\infty} \frac{n^k}{c_n^2} c_n^{-pk+2} = C \sum_{n=k}^{\infty} \frac{n^k}{c_n^{pk}} < \infty
\end{aligned}
$$

as in the previous calculation. If $pk = 2$, then

$$
\begin{aligned}
\sum_{n=k}^{\infty} c_n^{-2} E X_{nk}^2 I(X_{nk} \le c_n) &< C \sum_{n=k}^{\infty} \frac{n^k}{c_n^2} \int_1^{c_n} x^{-1} dx = C \sum_{n=k}^{\infty} \frac{n^k}{c_n^2} \lg c_n \\
&< C \sum_{n=k}^{\infty} \frac{n^k \lg n}{n^{2+2/p}} = C \sum_{n=k}^{\infty} \frac{\lg n}{n^2} < \infty.
\end{aligned}
$$

Lastly, if $pk > 2$, then

$$
\begin{aligned}
&\sum_{n=k}^{\infty} c_n^{-2} E X_{nk}^2 (X_{nk} \le c_n) \\
&= \sum_{n=k}^{\infty} \frac{p \cdot n!}{c_n^2 (n-k)!(k-1)!} \int_1^{c_n} (1-x^{-p})^{n-k} x^{-pk+1} dx \\
&= \sum_{n=k}^{\infty} \frac{p \cdot n!}{c_n^2 (n-k)!(k-1)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{-j} \int_1^{c_n} x^{-p(j+k)+1} dx \\
&= \sum_{n=k}^{\infty} \frac{p \cdot n!}{c_n^2 (n-k)!(k-1)!} \sum_{j=0}^{n-k} \frac{\binom{n-k}{j} (-1)^j}{p(j+k)-2} \\
&\quad + \sum_{n=k}^{\infty} \frac{p \cdot n!}{c_n^2 (n-k)!(k-1)!} \sum_{j=0}^{n-k} \frac{\binom{n-k}{j} (-1)^{j+1} c_n^{-p(j+k)+2}}{p(j+k)-2}.
\end{aligned}
$$

Next we use Lemmas 1 and 2, with $m = k - 2/p$. The first term is

$$
\begin{aligned}
&\sum_{n=k}^{\infty} \frac{n!}{c_n^2 (n-k)!(k-1)!} \sum_{j=0}^{n-k} \frac{\binom{n-k}{j} (-1)^j}{j+m} \\
&= \sum_{n=k}^{\infty} \Big(\frac{n!}{c_n^2 (n-k)!(k-1)!} \Big) \Big(\frac{(n-k)!\Gamma(m)}{\Gamma(n-k+m+1)} \Big) \\
&< C \sum_{n=k}^{\infty} \frac{\Gamma(n+1)}{c_n^2 \Gamma(n-k+m+1)} < C \sum_{n=k}^{\infty} \frac{n^{-m+k}}{c_n^2}
\end{aligned}
$$

$$= \quad C\sum_{n=k}^{\infty}\frac{n^{2/p}}{n^{2+2/p}} = C\sum_{n=k}^{\infty}\frac{1}{n^2} < \infty.$$

As for the second term we have

$$\left|\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+2}}{p(j+k)-2}\right|$$

$$< \quad C\sum_{n=k}^{\infty}n^k\sum_{j=0}^{n-k}\binom{n-k}{j}c_n^{-p(j+k)} = C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}}\sum_{j=0}^{n-k}\binom{n-k}{j}[c_n^{-p}]^j$$

$$= \quad C\sum_{n=k}^{\infty}\Big(\frac{n^k}{c_n^{pk}}\Big)\cdot\Big(1+\frac{1}{c_n^p}\Big)^{n-k} < C\sum_{n=k}^{\infty}\frac{1}{n^{pk}} < \infty$$

since $pk > 2$.

Finally, we will examine the last term in our partition

$$\begin{aligned} EX_{nk}I(X_{nk}\le c_n) &= \frac{p\cdot n!}{(n-k)!(k-1)!}\int_1^{c_n}(1-x^{-p})^{n-k}x^{-pk}dx \\ &= \frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_1^{c_n}x^{-p(j+k)}dx \\ &= \frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{p(j+k)-1} \\ &\quad +\frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+1}}{p(j+k)-1}. \end{aligned}$$

Using both our lemmas, with $m = k - 1/p$, the first term equals

$$\begin{aligned} &\frac{n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{j+m} \\ =\ &\Big(\frac{n!}{(n-k)!(k-1)!}\Big)\Big(\frac{(n-k)!\Gamma(m)}{\Gamma(n-k+m+1)}\Big) \\ =\ &\frac{\Gamma(k-1/p)\Gamma(n+1)}{(k-1)!\Gamma(n+1-1/p)} \sim \frac{\Gamma(k-1/p)}{(k-1)!}\cdot n^{1/p}. \end{aligned}$$

The other term in our first moment is bounded above by

$$\begin{aligned}
&\left|\frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+1}}{p(j+k)-1}\right| \\
<\ & Cn^k\sum_{j=0}^{n-k}\binom{n-k}{j}c_n^{-p(j+k)+1} = Cn^k c_n^{-pk+1}\sum_{j=0}^{n-k}\binom{n-k}{j}c_n^{-pj} \\
=\ & Cn^{-pk+1+1/p}\Big(1+\frac{1}{c_n^p}\Big)^{n-k} < Cn^{-pk+1+1/p}\Big(1+\frac{1}{c_n^p}\Big)^{c_n^p} \\
<\ & Cn^{-pk+1+1/p} = o(n^{1/p})
\end{aligned}$$

since $pk > 1$, thereby establishing

$$EX_{nk}I(X_{nk}\le c_n) \sim \frac{\Gamma(k-1/p)}{(k-1)!}\cdot n^{1/p}.$$

This, together with $a_n = n^{\alpha}L(n)$ and $b_n = n^{\alpha+1+1/p}L(n)$ it follows, with probability one, that

$$\begin{aligned}
&\frac{\sum_{n=k}^{N} n^{\alpha}L(n)X_{nk}}{N^{\alpha+1+1/p}L(N)} \sim \frac{\sum_{n=k}^{N} a_n EX_{nk}I(X_{nk}\le c_n)}{b_N} \\
\sim\ & \Big(\frac{\Gamma(k-1/p)}{(k-1)!}\Big)\cdot\Big(\frac{\sum_{n=k}^{N} n^{\alpha+1/p}L(n)}{N^{\alpha+1+1/p}L(N)}\Big) \to \frac{\Gamma(k-1/p)}{(\alpha+1+1/p)(k-1)!}
\end{aligned}$$

by using Theorem 1 from page 281 of Feller [4].

Next we reexamine our last result when $\alpha = -1-1/p$. Once again we obtain a Strong Law.

Theorem 2. *If $pk > 1$ and $a > -1$, then*

$$\lim_{N\to\infty}\frac{\sum_{n=k}^{N} n^{-1-1/p}(\lg n)^a X_{nk}}{(\lg N)^{a+1}} = \frac{\Gamma(k-1/p)}{(a+1)(k-1)!} \qquad \textit{almost surely.}$$

Proof. Here we set $a_n = n^{-1-1/p}(\lg n)^a$, $b_n = (\lg n)^{a+1}$ and $c_n = b_n/a_n = n^{1+1/p}\lg n$. As in the last proof we partition the series $b_N^{-1}\sum_{n=k}^{N} a_n X_{nk}$ into the following three terms:

$$\begin{aligned} b_N^{-1}\sum_{n=k}^{N} a_n X_{nk} &= b_N^{-1}\sum_{n=k}^{N} a_n[X_{nk}I(X_{nk}\le c_n) - EX_{nk}I(X_{nk}\le c_n)] \\ &\quad + b_N^{-1}\sum_{n=k}^{N} a_n X_{nk} I(X_{nk} > c_n) + b_N^{-1}\sum_{n=k}^{N} a_n EX_{nk}I(X_{nk}\le c_n). \end{aligned}$$

We eliminate the middle term of these three by again using the Borel-Cantelli lemma. Since $pk > 1$ we have

$$\begin{aligned} \sum_{n=k}^{\infty} P\{X_{nk} > c_n\} &= \sum_{n=k}^{\infty} \frac{p\cdot n!}{(n-k)!(k-1)!}\int_{c_n}^{\infty}(1-x^{-p})^{n-k}x^{-pk-1}dx \\ &< C\sum_{n=k}^{\infty} n^k \int_{c_n}^{\infty} x^{-pk-1}dx < C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}} \\ &= C\sum_{n=k}^{\infty}\frac{n^k}{(n^{1+1/p}\lg n)^{pk}} = C\sum_{n=k}^{\infty}\frac{n^k}{n^{pk+k}(\lg n)^{pk}} \\ &= C\sum_{n=k}^{\infty}\frac{1}{n^{pk}(\lg n)^{pk}} = C\sum_{n=k}^{\infty}\frac{1}{(n\lg n)^{pk}} < \infty. \end{aligned}$$

We next show that the first term vanishes almost surely. First, let $1 < pk < 2$, then

$$\begin{aligned} &\sum_{n=k}^{\infty} c_n^{-2}EX_{nk}^2 I(X_{nk}\le c_n) \\ =\ & \sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\int_1^{c_n}(1-x^{-p})^{n-k}x^{-pk+1}dx \\ <\ & C\sum_{n=k}^{\infty}\frac{n^k}{c_n^2}\int_1^{c_n}x^{-pk+1}dx < C\sum_{n=k}^{\infty}\frac{n^k}{c_n^2}c_n^{-pk+2} = C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}} < \infty \end{aligned}$$

as in the last calculation. If $pk = 2$, then

$$\sum_{n=k}^{\infty} c_n^{-2}EX_{nk}^2 I(X_{nk}\le c_n) < C\sum_{n=k}^{\infty}\frac{n^k}{c_n^2}\lg c_n < C\sum_{n=k}^{\infty}\frac{n^k\lg n}{n^{2+2/p}(\lg n)^2}$$

$$= \quad C\sum_{n=k}^{\infty}\frac{1}{n^2\lg n}<\infty.$$

Finally, if $pk>2$, then

$$\begin{aligned}
&\sum_{n=k}^{\infty}c_n^{-2}EX_{nk}^2I(X_{nk}\le c_n)\\
=\quad&\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\int_1^{c_n}(1-x^{-p})^{n-k}x^{-pk+1}dx\\
=\quad&\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_1^{c_n}x^{-p(j+k)+1}dx\\
=\quad&\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{p(j+k)-2}\\
&+\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+2}}{p(j+k)-2}.
\end{aligned}$$

Using both our lemmas, with $m=k-2/p$, the first term is

$$\begin{aligned}
&\sum_{n=k}^{\infty}\frac{n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{j+m}\\
=\quad&\sum_{n=k}^{\infty}\Big(\frac{n!}{c_n^2(n-k)!(k-1)!}\Big)\Big(\frac{(n-k)!\Gamma(m)}{\Gamma(n-k+m+1)}\Big)\\
<\quad&C\sum_{n=k}^{\infty}\frac{\Gamma(n+1)}{c_n^2\Gamma(n-k+m+1)}<C\sum_{n=k}^{\infty}\frac{n^{-m+k}}{c_n^2}\\
=\quad&C\sum_{n=k}^{\infty}\frac{n^{2/p}}{n^{2+2/p}(\lg n)^2}=C\sum_{n=k}^{\infty}\frac{1}{(n\lg n)^2}<\infty.
\end{aligned}$$

As for the second term in our truncated second moment, we have

$$\begin{aligned}
&\left|\sum_{n=k}^{\infty}\frac{p\cdot n!}{c_n^2(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+2}}{p(j+k)-2}\right|\\
<\quad&C\sum_{n=k}^{\infty}\frac{n^k}{c_n^2}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}}{c_n^{p(j+k)-2}}=C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}}\sum_{j=0}^{n-k}\binom{n-k}{j}[c_n^{-p}]^j
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}}[1+c_n^{-p}]^{n-k} < C\sum_{n=k}^{\infty}\frac{n^k}{c_n^{pk}}\\
&= C\sum_{n=k}^{\infty}\frac{n^k}{n^{pk+k}(\lg n)^{pk}} = C\sum_{n=k}^{\infty}\frac{1}{(n\lg n)^{pk}} < \infty
\end{aligned}
$$

establishing that the first term in our partition is almost surely negligible.

All that remains, is to determine where our truncated first moment is going

$$
\begin{aligned}
EX_{nk}I(X_{nk}\le c_n) &= \frac{p\cdot n!}{(n-k)!(k-1)!}\int_1^{c_n}(1-x^{-p})^{n-k}x^{-pk}dx\\
&= \frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_1^{c_n}x^{-p(j+k)}dx\\
&= \frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{p(j+k)-1}\\
&\quad +\frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+1}}{p(j+k)-1}.
\end{aligned}
$$

Using our lemmas, with $m = k - 1/p$, the first term equals

$$
\begin{aligned}
\frac{n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{j+m} &= \Big(\frac{n!}{(n-k)!(k-1)!}\Big)\Big(\frac{(n-k)!\Gamma(m)}{\Gamma(n-k+m+1)}\Big)\\
&= \frac{\Gamma(k-1/p)\Gamma(n+1)}{(k-1)!\Gamma(n+1-1/p)}\\
&\sim \frac{\Gamma(k-1/p)}{(k-1)!}\cdot n^{1/p}.
\end{aligned}
$$

The other term in our first moment is bounded above by

$$
\begin{aligned}
&\left|\frac{p\cdot n!}{(n-k)!(k-1)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^{j+1}c_n^{-p(j+k)+1}}{p(j+k)-1}\right|\\
<\ & Cn^k\sum_{j=0}^{n-k}\binom{n-k}{j}c_n^{-p(j+k)+1} < Cn^kc_n^{-pk+1} = \frac{Cn^kc_n}{c_n^{pk}}
\end{aligned}
$$

$$= \frac{Cn^k n^{1+1/p}\lg n}{n^{pk+k}(\lg n)^{pk}} = \frac{Cn^{1/p}}{n^{pk-1}(\lg n)^{pk-1}} = \frac{Cn^{1/p}}{(n\lg n)^{pk-1}} = o(n^{1/p})$$

since $pk > 1$. Thus

$$EX_{nk}I(X_{nk} \le c_n) \sim \frac{\Gamma(k-1/p)}{(k-1)!} \cdot n^{1/p}.$$

Therefore, with probability one

$$\begin{aligned} \frac{\sum_{n=k}^N a_n X_{nk}}{b_N} &\sim \frac{\sum_{n=k}^N a_n EX_{nk}I(X_{nk} \le c_n)}{b_N} \\ &\sim \frac{\Gamma(k-1/p)}{(k-1)!} \cdot \frac{\sum_{n=k}^N n^{-1-1/p}(\lg n)^a n^{1/p}}{(\lg N)^{a+1}} \\ &= \Big(\frac{\Gamma(k-1/p)}{(k-1)!}\Big) \cdot \Big(\frac{\sum_{n=k}^N n^{-1}(\lg n)^a}{(\lg N)^{a+1}}\Big) \to \frac{\Gamma(k-1/p)}{(a+1)(k-1)!} \end{aligned}$$

as $N \to \infty$, which completes this proof.

4. pk equal to one. It is surprising that in this section we are still able to obtain a Strong Law of Large Numbers. (For if $pk < 1$ we cannot even obtain a similar Weak Law, see Section 5.) However, in this case our weights, a_n, cannot be one. Theorem 3 shows why we need to study weighted sums of random variables. In this situation, from Adler [2], we have

Theorem 3. *If* $pk = 1$ *and* $a > -2$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=k}^N (\lg n)^a n^{-k-1} X_{nk}}{(\lg N)^{a+2}} = \frac{1}{(a+2)k!} \qquad \textit{almost surely.}$$

However, we are not so fortunate when $pk < 1$.

5. pk less than one. In this situation we show that even a "fair"($L > 0$) Weak Law of Large Numbers fails to hold under the mildest of all assumptions. Clearly (2) holds in all our comparable Strong Laws.

Theorem 4. *If* $pk < 1$ *and* a_n *and* b_N *are positive constants where*

$$\max_{k\le n\le N} na_n^p = o(b_N^p) \tag{2}$$

then the only finite limit of our normalized sums is zero, i.e.,

$$\frac{\sum_{n=k}^N a_n X_{nk}}{b_N} \xrightarrow{P} 0.$$

Proof. Let m_n be the median from the density of X_{nk}. We first show that

$$\max_{k\le n\le N}\Big(\frac{a_n m_n}{b_N}\Big) \to 0 \tag{3}$$

as $N\to\infty$. By definition

$$\begin{aligned}\frac{1}{2} &= \frac{p\cdot n!}{(n-k)!(k-1)!}\int_{m_n}^{\infty}(1-x^{-p})^{n-k}x^{-pk-1}dx\\ &< \frac{pn^k}{(k-1)!}\int_{m_n}^{\infty}x^{-pk-1}dx < \frac{n^k}{k!}m_n^{-pk}.\end{aligned}$$

Thus $m_n < Cn^{1/p}$. So

$$\frac{\max_{k\le n\le N} a_n m_n}{b_N} < \frac{C\max_{k\le n\le N} a_n n^{1/p}}{b_N} \to 0$$

by (2), establishing (3). Assuming that a Weak Law holds we have, by the Degenerate Convegence Theorem, see Chow and Teicher [3], page 338

$$\begin{aligned}0 &\leftarrow \sum_{n=k}^N P\{X_{nk} > b_N/a_n\}\\ &= \frac{p}{(k-1)!}\sum_{n=k}^N\frac{n!}{(n-k)!}\int_{b_N/a_n}^{\infty}(1-x^{-p})^{n-k}x^{-pk-1}dx\\ &= \frac{p}{(k-1)!}\sum_{n=k}^N\frac{n!}{(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_{b_N/a_n}^{\infty}x^{-p(j+k)-1}dx\\ &= \frac{p}{(k-1)!}\sum_{n=k}^N\frac{n!}{(n-k)!}\sum_{j=0}^{n-k}\frac{\binom{n-k}{j}(-1)^j(a_n/b_N)^{p(j+k)}}{p(j+k)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(k-1)!}\sum_{n=k}^{N}\frac{n!}{(n-k)!}\Big(\frac{a_n}{b_N}\Big)^{pk}\Big[\frac{1}{k}+\sum_{j=1}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{j+k}\Big(\frac{a_n}{b_N}\Big)^{pj}\Big]\\
&> C\sum_{n=k}^{N}\frac{n!}{(n-k)!}\Big(\frac{a_n}{b_N}\Big)^{pk}
\end{aligned}
$$

since the second sum is negligible, as we are about to show. Let $0<\epsilon<1/2$. For all large N we have from (2) that $na_n^p<\epsilon b_n^p$, whence

$$
\begin{aligned}
\left|\sum_{j=1}^{n-k}\frac{\binom{n-k}{j}(-1)^j}{j+k}\Big(\frac{a_n}{b_N}\Big)^{pj}\right| &< \sum_{j=1}^{n-k}\binom{n-k}{j}\Big(\frac{a_n}{b_N}\Big)^{pj}<\sum_{j=1}^{n-k}(n-k)^j\Big(\frac{a_n}{b_N}\Big)^{pj}\\
&< \sum_{j=1}^{\infty}n^j\Big(\frac{a_n}{b_N}\Big)^{pj}=\sum_{j=1}^{\infty}\Big(\frac{na_n^p}{b_N^p}\Big)^j<\sum_{j=1}^{\infty}\epsilon^j<2\epsilon.
\end{aligned}
$$

Therefore

$$
\sum_{n=k}^{N}\frac{n!}{(n-k)!}\Big(\frac{a_n}{b_N}\Big)^{pk}\to 0. \tag{4}
$$

Finally, by once again utilizing the Degenerate Convergence Theorem, we have as the weak limit of (1)

$$
\begin{aligned}
&\lim_{N\to\infty}\sum_{n=k}^{N}\frac{a_n}{b_N}EX_{nk}I(X_{nk}\le b_N/a_n)<\lim_{N\to\infty}\frac{C}{b_N}\sum_{n=k}^{N}\frac{a_n n!}{(n-k)!}\int_1^{b_N/a_n}x^{-pk}dx\\
&< \lim_{N\to\infty}\frac{C}{b_N}\sum_{n=k}^{N}\frac{a_n n!}{(n-k)!}\Big(\frac{b_N}{a_n}\Big)^{-pk+1}=\lim_{N\to\infty}\frac{C}{b_N^{pk}}\sum_{n=k}^{N}\frac{a_n^{pk}n!}{(n-k)!}=0
\end{aligned}
$$

by (4), which completes the proof.

6. Discussion. A comment should be made as to why different results occur as to whether pk is greater, equal or less than one. If pk exceeds one then EX_{nk} is finite and typical laws of large number do exist as shown in this paper. When $pk=1$, then EX_{nk} is infinite, but barely which sometimes allows us to obtain unusual laws of large numbers (see Adler [2]). We say a random variable's expectation is barely infinite if $EX=\infty$ while $EX^{1-\epsilon}<$

∞ for any $\epsilon > 0$. Now, when $pk < 1$ we have $EX_{nk} = \infty$, but not barely, which surprisingly does not allow us to establish any kind of law of large numbers, whether it be of the strong or weak type.

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