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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF HIGHER ORDER NEUTRAL EQUATIONS

BY

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Abstract. In this paper sufficient conditions are obtained for every solution of

$$(*)\quad (y(t)-p(t)y(t-\tau))^{(n)}+Q(t)G(y(t-\sigma))=f(t),\quad t\geq 0,$$

to oscillate or tend to zero as $t\to\infty$, for both n odd or even. Here $0\leq p(t)\leq p$ or $-p\leq p(t)\leq 0$, where p is a positive scalar. The results of this paper hold for linear, super linear or sublinear equations, and answer an open problem suggested by Ladas and Gyori in [1]. The results of the paper are also true for the homogeneous equation associated with $(*)$, and generalize/improve some known results.

1. Introduction. In the present work the author has obtained sufficient conditions for every solution of

$$(E)\qquad (y(t)-p(t)y(t-\tau))^{(n)}+Q(t)G(y(t-\sigma))=f(t).$$

to oscillate or tend to zero as $t\to\infty$, where p and $f\in C([0,\infty),R)$, $Q\in C([0,\infty),[0,\infty))$, $G\in C(R,R)$, $\tau>0$ and $\sigma\geq 0$. Following assumptions are needed in the sequel.

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(H_1) There exists $F \in C^{(n)}([0,\infty), R)$ such that $F^{(n)}(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$

(H_2) G is non-decreasing and $u\, G(u) > 0$ for $u \neq 0$.

(H_3) For $u > 0$, $v > 0$, $\exists$ a scalar $\delta > 0$ such that $G(u) + G(v) \geq \delta\, G(u+v)$

(H_4) $lim_{|u|\to\infty} G(u)/u \geq \alpha > 0$,where α is a scalar.

(H_5) For $u > 0$, $v > 0$, $G(u)G(v) \geq G(uv)$

(H_6) $G(-u) = -G(u)$

(H_7) $\int_0^\infty t^{n-2} Q(t)dt = \infty$, $n \geq 2$.

(H_8) $\int_0^\infty Q(t)dt = \infty$

(H_9) Suppose that, for every sequence $< \sigma_i > \subset (0,\infty)$, $\sigma_i \to \infty$ as $i \to \infty$ and for every $\beta > 0$ such that the intervals $(\sigma_i - \beta, \sigma_i + \beta)$, $i = 1, 2, \ldots$, are non overlapping,

$$\sum_{i=1}^{\infty} \int_{\sigma_i - \beta}^{\sigma_i + \beta} t^{n-1} Q(t)dt = \infty, \quad \text{for } n \geq 1.$$

(H_{10}) In (H_9) replace t^{n-1} by t^{n-2} i.e

$$\sum_{i=1}^{\infty} \int_{\sigma_i - \beta}^{\sigma_i + \beta} t^{n-2} Q(t)dt = \infty, \quad \text{for } n \geq 2.$$

In recent years a good deal of work is done on the oscillation theory of higher order neutral delay-differential equations. Most of these results are concerned with (E) where $f(t) \equiv 0$ and $G(u) \equiv u$. It seems that little work is done for the oscillatory and asymptotic behaviour of solutions of (E). In particular still less work is done, when $p \geq p(t) \geq 1$. The author is motivated for the present work due to this observation and an open problem of [1,pp-287]. The problem 10.10.2 of above reference suggested by Ladas and Gyori is "Extend the results of section 10.4 to equations where the coefficient $p(t)$ lies in different ranges". The following ranges for $p(t)$ are considered in section 10.4 of [1].

(A_1) $1 \le p(t) \le p_1$ (A_2) $0 \le p(t) \le p_2 < 1$

(A_3) $-1 < -p_3 \le p(t) \le 0$ (A_4) $p(t) \equiv -1$

(A_5) $0 < p(t) \le 1$.

Where p_i is a positive scalar for i=1, 2, 3. In this paper the following two ranges are considered for $p(t)$ which are different from the above mentioned ranges.

(B_1) $0 \le p(t) \le p$ (B_2) $-p \le p(t) \le 0$

where p is a positive scalar.

The present study deals with Eq. (E) with $n \ge 2$ (also true for $n = 1$, with little modification) and super linear assumption (H_4). It may be noted that (H_4) includes linear case. The prototype of G satisfying $(H_2)-(H_6)$ is

$$G(u) = (\beta + |u|^{\lambda})|u|^{\mu}\operatorname{sgn} u,\ \beta \ge 1,\ \lambda \ge 0,\ \mu \ge 0 \text{ and } \lambda + \mu \ge 1.$$

See [8, p. 292]. This work also hold for homogeneous neutral delay equations of order n.

By a solution of (E) we mean a real-valued continuous function y on $[T_y-\rho, \infty)$ for some $T_y \ge 0$, where $\rho = \max\{\tau, \sigma\}$, such that $y(t)-p(t)\,y(t-\tau)$ is n-time continuously differentiable and (E) is satisfied for $t \in [T_y, \infty)$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros, otherwise, it is called non-oscillatory.

In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large t.

2. Main Results. First we state some Lemmas which are needed in the sequel,

Lemma 2.1. *$Q \in C([0,\infty),[0,\infty))$ and $Q(t) \not\equiv 0$ on any interval of the form $[T,\infty)$, $T \geq 0$, and $G \in C(R,R)$ with $u\ G(u) > 0$ for $u \neq 0$. Let $y \in C([0,\infty),\ R)$ with $y(t) > 0$ for $t \geq t_0 \geq 0$. If $w \in C^{(n)}([0,\infty),R)$, with*

$$w^{(n)}(t) = -Q(t)G(y(t-\sigma)), \quad t \geq t_0 + \sigma, \quad \sigma \geq 0, \tag{1}$$

and there exists an integer $n^* \in \{0,1,2,\ldots,n-1\}$ such that $lim_{t\to\infty} w^{n^*}(t)$ exists and $lim_{t\to\infty} w^i(t) = 0$ for $i \in \{n^*+1,\ldots,n-1\}$,then

$$w^{n^*}(t) = w^{n^*}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!}\int_t^\infty (s-t)^{n-n^*-1}Q(s)G(y(s-\sigma))ds \tag{2}$$

for large t.

If $y(t) < 0$ for $t \geq t_0$ then also (2) holds.

The proof follows by integrating (1), $n-n^*$ times and it is found in [5].

Lemma 2.2. *Suppose that $p(t)$ is in the range (B_1). Let (H_1), (H_2), (H_4) and (H_7) hold. If $y(t)$ is a positive solution of (E) for $t \geq t_0 > 0$ then either $w(t) = -\infty$ or $lim\ w(t) = 0, (-1)^{n+k}w^{(k)}(t) < 0$ for $k = 0,1,2,\ldots,n-1$, for large t, where*

$$w(t) = y(t) - p(t)y(t-\tau) - F(t).$$

If $y(t) < 0$ for $t \geq t_0$ then either $lim_{t\to\infty} w(t) = \infty$, or $lim_{t\to\infty} w(t) = 0$ and $(-1)^{n+k}w^{(k)(t)} > 0$ for $k = 0,1,2,\ldots,n-1$.

The proof is simple and it follows directly from Lemma 2.5 of [5].

Remak 2.1. Lemma 2.2 hold for $n \geq 2$. However, if $n = 1$, then one can replace (H_7) by (H_8) and see that it is true.

Theorem 2.3. *Let $p(t)$ be in the range (B_1). Suppose that (H_1), (H_2), (H_4), (H_7), and (H_9) hold. Then every bounded solution of (E) oscillates* *or tends to zero as* $t \to \infty$ *and every unbounded solution of* (E) *oscillates or tends to* $\pm\infty$.

Proof. Let $y(t)$ be an unbounded solution of (E). If $y(t)$ is oscillatory, then there is nothing of prove. If $y(t)$ is non-oscillatory, then $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > 0$. Let $y(t) > 0$, $t \geq t_0$. Setting

$$z(t) = y(t) - p(t)y(t-\tau) \text{ and } w(t) = z(t) - F(t) \text{ for } t > t_1 > t_0 + \rho, \tag{3}$$

we obtain

$$w^{(n)}(t) = -Q(t)G(y(t-\sigma)) \leq 0. \tag{4}$$

From Lemma 2.2 if follows that either $lim_{t\to\infty} w(t) = -\infty$ or $lim_{t\to\infty} w(t) = 0$ and $(-1)^{n+k} w^{(k)}(t) < 0$ for k=0, 1, 2, $\ldots$, $n-1$ If the latter holds, then since y(t) is unbounded, there exits a sequence $< t_n > \subset [t_2, \infty)$ where $t_2 > t_1$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$. Let $M > 0$. Then $y(t_n) > M$ for $n \geq N_1 > 0$. From the continuity of y it follows that there exists $\delta_n > 0$ with $lim\ inf_{n\to\infty} \delta_n > 0$ such that $y(t) > M$ for $t \in (t_n - \delta_n, t_n + \delta_n)$. Then choosing n large enough such that $\delta_n > \delta > 0$ for $n \geq N > N_1$, we obtain

$$\begin{aligned}
\int_{t_2}^{\infty} t^{n-1} Q(t) G(y(t-\sigma)) dt &\geq \sum_{n=N}^{\infty} \int_{t_n - \delta_n + \sigma}^{t_n + \delta_n + \sigma} t^{n-1} Q(t) G(y(t-\sigma)) dt \\
&\geq G(M) \sum_{n=N}^{\infty} \int_{t_n - \delta_n + \sigma}^{t_n + \delta_n + \sigma} t^{n-1} Q(t) G(y(t-\sigma)) dt \\
&> G(M) \sum_{n=N}^{\infty} \int_{t_n - \delta + \sigma}^{t_n + \delta + \sigma} t^{n-1} Q(t) dt.
\end{aligned}$$

Hence from (H_9), it follows that

$$\int_{t_2}^{\infty} t^{n-1} Q(t) G(y(t-\sigma)) dt = \infty \tag{5}$$

On the other hand since $\lim_{t\to\infty} w(t) = 0$; by using Lemma 2.1 for $n^* = 0$, we obtain for large t

$$w(t) = -\frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)G(y(s-\sigma))ds. \tag{6}$$

From (6) it follows that.

$$\int_{t_2}^\infty t^{n-1}Q(t)G(y(t-\sigma))dt < \infty, \tag{7}$$

a contradiction. Hence the only possibility left is $\lim_{t\to\infty} w(t) = -\infty$. If $p(t) = 0$, then $w(t) = y(t) - F(t) \geq -F(t)$, which implies $F(t) \geq -w(t)$. Then $\lim_{t\to\infty} F(t) = \infty$ a contradiction to (H_1). If $p(t) > 0$, then from (3) we get $z(t) \geq -p(t)y(t-\tau) \geq -py(t-\tau)$. Hence $y(t-\tau) \geq \frac{z(t)}{(-p)}$, which implies $\liminf_{t\to\infty} y(t) = \infty$ because $\lim_{t\to\infty} z(t) = -\infty$ by (H_1). Hence $\lim_{t\to\infty} y(t) = \infty$.

Next let us assume that $y(t)$ is a bounded solution of (E) for $t > t_0 > 0$. Suppose $y(t)$ is non oscillatory. Then $y(t) > 0$ or $y(t) < 0$ for large t. Let $y(t) > 0$ for $t > t_1$. Then using Lemma 2.2 and boundedness of $y(t)$ we obtain $\lim_{t\to\infty} w(t) = 0$. Hence using Lemma 2.1 for $n^* = 0$, we obtain (6). Consequently (7) holds. Then we claim that $\limsup_{t\to\infty} y(t) = 0$. If not then $\limsup_{t\to\infty} y(t) = \alpha$, $\alpha > 0$. Then there exists a sequence $< t_n >$ such that $y(t_n) > M > 0$ for large n. proceeding as above we arrive at (5), which contradicts (7). Hence $\lim_{t\to\infty} y(t) = 0$. The proof for the case $y(t) < 0$ is similar. Hence the theorem is proved.

Remark 2.2. Since $(H_{10}) \Rightarrow (H_9)$ and (H_7) therefore we can assume (H_{10}) in place of (H_9) and (H_7) in Theorem 2.3. It may be noted that Theorem 2.3 holds for $n \geq 2$, but it also holds for $n = 1$, if we assume (H_8) in place of (H_7).

Remark 2.3. Theorem 2.3 improves Theorem 2.9 of [6] and generalizes Theorem 2.2 in [4].

Remark 2.4. Theorem 2.3 is true for both n odd and even. It holds when $f(t) \equiv 0$ and $G(u) \equiv u$.

Example 1. Consider

$$(y(t) - py(t - \ln 2))^{(n)} + (p - 2 + e^{-2t})y(t - \ln 2) = (e^{-t})/2$$

$t \geq 0$, where $p > 2$ and $n \geq 2$. If $F(t) = \frac{1}{2}(-1)^n e^{-t}$ then $F^{(n)}(t) = \frac{1}{2}e^{(-t)} = f(t)$ and $F(t) \to 0$ as $t \to \infty$. Since $Q(t) = p - 2 + e^{-2t} > p - 2 > 0$ then all the conditions of Theorem 2.3 are satisfied. Clearly, $y(t) = e^t$ is a solution of the equation tending to $+\infty$ as $t \to \infty$.

Example 2. From Theorem 2.3 it follows that all bounded solutions of

$$(y(t) - 2y(t - \pi))^{(Iv)} + 3y(t - \pi) = 0$$

oscillate or tend to zero. In particular $y(t) = \sin t$ is a bounded oscillatory solution of the equation.

Theorem 2.4. *Let (H_1), (H_2), (H_3), (H_5), (H_6) and (H_8) hold. Suppose that $Q(t)$ is monotonic decreasing. If $p(t)$ lies in the range (B_2), then every solution of (E) oscillates or tends to zero as $t \to \infty$.*

Proof. If $y(t)$ is a non-oscillatory solution of (E), then $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > 0$. Let $y(t) > 0$ for $t > t_0$. The case $y(t) < 0$ for $t > t_0$ may be dealt with similarly. Setting $z(t)$ and $w(t)$ as in (3) for $t > t_1 > t_0 + \rho$, we obtain $z(t) > 0$ and (4). Hence $w, w', w'', \ldots w^{(n-1)}$ are monotonic and each is of constant sign for large t. Thus $\lim_{t\to\infty} w(t) = \ell$ where $-\infty \leq l \leq \infty$. If $-\infty \leq \ell < 0$ then $z(t) < 0$ for large t, a contradiction. If $\ell = 0$ then $z(t) > y(t)$ implies that $\lim_{t\to\infty} y(t) = 0$. Suppose that $0 < \ell \leq \infty$. Then $w^{(n-1)}(t) > 0$ for large t and hence $\lim_{t\to\infty} w^{(n-1)}(t)$ exists finitely. Further, $z(t) > \lambda > 0$ for $t > t_2 > t_1$. Integrating (4) from t_2

to $s(s > t_2)$ and then taking limit as $s \to \infty$, we obtain

$$\int_{t_2}^{\infty} Q(s)G(y(s-\sigma))ds < \infty \tag{8}$$

On the other hand, for $t_3 > t_2 + \rho$,

$$\int_{t_3}^{\infty} Q(s)G(z(s-\sigma))ds \geq G(\lambda)\int_{t_3}^{\infty} Q(s)ds$$

implies that

$$\int_{t_3}^{\infty} Q(s)G(z(s-\sigma))ds = \infty$$

due to (H_8). Hence using (H_3) and (H_5) we obtain

$$\begin{aligned}
\infty &= \int_{t_3}^{\infty} Q(s)G(y(s-\sigma) - p(s-\sigma)y(s-\tau-\sigma))ds \\
&\leq \frac{1}{\delta}\int_{t_3}^{\infty} Q(s)\{G(y(s-\sigma)) + G(-p(s-\sigma)y(s-\tau-\sigma))\}ds \\
&\leq \frac{1}{\delta}\int_{t_3}^{\infty} Q(s)G(y(s-\sigma))ds + \frac{1}{\delta}\int_{t_3}^{\infty} Q(s)G(-p(s-\sigma))G(y(s-\tau-\sigma))ds \\
&\leq \frac{1}{\delta}\int_{t_3}^{\infty} Q(s)G(y(s-\sigma))ds + \frac{G(p)}{\delta}\int_{t}^{\infty} Q(s)G(y(s-\tau-\sigma))ds
\end{aligned} \tag{9}$$

From (8) and (9) it follows that

$$\int_{t_3}^{\delta} Q(t)G(y(t-\sigma-\tau))dt = \infty,$$

that is (since $Q(t)$ is decreasing),

$$\infty = \int_{t_3-\tau}^{\infty} Q(s+\tau)G(y(s-\sigma))ds < \int_{t_3-\tau}^{\infty} Q(s)G(y(s-\sigma))ds < \infty$$

a contradiction. Hence $\ell = 0$ is the only possibility. If $y(t) < 0$ for $t > t_0$ then setting $x(t) = -y(t)$ for $t \geq t_0$ we obtain

$$(x(t) - p(t)x(t-\tau))^{(n)} + Q(t)\bar{G}(x(t-\sigma)) = \bar{f}(t)$$

where $\bar{f}(t) = -f(t)$ and $\bar{G}(u) = -G(-u) = G(u)$ by (H_6) and $\bar{F}(t) = -F(t)$. Then (H_1) is satisfied by $\bar{F}$. Also the conditions satisfied by G are satisfied by $\bar{G}$. Hence $\lim_{t\to\infty} x(t) = 0$, that is $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved.

Corollary 2.5. *If all conditions of Theorem* 2.4 *are satisfied then every unbounded solution of* (E) *oscillates.*

Remark 2.5. Theorem 2.4 holds for linear, sublinear and super linear G. It is true for $n \geq 1$ (odd or even). Also it holds when $f(t) \equiv 0$.

Example 3.

$$(y(t)-py(t-\ln 2))^{(Iv)}+((2p-1)\exp(-(1+2t)/3)+1)y^{\frac{1}{3}}(t-1)=\exp((1-t)/3),\ t\geq t_0,$$

where $p < 0$ and $t_0 > 0$ such that $\exp(\frac{1+2t_0}{3}) > 1 - 2p$.

Here $F(t) = 81\ \exp((1-t)/3)$ and $Q(t)$ is monotonic decreasing, where $Q(t) = 1+(2p-1)\exp(-(1+2t)/3) > 0$ for $t \geq t_0$. From Theorem 2.4 it follows that every solution of the equation oscillates or tends to zero as $t \to \infty$. In particular $y(t) = e^{-t}$ is a solution of the equation which tends to zero as $t \to \infty$.

Remark 2.6. Theorem 2.4. improves and generalizes Theorem 2.5 in [6], and generalizes Theorem 2.1 in [4].

Remark 2.7. In [7] the author has solved one open problem with an extra condition. Indeed, he showed that every nonoscillatory solution of

$$(y(t) + y(t-\tau))' + Q(t)y(t-\sigma) = 0$$

tends to zero as $t \to \infty$ if (H_8) holds and $Q(t+\tau/n) \leq Q(t)$ for $t \in [0,\infty)$ where n is any fixed, positive integer. Theorem 2.4 of this paper improves and generalizes the work in [7] not only to nonlinear nonhomogeneous equations but also to a greater range of $p(t)$.

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