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THE FORWARD SELF-SIMILAR SOLUTION FOR A NONLINEAR PARABOLIC EQUATION

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Abstract. The Cauchy problem for the nonlinear parabolic equation $u_t = u(\Delta u + u^p)$, $p > 1$, is studied. We first prove that there are forward self-similar positive solutions for this equation. Then a more precise asymptotic behavior of these solutions is obtained. From this result it follows that global (in time) solutions for the Cauchy problem exist.

1. Introduction. In this paper, we study the Cauchy problem for the following nonlinear parabolic equation

$$u_t = u(\Delta u + u^p), x \in \mathbf{R}^n, \qquad t > 0, \tag{1.1}$$

where $p > 1$. We are interested in looking for the forward self-similar positive solutions of (1.1) in the form

$$u(x,t) = t^{-\alpha}\phi(\frac{|x|}{t^{\beta}}), \tag{1.2}$$

where the similarity exponents are necessarily given by

$$\alpha = \frac{1}{p}, \quad \beta = \frac{p-1}{2p}. \tag{1.3}$$

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Set $\xi = |x|/t^{\beta}$. It follows that u satisfies (1.1) if and only if ϕ satisfies the equation

$$\phi'' + \frac{n-1}{\xi}\phi' + \phi^p + \alpha + \beta\xi\phi^{-1}\phi' = 0, \quad \xi > 0, \tag{1.4}$$

and $\phi'(0) = 0$.

We remark that (1.1) is a special case of the equation

$$u_t = u^{\sigma}(\Delta u + u^p). \tag{1.5}$$

This equation for $\sigma < 1$ has been studied fairly extensively over past years. Indeed, $\sigma = 0$ is the case of standard heat equation. For $\sigma < 1$, by making the transformation $v = u^{1-\sigma}$, (1.5) is equivalent to the equation

$$v_t = (1-\sigma)[\Delta v^{1/(1-\sigma)} + v^{p/(1-\sigma)}] \tag{1.6}$$

which is the case of porous medium equation for $\sigma \in (0, 1)$; or, the case of fast diffusion equation for $\sigma < 0$.

For results related to the global (in time) existence for the Cauchy problem (among others), we refer the readers to [4] ([5]) and the references cited therein for the case $0 < \sigma < 1$ ($\sigma = 0$, respectively). The case of $\sigma < 0$ was studied in [6]. The purpose of this paper is to study the case $\sigma = 1$.

We first prove in Section 2 that the solution of initial value problem for the equation (1.4) with the initial value

$$\phi'(0) = 0, \quad \phi(0) = \eta$$

exists globally for any given $\eta > 0$. Moreover, it is monotone decreasing to zero as $\xi \to \infty$.

In Section 3, we shall study the asymptotic bahaviour as $\xi \to \infty$ of positive solution of (1.4). We prove that for any positive solution ϕ of (1.4) the limit

$$\lim_{\xi\to\infty}[\xi^{\alpha/\beta}\phi(\xi)] = A \tag{1.7}$$

exists and $A > 0$.

It follows that the Cauchy problem for (1.1) has a global (in time) solution v given by (1.2) such that $v(x,0) = A|x|^{-\alpha/\beta}$ and

$$\lim_{|x|\to\infty} |x|^{\alpha/\beta} v(x,t) = A$$

for each fixed $t > 0$. Note that $\alpha/\beta = 2/(p-1)$. Therefore, for $p > 1 + 2/n$ there is a global solution for the Cauchy problem for (1.1) in the class of locally integrable functions over $\mathbf{R}^n$ for each $t \geq 0$. For $p = 1 + 2/n$, the solution we found is not integrable at both $x = 0$ for $t = 0$ and $|x| = \infty$ for each $t \geq 0$. For $1 < p < 1 + 2/n$, the initial value is not integrable at $x = 0$, but it is integrable at $|x| = \infty$. In this case, the solution is the so-called very singular solution (cf. [1]), since

$$\int_{|x|\leq R} v(x,t)dx = t^{n\beta-\alpha}\omega_n \int_0^{Rt^{-\beta}} r^{n-1}\phi(r)dr \to \infty$$

as $t \downarrow 0$, $\forall R > 0$, where ω_n is the surface area of n-dimensional unit sphere.

Finally, we remark that (1.1) can also be rewritten as

$$u_t = \Delta(\frac{1}{2}u^2) - |\nabla u|^2 + u^{p+1}. \tag{1.8}$$

The equation (1.8) without the gradient term, i.e., the equation

$$u_t = \Delta(\frac{1}{2}u^2) + u^{p+1} \tag{1.9}$$

is a special case of (1.6) with $\sigma = 1/2$. It is well-known that $p^* = 1 + 2/n$ is the so-called critical exponent for (1.9) (cf. [4] and its references) in which (1.9) has a bounded global solution if and only if $p > p^*$. On the other hand, for any fixed $\delta > 0$, there are global solutions of (1.8) in the form

$$u(x,t) = (t+\delta)^{-\alpha}\phi(\frac{|x|}{(t+\delta)^{\beta}}), t \geq 0.$$

These solutions are all bounded smooth in $\mathbf{R}^n \times [0,\infty)$ such that they tend to zero uniformly as $t \to \infty$ for any $p > 1$. This observation shows that there is no critical exponent for (1.8). We also observe that any nontrivial spatial homogeneous solution of (1.8) must blow up in finite time. Indeed, if $u(x,t) = u(t)$ is a solution of (1.8) with $u(0) > 0$, then $u(T) = \infty$ for $T = u(0)^{-p}/p < \infty$.

2. Existence. In this section, we shall study the positive solution of the following initial value problem (P):

$$\phi'' + \frac{n-1}{\xi}\phi' + \phi^p + \alpha + \beta\xi\phi^{-1}\phi' = 0, \xi > 0, \tag{2.1}$$

$$\phi'(0) = 0, \quad \phi(0) = \eta, \tag{2.2}$$

where $\eta > 0$ is given. Note that there is no constant solution of (P). From the local existence and uniqueness theorem of initial value problem it follows that there is a unique positive local solution ϕ of (P) for each given $\eta > 0$. Let $[0,R)$ be the maximal existence interval of ϕ such that $\phi > 0$, where $0 < R \leq \infty$. Define

$$\rho(y) = \exp\{\beta\int_0^y \xi\phi^{-1}(\xi)d\xi\}. \tag{2.3}$$

Then it follows from (2.1) that

$$(\xi^{n-1}\rho(\xi)\phi'(\xi))' = -\xi^{n-1}\rho(\xi)[\phi^p + \alpha] \tag{2.4}$$

and so

$$\phi'(\xi) = -\frac{1}{\xi^{n-1}\rho(\xi)}\int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha]dy, \xi > 0. \tag{2.5}$$

Hence ϕ is monotone decreasing in $[0, R)$.

The following result shows that ϕ must be globally defined.

Proposition 2.1. *For any given $\eta > 0$, the local solution ϕ of (P) can be continued globally so that $R = \infty$.*

Proof. Suppose that $R < \infty$. Then $\phi(\xi) \to 0$ as $\xi \to R^-$. Integrating the equation (2.1) from $R/2$ to $z > R/2$, using the fact $\phi' < 0$, we obtain

$$\begin{aligned}
\phi'(z) &= \phi'(\frac{R}{2}) - (n-1)\int_{R/2}^{z} \frac{\phi'}{\xi} d\xi - \beta \int_{R/2}^{z} \xi \frac{\phi'}{\phi} d\xi - \int_{R/2}^{z} \phi^p d\xi - \alpha(z - R/2) \\
&\geq \phi'(\frac{R}{2}) - \frac{n-1}{z}\int_{R/2}^{z} \phi' d\xi - \frac{\beta R}{2}\int_{R/2}^{z} \frac{\phi'}{\phi} d\xi - [\phi^p(\frac{R}{2}) + \alpha](z - R/2) \\
&= \phi'(\frac{R}{2}) - \frac{n-1}{z}[\phi(z) - \phi(\frac{R}{2})] - \frac{\beta R}{2} \ln \frac{\phi(z)}{\phi(\frac{R}{2})} \\
&\quad -[\phi^p(R/2) + \alpha](z - R/2) \\
&\to \infty
\end{aligned}$$

as $z \to R^-$, a contradiction to the fact that ϕ is monotone decreasing in $(0, R)$. Therefore, $R = \infty$.

Next, we define

$$H(\xi) = \frac{1}{2}[\phi'(\xi)]^2 + G(\phi(\xi)), \xi > 0,$$

where

$$G(\phi) = \int_0^{\phi} (s^p + \alpha) ds, \phi \geq 0.$$

Proposition 2.2. *$\phi(\xi) \to 0$ and $\phi'(\xi) \to 0$ as $\xi \to \infty$.*

Proof. It is clear that the limit

$$l = \lim_{\xi \to \infty} \phi(\xi)$$

exists and $l \geq 0$. Since

$$H'(\xi) = -[\frac{n-1}{\xi} + \beta\xi\phi^{-1}][\phi'(\xi)]^2 \leq 0,$$

the limit

$$L = \lim_{\xi\to\infty} H(\xi)$$

exists and $L \geq 0$. Consequently, the limit

$$K = \lim_{\xi\to\infty} [\phi'(\xi)]^2$$

exists and $K \geq 0$.

Suppose that $l > 0$. Then from

$$\int_0^\infty \phi'(\xi)d\xi = l - \eta,$$

it follows that there is a sequence $\xi_k \to \infty$ such that $\phi'(\xi_k) \to 0$ as $k \to \infty$. Dividing the equation (2.1) by ξ and integrating it from 1 to ξ_k, $k \geq 1$, we obtain

$$\int_1^{\xi_k} \left(\frac{\phi''}{\xi} + \frac{n-1}{\xi^2}\phi'\right) d\xi = \frac{\phi'(\xi_k)}{\xi_k} - \phi'(1) + \int_1^{\xi_k} \frac{n}{\xi^2}\phi' d\xi,$$
$$\int_1^{\xi_k} \beta\frac{\phi'}{\phi} d\xi = \beta \ln \frac{\phi(\xi_k)}{\phi(1)}.$$

It is easy to show that the above two integrals are uniformly bounded for all k. On the other hand, we have

$$\int_1^{\xi_k} \frac{\alpha + \phi^p}{\xi} d\xi \geq \int_1^{\xi_k} \frac{\alpha}{\xi} d\xi$$

which tends to ∞ as $k \to \infty$, a contradiction (cf. [2, Proposition 5). Therefore, $l = 0$.

Hence $K = 2L$. If $K > 0$, then

$$\phi'(\xi) \to -\sqrt{K} < 0$$

as $\xi \to \infty$. Hence, by an integration of ϕ', we obtain that $\phi(\xi) \to -\infty$ as $\xi \to \infty$, a contradiction to Proposition 2.1. This completes the proof of the proposition.

3. Asymptotic behavior. We shall apply the method used in [3] to study more precisely the asymptotic behavior of solution ϕ of (P) as $\xi \to \infty$. From now on, ϕ will be the solution of (P) with $\phi(0) = \eta$. Recall that ϕ is monotone decreasing to zero and $\phi' \to 0$. From the definition of ρ, we have

$$\rho(y) \geq \exp\{\beta\eta^{-1}y^2/2\}. \tag{3.1}$$

The following lemma shows that ϕ decays to zero at most polynomially.

Lemma 3.1. *There holds*

$$\lim_{\xi\to\infty}[\xi^q\phi(\xi)] = \infty \tag{3.2}$$

for any $q > (\alpha + \eta^p)/\beta$.

Proof. Dividing the equation (2.1) by ξ, we obtain

$$\frac{\phi''}{\xi} + \frac{n-1}{\xi^2}\phi' = -\left(\beta\frac{\phi'}{\phi} + \frac{\alpha+\phi^p}{\xi}\right). \tag{3.3}$$

Given any $q > (\alpha + \eta^p)/\beta$. Suppose that (3.2) does not hold. Then

$$\liminf_{\xi\to\infty}\,[\xi^q\phi(\xi)] < \infty$$

and so there is a positive constant K and a sequence $\{\xi_k\}$ such that $\xi_k \to \infty$ as $k \to \infty$ and $\xi_k^q\phi(\xi_k) \leq K$ for all $k \geq 1$.

Now, integrating (3.3) from 1 to ξ_k, we obtain

$$\int_1^{\xi_k}\left(\frac{\phi''}{\xi}+\frac{n-1}{\xi^2}\phi'\right)d\xi=\frac{\phi'(\xi_k)}{\xi_k}-\phi'(1)+\int_1^{\xi_k}\frac{n}{\xi^2}\phi' d\xi\le -\phi'(1)$$

for all k, and

$$\begin{aligned}
&\int_1^{\xi_k}\left(\beta\frac{\phi'}{\phi}+\frac{\alpha+\phi^p}{\xi}\right)d\xi\\
\le\ &\int_1^{\xi_k}\left(\beta\frac{\phi'}{\phi}+\frac{\alpha+\eta^p}{\xi}\right)d\xi\\
=\ &\ln\left\{\phi(\xi_k)^\beta\phi(1)^{-\beta}\xi_k^{\alpha+\eta^p}\right\}\\
=\ &\ln\left\{[\xi_k^q\phi(\xi_k)]^\beta\xi_k^{\alpha+\eta^p-\beta q}\phi(1)^{-\beta}\right\}\\
\to\ &-\infty
\end{aligned}$$

as $k\to\infty$. This is a contradiction and the lemma is proved.

From (3.1) and (3.2), we have $\rho(\xi)\phi(\xi)\to\infty$ as $\xi\to\infty$. Using (2.5) and applying l'Hôpital's Rule, we obtain

$$\begin{aligned}
\lim_{\xi\to\infty}\frac{\phi'(\xi)}{\phi(\xi)} &= \lim_{\xi\to\infty}\frac{-\int_0^\xi y^{n-1}\rho(y)[\phi^p(y)+\alpha]dy}{\xi^{n-1}\rho(\xi)\phi(\xi)}\\
&= \lim_{\xi\to\infty}\frac{-[\alpha+\phi^p]}{(n-1)\xi^{-1}+\beta\xi+\phi'}\\
&= 0.
\end{aligned}\tag{3.4}$$

Similarly, we have

$$\begin{aligned}
\lim_{\xi\to\infty}\frac{\xi\phi'(\xi)}{\phi(\xi)} &= \lim_{\xi\to\infty}\frac{-\int_0^\xi y^{n-1}\rho(y)[\phi^p(y)+\alpha]dy}{\xi^{n-2}\rho(\xi)\phi(\xi)}\\
&= \lim_{\xi\to\infty}\frac{-[\alpha+\phi^p]}{(n-2)\xi^{-2}+\beta+\xi^{-1}\phi'}\\
&= -\frac{\alpha}{\beta}.
\end{aligned}\tag{3.5}$$

It follows from (3.5) that for any $\epsilon>0$ there are positive constants K,R

depending only on ϵ such that

$$\phi(\xi) \leq K\xi^{-\alpha/\beta+\epsilon}, \forall \xi \geq R. \tag{3.6}$$

Let $\psi = \phi'/\phi$. Then ψ satisfies the equation

$$\psi' + \frac{n-1}{\xi}\psi + \beta\xi\phi^{-1}\psi = -[\frac{\alpha}{\phi} + \phi^{p-1} + \psi^2].$$

It follows that

$$\psi(\xi) = \frac{-1}{\xi^{n-1}\rho(\xi)}\int_0^\xi y^{n-1}\rho(y)[\frac{\alpha}{\phi(y)} + \phi^{p-1}(y) + \psi^2(y)]dy. \tag{3.7}$$

We now are ready to state and prove the main theorem of this section as follows.

Theorem 3.2. *The limit*

$$\lim_{\xi\to\infty}[\xi^{\alpha/\beta}\phi(\xi)]$$

exists and is positive.

Proof. From (3.7), we can write

$$\begin{aligned} & [\xi\psi(\xi) + \frac{\alpha}{\beta}]\xi^\lambda \\ = & \frac{-\int_0^\xi y^{n-1}\rho(y)[\alpha/\phi(y) + \phi^{p-1}(y) + \psi^2(y)]dy + \alpha\xi^{n-2}\rho(\xi)/\beta}{\xi^{n-\lambda-2}\rho(\xi)}, \end{aligned} \tag{3.8}$$

where λ is a constant with $\lambda \in (0, 2)$. Applying l'Hôpital's Rule to (3.8), using (3.6) with suitable ϵ, we deduce that

$$\lim_{\xi\to\infty}[\xi\psi(\xi) + \frac{\alpha}{\beta}]\xi^\lambda = 0 \tag{3.9}$$

for any $\lambda \in (0, 2)$. From (3.9) and by an integration, we obtain that

$$\phi(\xi) = A\xi^{-\alpha/\beta}[1 + o(\xi^{-\lambda})]$$

as $\xi \to \infty$ for some positive constant A. Hence the theorem follows.

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