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KRONECKER PRODUCT SYSTEM OF FIRST ORDER RECTANGULAR MATRIX DIFFERENTIAL EQUATIONS — EXISTENCE AND UNIQUENESS

BY

K. N. MURTY, K. R. PRASAD AND R. SURYANARAYANA

Abstract. In this paper, we establish the existence and uniqueness solutions of the two-point boundary value problem associated with a system of first order rectangular matrix differential equations involving kronecker products. The solution of the two-point boundary value problem is presented in terms of the Green's matrix. The properties of the Green's matrix are also studied.

1. Introduction. Boundary value problems play an important role in a variety of real world problems. In finding solutions to two-point boundary value problems involving kronecker products, the construction of a Green's matrix is vital. It is sufficiently known about the construction of a Green's matrix for problems involving nonsingular square matrices. However the theory for rectangular matrices involves significant difficulties as the inverse of the matrix in the usual sense, does not exist. In this paper, we establish the solutions of boundary value problems associated with kronecker product system of first order rectangular matrix differential equations. By a suitable transformation, the rectangular matrices are transformaed into non-singular square matrices and solutions are finally expressed in terms of the rectangular matrices. In 1992, Murty, K. N., Fausett, D. W. and Fausett, L. V.

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[3] established the solutions of two-point boundary value problems involving a kronecker product system of first order differential equations. In 1992, Murty, K. N, Prasad, K. R. and Rao, Y. S. [4] established the existence and uniqueness of solutions to a kronecker product three point (multi point) boundary value problems associated with a system of first order matrix differential operator involving kronecker products.

In this paper, we consider the following kronecker product two-point boundary value problem:

$$(P(t)\otimes Q(t))y'(t)+(R(t)\otimes S(t))y(t) = f(t,y(t)),\ a\le t\le b \tag{1.1}$$

$$(M_1\otimes N_1)y(a)+(M_2\otimes N_2)y(b) = \alpha, \tag{1.2}$$

where $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ are rectangular matrices of order $(m\times n)$, $y(t)$ is of order $(n^2\times 1)$, $f:[a,b]\times R^{n^2}\to R^{m^2}$ and the components of $P(t)$, $Q(t)$, $R(t)$, $S(t)$ and f are continuous on $[a,b]$, we assume that $f(t,0)\equiv 0$ for all $t\in[a,b]$ and f satisfies a Lipschitz condition on $[a,b]$. We also assume that the rows of $P(t)$ and $Q(t)$ are linearly independent on $[a,b]$ and the system (1.1) is consistent. M_1, N_1, M_2 and N_2 are matrices of order $(m\times n)$ and α is a column matrix of order $(m^2\times 1)$.

This paper is organised as follows: In section 2, we develop the general solution of the homogeneous kronecker product system corresponding to (1.1) in terms of a fundamental matrix. We then establish the variation of parameters formula to find the solution of non-homogeneous kronecker product system (1.1). Section 3, presents a criteria for the existence and uniqueness of solutions to a two-point boundary value problem. We establish the general solution of the two-point kronecker product boundary value problem in terms of an integral representation involving the Green's matrix and we also verify the properties of the Green's matrix. The results obtained in this paper are exemplified at the end of this paper by a suitable example.

2. General Solution of the Non-linear Kronecker Product System. In this section, the general solution of the homogeneous kronecker product system

$$(P(t)\otimes Q(t))y'(t)+(R(t)\otimes S(t))y(t)=0 \tag{2.1}$$

is obtained and thereby establishes the general solution of the non-linear kronecker product system (1.1) using variation of parameters method. Let $y(t)=(P^T(t)\otimes Q^T(t))z(t)$. Then the transformed equation of (2.1) is of the form

$$\begin{aligned}&(P(t)P^T(t)\otimes Q(t)Q^T(t))z'(t)+[(P(t)\otimes Q(t))\cdot(P^T(t)\otimes Q^T(t))'\\ &+(R(t)P^T(t)\otimes S(t)Q^T(t))]z(t)=0.\end{aligned}$$

Since $P(t)P^T(t)\otimes Q(t)Q^T(t)$ is non-singular, follows that

$$\begin{aligned}z'(t) &= -(P(t)P^T(t)\otimes Q(t)Q^T(t))^{-1}[(P(t)\otimes Q(t))\\ &\quad\cdot(P^T(t)\otimes Q^T(t))'+(R(t)P^T(t)\otimes S(t)Q^T(t))]z(t),\end{aligned}$$

$$\text{i.e.}\qquad z'(t)=-A^{-1}(t)B(t)z(t), \tag{2.2}$$

where

$$\begin{aligned}A(t) &= (P(t)P^T(t)\otimes Q(t)Q^T(t)),\\ B(t) &= [(P(t)\otimes Q(t))\cdot(P^T(t)\otimes Q^T(t))'+(R(t)P^T(t)\otimes S(t)Q^T(t))]\end{aligned}$$

and $P^T(t)$, $Q^T(t)$ are the transposes of the matrices $P(t)$ and $Q(t)$.

Theorem 2.1. *If the system of equations* (2.1) *is consistent, then any solution of* (2.1) *is of the form* $(P^T(t)\otimes Q^T(t))\Phi(t)c$, *where* $\Phi(t)$ *is a fundamental matrix of* (2.2) *and* c *is a constant vector of order* $(m^2\times 1)$.

Proof. The transformation $y(t) = (P^T(t) \otimes Q^T(t))z(t)$ transforms (2.1) into (2.2). Since $\Phi(t)$ is a fundamental matrix of (2.2) it follows that any solution $z(t)$ is of the form $z(t) = \Phi(t)c$, where c is a constant vector of order $(m^2 \times 1)$. Hence $y(t) = (P^T(t) \otimes Q^T(t))\Phi(t)c$.

Theorem 2.2. *A particular solution* $\bar{y}(t)$ *of* (1.1), *is of the form*

$$\bar{y}(t) = (P^T(t) \otimes Q^T(t))\Phi(t) \int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds.$$

Proof. The transformation $y(t) = (P^T(t) \otimes Q^T(t))z(t)$ transforms the equation (1.1) into

$$z'(t) + A^{-1}(t)B(t)z(t) = A^{-1}(t)f(t, (P^T(t) \otimes Q^T(t))z(t)). \tag{2.3}$$

Now we seek a particular solution of (2.3) in the form $\bar{z}(t) = \Phi(t)K(t)$. Then $\Phi'(t)K(t) + \Phi(t)K'(t) + A^{-1}(t)B(t)\Phi(t)K(t) = A^{-1}(t)f(t, (P^T(t) \otimes Q^T(t))z(t))$.

$$\begin{aligned}
\Leftrightarrow \Phi(t)K'(t) &= A^{-1}(t)f(t, (P^T(t) \otimes Q^T(t))z(t)) \\
\Leftrightarrow \qquad K'(t) &= \Phi^{-1}(t)A^{-1}(t)f(t, (P^T(t) \otimes Q^T(t))z(t)) \\
K(t) &= \int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} \\
&\qquad \cdot f(s, (P^T(s) \otimes Q^T(s))z(s))ds.
\end{aligned}$$

Hence a particular solution of (2.3) is given by

$$\bar{z}(t) = \Phi(t) \int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, (P^T(s) \otimes Q^T(s))z(s))ds.$$

And hence a particular solution of (1.1) is of the form

$$\begin{aligned}
\bar{y}(t) &= (P^T(t) \otimes Q^T(t))\Phi(t) \\
&\quad \cdot \int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1}(s) f(s, y(s))ds.
\end{aligned}$$

Theorem 2.3. *Any solution of* (1.1) *is of the form* $y(t) = (P^T(t) \otimes Q^T(t))\Phi(t)c + \bar{y}(t)$, *where* $\bar{y}(t)$ *is a particular solution of* (1.1) *and is given by*

$$\bar{y}(t)=(P^T(t)\otimes Q^T(t))\Phi(t)\int_a^t \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}f(s,y(s))ds.$$

3. Existence and Uniqueness of Solutions to Boundary Value Problems. In this section, we obtain our main result on existence and uniqueness of solutions associated with the kronecker product two point boundary value problem in terms of an integral equation involving Green's matrix.

Definition 3.1. If $(P^T(t) \otimes Q^T(t))\Phi(t)c$ is a fundamental matrix of (1.1), then the matrix D defined by $D = (M_1 \otimes N_1)(P^T(a) \otimes Q^T(a))\Phi(a) + (M_2 \otimes N_2)(P^T(b) \otimes Q^T(b))\Phi(b)$ is called the characteristic matrix for the kronecker product boundary value problem (1.1) and (1.2).

Definition 3.2. The dimension of the solution space of the kronecker product boundary value problem is the index of compatibility of the problem. A kronecker product boundary value problem is said to be incompatible if its index of compatibility is zero.

Theorem 3.1. *Suppose the kronecker product homogeneous two-point boundary value problem is incompatible and their exists a constant* K *such that*

$$||f(t,y_1) - f(t,y_2)|| \leq K||(y_1 - y_2)|| \quad \textit{(Lipschitz condition)}$$

for all (t, y_1), $(t, y_2) \in [a, b] \times R^{n^2}$ *and a constant* $M > 0$ *such that* $||G(t, s)|| \leq M$ *and further suppose that* $MK(b - a) < 1$. *Then there exists a unique* *solution of the kronecker product two-point boundary value problem* (1.1) & (1.2).

Proof. From theorems (2.2) and (2.3), any solution of (1.1) is of the form

$$\begin{aligned} y(t) &= (P^T(t) \otimes Q^T(t))\Phi(t)c + (P^T(t) \otimes Q^T(t)) \\ &\cdot\Phi(t)\int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds. \end{aligned}$$

Substituting the general form of $y(t)$ in the matrix boundary condition (1.2), we get,

$$\begin{aligned} &(M_1P^T(a) \otimes N_1Q^T(a))\Phi(a)c + (M_2P^T(b) \otimes N_2Q^T(b))\Phi(b)c \\ &\quad +(M_2P^T(b) \otimes N_2Q^T(b))\Phi(b)\int_a^b \Phi^{-1}(s) \\ &\qquad \Big(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s)\Big)ds = \alpha, \\ c &= D^{-1}\alpha - D^{-1}(M_2P^T(b) \otimes N_2Q^T(b)) \\ &\cdot\Phi(b)\int_a^b \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds, \end{aligned}$$

where $D = (M_1P^T(a) \otimes N_1Q^T(a))\Phi(a) + (M_2P^T(b) \otimes N_2Q^T(b))\phi(b)$.

Substituting the form of c in the general solution of $y(t)$ in (1.1), we get

$$\begin{aligned} y(t) &= (P^T(t) \otimes Q^T(t))\Phi(t)D^{-1}\alpha - (P^T(t) \otimes Q^T(t))\Phi(t)D^{-1} \\ &\cdot(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\int_a^b \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds \\ &+(P^T(t)\otimes Q^T(t)\Phi(t)\int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds. \\ &= (P^T(t) \otimes Q^T(t))\Phi(t)D^{-1}\alpha + (P^T(t) \otimes Q^T(t))\Phi(t) \\ &\quad \cdot[-D^{-1}(M_2P^T(b) \otimes N_2Q^T(b))\Phi(b) + I] \\ &\quad \cdot\int_a^t \Phi^{-1}(s)(P(s)P^T(s) \otimes Q(s)Q^T(s))^{-1} f(s, y(s))ds \\ &\quad -(P^T(t) \otimes Q^T(t))\Phi(t)D^{-1}(M_2P^T(b) \otimes N_2Q^T(b))\Phi(b) \end{aligned}$$

$$
\begin{aligned}
&\cdot\int_t^b \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}f(s,y(s))ds\\
&+(P^T(t)\otimes Q^T(t))\Phi(t)\int_a^t \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}f(s,y(s))ds.\\
= \;& (P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}\alpha+(P^T(t)\otimes Q^T(t))\Phi(t)\\
&\cdot D^{-1}(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)\\
&\cdot\int_a^t \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}f(s,y(s))ds\\
&-(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\\
&\cdot\int_a^b \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}f(s,y(s))ds.\\
= \;& \int_a^b G(t,s)f(s,y(s))ds+(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}\alpha,
\end{aligned}
$$

where $G(t,s)$ the Green's matrix, is given by

$$
G(t,s)=\begin{cases}
(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)\Phi^{-1}(s)\\
(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \qquad a\le s<t\le b,\\
\\
-(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\Phi^{-1}(s)\\
(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \qquad a\le t<s\le b
\end{cases}
$$

Let S be a closed subset of a Banach space B. Define an operator $H:S\to S$ by

$$
H(y^{(i)}(t))=\int_a^b G(t,s)f(s,y^{(i-1)}(s))ds+(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}\alpha.
$$

Then

$$
\begin{aligned}
&||H(y^{(i)}(t))-H(y^{(i-1)}(t))||\\
\le\;& \int_a^b ||G(t,s)||\Big[||f(s,y^{(i-1)}(s))-f(s,y^{(i-2)}(s))||\Big]ds.\\
\le\;& MK||y^{(i-1)}(s)-y^{(i-2)}(s)||(b-a)\\
&\quad\vdots\\
\le\;& M^{(i-1)}K^{(i-1)}(b-a)^{(i-1)}||y^{(1)}(s)-y^{(0)}(s)||
\end{aligned}
$$

where M, K are positive constants.

Thus if $MK(b-a) < 1$, H is a contraction operator. Hence by the Banach fixed point theorem, H has a unique fixed point and this fixed point is the unique solution of the two-point kronecker product boundary value problem (1.1) and (1.2).

Theorem 3.2. *The Green's matrix $G(t,s)$ has the following properties:*

(i) *The components of $G(t,s)$ regarded as functions of t with s fixed have continuous first derivatives everywhere except at $t=s$. At the point $t=s$, G has an upward jump-discontinuity of magnitude $(P^T(t)\otimes Q^T(t))(P(t)P^T(t)\otimes Q(t)Q^T(t))^{-1}$. i.e.,*

$$G(s^+,s)-G(s^-,s)=(P^T(s)\otimes Q^T(s))(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}.$$

(ii) *$G(t,s)$ is a formal solution of the kronecker product homogeneous boundary value problem* (2.1) *satisfying* (1.2). *G fails to be a true solution because of its discontinuity at $t=s$.*

(iii) *$G(t,s)$ satisfying properties* (i) *and* (ii) *is unique.*

Proof. For fixed s define $G(t,s)$ as

$$G(t,s)=\begin{cases}(P^T(t)\otimes Q^T(t))\Phi(t)H_+, & a\le s<t\le b,\\ (P^T(t)\otimes Q^T(t))\Phi(t)H_-, & a\le t<s\le b\end{cases}$$

where H_+ and H_- are free from t. Therefore the components of $G(t,s)$ have continuous first derviatives with respect to t on each of the intervals $[a,s)$ and $(s,b]$. Further

$$\begin{aligned}&H_+-H_-\\ =\;& D^{-1}(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}\end{aligned}$$

$$
\begin{aligned}
& +D^{-1}(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \\
= \; & D^{-1}[(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)+(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)] \\
& \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}. \\
= \; & D^{-1}D\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}. \\
= \; & \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& G(s^+,s)-G(s^-,s) \\
= \; & (P^T(s)\otimes Q^T(s))\Phi(s)H_+-(P^T(s)\otimes Q^T(s))\Phi(s)H_- \\
= \; & (P^T(s)\otimes Q^T(s))\Phi(s)[H_+-H_-] \\
= \; & (P^T(s)\otimes Q^T(s))\Phi(s)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \\
= \; & (P^T(s)\otimes Q^T(s))(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}.
\end{aligned}
$$

ii)The representation of $G(t,s)$ clearly shows that $G(t,s)$ is a matrix solution of the kronecker product homogeneous system (2.1) on $[a,s)$ and $(s,b]$. Now to show that $G(t,s)$ satisfies the given matrix boundary condition (1.2); we have

$$
\begin{aligned}
& (M_1\otimes N_1)G(a,s)+(M_2\otimes N_2)G(b,s) \\
= \; & (M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)H_-+(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)H_+.
\end{aligned}
$$

Since $(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)=D-(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)$, we have

$$
\begin{aligned}
& (M_1\otimes N_1)G(a,s)+(M_2\otimes N_2)G(b,s) \\
= \; & [D-(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)]H_-+(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)H_+ \\
= \; & DH_-+(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)[H_--H_+] \\
= \; & DH_-+(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \\
= \; & -DD^{-1}(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1} \\
& +(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)\Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}
\end{aligned}
$$

$= \; 0.$

Thus G is a formal solution of the homogeneous kronecker product boundary value problem.

iii) Now to prove that G is unique, let $G_1(t,s)$ and $G_2(t,s)$ be continuous matrices with properties (i), (ii). Write

$H(t,s) = G_1(t,s) - G_2(t,s)$. Clearly G is continuous on $[a,s)$ and $(s,b]$, and H satisfies the kronecker product homogeneous system (2.1) on $[a,s)$ and $(s,b]$. At the point $t = s$.

$$\begin{aligned} H(s^+,s) - H(s^-,s) &= G_1(s^+,s) - G_2(s^+,s) - G_1(s^-,s) + G_2(s^-,s) \\ &= [G_1(s^+,s) - G_1(s^-,s)] - [G_2(s^+,s) - G_2(s^-,s)] = 0 \end{aligned}$$

Therefore H has a removable discontinuity at $t = s$. By defining H appropriately at this point, we ensure that it is continuous for all $t \in [a,b]$. Since the matrix boundary condition is linear and H is a linear combination of G_1 and G_2, we have

$$(M_1 \otimes N_1)H(a,s) + (M_2 \otimes N_2)H(b,s) = 0$$

i.e., $$(M_1 \otimes N_1)[G_1(a,s) - G_1(a,s)] + (M_2 \otimes N_2)[G_2(b,s) - G_2(b,s)] = 0.$$

Since H is a solution of (2.1), it satisfies the homogeneous matrix boundary condition and from our inital assumption that the homogeneous two point boundary value problem has only a trivial solution, it follows that $H(t,s) = 0$. i.e., $G_1(t,s) - G_2(t,s) = 0$ implies $G_1(t,s) = G_2(t,s)$. Thus G is unique.

Example 3.1. Consider the boundary value problem

$$(P \otimes Q)y' + (R \otimes S)y = f(t, y(t)) \tag{3.1}$$

$$(M_1 \otimes N_1)y(a) + (M_2 \otimes N_2)y(b) = 0, \tag{3.2}$$

where

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \\ S &= \begin{bmatrix} -1 & 0 & 0 \\ -3 & 2 & 0 \end{bmatrix}, \; M_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \\ N_1 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; N_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Now the transformation $y(t) = (P^T \otimes Q^T)z(t)$ transforms the equation (3.1) into

$$Az' + Bz = 0 \tag{3.3}$$

where

$$\begin{aligned} A &= PP^T \otimes QQ^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ B &= (P \otimes Q)(P^T \otimes Q^T)' + (RP^T \otimes SQ^T) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 6 & -9 \end{bmatrix}. \\ \text{i.e. } z' &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -6 & 9 \end{bmatrix} z. \end{aligned}$$

Now the fundamental matrix $\Phi(t)$ for the system (3.3) is given by

$$\Phi(t) = \begin{bmatrix} e^t & e^{2t} & 0 & 0 \\ e^t & 2e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & e^{6t} \\ 0 & 0 & e^{3t} & 2e^{6t} \end{bmatrix}.$$

The characteristic matrix D is given by

$$\begin{aligned} D &= (M_1 \otimes N_1)(P^T \otimes Q^T)\Phi(0) + (M_2 \otimes N_2)(P^T \otimes Q^T)\Phi(1) \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 4 & 12 & 1 & 2 \\ -1 & -1 & 0 & 0 \\ -1 & -2 & 16 & 192 \end{bmatrix} \text{ and } D^{-1} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & Q \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{where } A &= \frac{160}{1409},\ B=\frac{-176}{1409},\ C=\frac{-2114}{1409},\ D=\frac{1}{1409},\ E=\frac{-160}{1409}, \\ F &= \frac{176}{1409},\ G=\frac{705}{1409},\ H=\frac{-1}{1409},\ I=\frac{1538}{1409},\ J=\frac{-1}{1409},\ K=\frac{4}{1409}, \\ L &= \frac{-8}{1409},\ M=\frac{-129}{1409},\ N=\frac{1}{1409},\ P=\frac{-4}{1409},\ Q=\frac{8}{1409}. \end{aligned}$$

Now the solution will be in the form

$$\begin{aligned} y(t) &= \int_0^1 G(t,s)f(s,y(s))ds \\ &\text{where } G(t,s) = [a_{ij}]_{9\times 4}, \qquad a \le s < t \le b \end{aligned}$$

and are given by

$$\begin{aligned} a_{11} &= -2(C+D)e^{t-s}+(C+2D)e^{t-2s}-4(G+H)e^{2t-s}+2(G+2H)e^{2t-2s}, \\ a_{12} &= (C+D)e^{t-s}-(C+2D)e^{t-2s}+2(G+H)e^{2t-s}-2(G+2H)e^{2t-2s}, \\ a_{13} &= (2A+2B)e^{t-3s}-(A+2B)e^{t-6s}+4(E+F)e^{2t-3s}-2(E+2F)e^{2t-3s}, \\ a_{14} &= -(A+B)e^{t-3s}+(A+2B)e^{t-6s}-2(E+F)e^{2t-3s}+2(E+2F)e^{2t-6s}, \\ a_{21} &= -2(C+D)e^{t-s}+(C+2D)e^{t-2ts}-2(G+H)e^{2t-s}+(G+2H)e^{2t-2s}, \\ a_{22} &= (C+D)e^{t-s}-(C+2D)e^{t-2s}+(G+H)e^{2t-s}-(G+2H)e^{2t-2s}, \\ a_{23} &= 2(A+B)e^{t-3s}-(A+2B)e^{t-6s}+2(E+F)e^{2t-3s}-(E+2F)e^{2t-6s}, \\ a_{24} &= -(A+B)e^{t-3s}+(A+2B)e^{t-6s}-(E+F)e^{2t-3s}+(E+2F)e^{2t-6s}, \\ a_{ij} &= 0,\ i=3,\ j=1,2,3,4. \\ a_{41} &= -2(k+L)e^{3t-s}+(k+2L)e^{3t-2s}-4(P+Q)e^{6t-s}+2(P+2Q)e^{6t-2s}, \end{aligned}$$

$$
\begin{aligned}
a_{42} &= (k+L)e^{3t-s}+(k+2L)e^{3t-2s}+2(P+Q)e^{6t-s}-2(P+2Q)e^{6t-2s},\\
a_{43} &= 2(I+J)e^{3t-3s}-(I+2J)e^{3t-6s}-4(M+N)e^{6t-3s}-2(M+2N)e^{6t-6s},\\
a_{44} &= -(I+J)e^{3t-3s}+(I+2J)e^{3t-6s}-2(M+N)e^{6t-3s}+2(M+2N)e^{6t-6s},\\
a_{51} &= -2(k+L)e^{3t-s}+(k+2L)e^{3t-2s}-2(P+Q)e^{6t-s}+(P+2Q)e^{6t-2s},\\
a_{52} &= (k+L)e^{3t-s}-(k+2L)e^{3t-2s}+(P+Q)e^{6t-s}-(P+2Q)e^{6t-2s},\\
a_{53} &= 2(I+J)e^{3t-3s}-(I+2J)e^{3t-6s}+2(M+N)e^{6t-3s}-(M+2N)e^{6t-6s},\\
a_{54} &= -(I+J)e^{3t-3s}+(I+2J)e^{3t-6s}+(M+N)e^{6t-3s}+(M+N)e^{6t-6s},\\
a_{ij} &= 0,\ i=6,7,8,9.\ j=1,2,3,4.
\end{aligned}
$$

And $G(t,s)=[b_{ij}]_{9\times 4}$, $a\le t<s\le b$,

where
$$
\begin{aligned}
b_{11} &= 8Be^{t-s}-12Be^{t-2s}+10Fe^{2t-s}-24Fe^{2t-2s},\\
b_{12} &= -4Be^{t-s}-12Be^{t-2s}-8Fe^{2t-s}-24Fe^{2t-2s},\\
b_{13} &= 32De^{t-3s}-192De^{t-6s}+64He^{2t-3s}-384e^{2t-6s},\\
b_{14} &= -16De^{t-3s}+192De^{t-6s}-32He^{2t-3s}+384e^{2t-6s},\\
b_{21} &= 8(B+F)e^{t-s}-12(B+F)e^{t-2s},\\
b_{22} &= -4(B+F)e^{t-s}-12(B+F)e^{t-2s},\\
b_{23} &= 32(D+H)e^{t-3s}-192(D+H)e^{t-6s},\\
b_{24} &= -16(D+H)e^{t-3s}+192(D+H)e^{t-6s},\\
b_{ij} &= 0,\ i=3,\ j=1,2,3,4,\\
b_{41} &= 8Je^{3t-s}-12Je^{3t-2s}+16Ne^{3t-s}-24Ne^{6t-2s},\\
b_{42} &= -4Je^{3t-s}-12Je^{3t-2s}-8Ne^{6t-s}-24Ne^{6t-2s},\\
b_{43} &= 32Le^{3t-3s}-192Le^{3t-6s}+64Qe^{6t-3s}-384Qe^{6t-6s},\\
b_{44} &= -16Le^{3t-3s}+192Le^{3t-6s}-32Qe^{6t-3s}+384Qe^{6t-6s},\\
b_{51} &= 8Je^{3t-s}-12Je^{3t-2s}+8Ne^{6t-2s}-12Ne^{6t-2s},\\
b_{52} &= -4Je^{3t-s}-12Je^{3t-2s}-4Ne^{6t-s}-12Ne^{6t-2s},\\
b_{53} &= 32Le^{3t-3s}-192e^{3t-6s}+32Qe^{6t-3s}-192Qe^{6t-6s},
\end{aligned}
$$

$$\begin{aligned} b_{54} &= -16Le^{3t-3s} + 192Le^{3t-6s} - 16Qe^{6t-3s} + 192Qe^{6t-6s}, \\ b_{ij} &= 0,\ i = 6,7,8,9.\ j = 1,2,3,4. \end{aligned}$$

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Department of Applied Mathematics, Andhra University Visakhapatnam-530 003, India.