BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 30, Number 3, September 2002

GROWTH ESTIMATES OF ENTIRE HARMONIC FUNCTION IN R^N, $N \geq 2$

BY

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Abstract. Let h be a harmonic function on R^N. Then there exists an entire function f on C such that $f(u) = h(u, 0, \cdots, 0)$ for all real u. The results connecting the (p, q)-order and generalized (p, q)-type in terms of $H_m(e^*)$, of f to that of h have been proved. Finally, we have obtained a saturation theorem for h which can be extened to an entire harmonic function of (p, q)-order 0 or 1 and for entire harmonic functions of minimal generalized (p, q)-type. All these results also have been characterized in terms of derivatives of h.

1. Introduction. If h is a function that is harmonic on the whole of the Euclidean space R^N, $N \geq 2$, then there is a unique entire (holomorphic) function f on the complex plane C such that $f(u) = h(u, 0, \cdots, 0)$ for all real u. This fact has been used to deduce theorems for hormonic function on R^N from classical results about entire functions [5, 12]. The space of functions that are harmonic on R^N is denoted by $\aleph_N$ and the space of entire functions on C is denoted by $\mathcal{E}$. If $f \in \aleph_N$ (respectively $\mathcal{E}$), then we write $M_\infty(f, r)$ for the maximum value of $|f|$ on the sphere of radius r centred at otigin.

Let $\aleph_{m,N}$ denote the vector space of all homogenous harmonic polynomials of degree m on R^N. Suppose that $h \in \aleph_N$. Then h has a unique expansion

Received by the editors March 29, 2001 and in revised form July 13, 2001.

AMS 1991 Subject Classification: Primary 30E10; Secondary 41A10.

Key words and phrases: Entire harmonic functions, index-pair, Euclidean space, m-gradient, proximate order, slow and fast growth.

of the form $h = \sum_{j=0}^{\infty} H_j$, where $H_j \in \aleph_{j,N}$ such that the series $\sum_{j=0}^{\infty} |H_j|$ is locally uniformly convergent on $R^N(1, p, 84)$. We then say $\sum_{j=0}^{\infty} H_j$ is the polynomial expansion of h. Write e^* for the vector $(1, 0, \cdots, 0)$ in R^N, we have

$$h(ue^*) = \sum_{j=0}^{\infty} H_j(ue^*) = \sum_{j=0}^{\infty} H_j(e^*)u^j \qquad \text{for all real } u.$$

Let

$$f(z) = \sum_{j=0}^{\infty} H_j(e^*)z^j. \tag{1.1}$$

The power series converges for all real numbers and hence all complex z, so $f \in \mathcal{E}$.

The concepts of index-pair (p, q) $p \geq q \geq 1$ and (p, q)-order, $(p\text{-}q)$-type etc. of an entire function were introduced by Juneja et al. ([6], [7]). Thus if we denote by $\log^{[p]} x$ the quantity $\log \log \cdots \log x$, where logarithm is taken p times, then an entire function f is said to be of (p, q)-order $\rho(p, q)$ if it is of index-pair (p, q) such that

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_\infty(f, r)}{\log r} = \rho(p, q),$$

and the function f having (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ is said to be of (p, q)-type $T(p, q)$ if

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_\infty(f, r)}{(\log^{[q-1]} r)^{\rho(p,q)}} = T(p, q), \qquad 0 \leq T(p, q) \leq \infty,$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$.

Definition. An entire function $f(z)$ is said to have index-pair (p, q), $p \geq q \geq 1$, if $b < \rho(p, q) < \infty$ and $\rho(p - 1, q - 1)$ is not a non zero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. If $\rho(p, q)$ is never greater than 1 and $\rho(p', p') = 1$ for some integer $p' \geq 1$ than the index - pair of $f(z)$ is defined as (m, m) where $m = \inf\{p : \rho(p', p') = 1\}$. If $\rho(p, q)$ is never non zero finite and $\rho(p'', 1) = 0$ for some integer $p'' \geq 1$ then the index-pair of $f(z)$ is defined as $(n, 1)$ where $n = \inf\{p'' : \rho(p'', 1) = 0\}$. If $\rho(p, q)$ is always infinite then the index - pair of $f(z)$ is defined to be (∞, ∞) if $f(z)$ has the index-pair (p, q) then $\rho = \rho(p, q)$ is called its (p, q) order.

Nandan et al. [11] has extended the idea of proximate order to entire functions with (p, q)-growth as follows.

A positive function $\rho_{p,q}(r)$ defined on (r_0, ∞), $r_0 > \exp^{[q-1]} 1$, is said to be of the proximate order of an entire function with index-pair (p, q) if

(i) $\rho_{p,q}(r) \to \rho(p, q)$ as $r \to \infty$, $b < \rho(p, q) < \infty$,

(ii) $\Delta_{[q]}(r)\rho'_{p,q}(r) = 0$ as $r \to \infty$, $\rho'_{p,q}(r)$ denotes the derivative of $\rho_{p,q}(r)$.

It is known [11] that $(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$, where $A = 1$ if $(p, q) = (2, 2)$ and $A = 0$ otherwire.

Hence, we can define the function $\phi(x)$ for $x > x_0$ to be the unique solution of the equaion,

$$x = (\log^{[q-1]} r)^{\rho_{p,q}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r.$$

Let f be an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ such that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_\infty(f, r)}{(\log^{[q-1]} r)^{\rho_{p,q}(r)}} = T^*(p, q), \qquad 0 \leq T^*(p, q) \leq \infty.$$

If the quantity $T^*(p, q)$ is different from zero and infinite then $\rho_{p,q}(r)$ is said to be the proximate order of a given entire function f and $T^*(p, q)$ as its generalized (p, q)-type. Clearly, proximate order and corresponding generalized (p, q)-type of f are not uniquely determined. For example, if we add $c/\log^{[q]} r$, $0 < c < \infty$, to the proximate order $\rho_{p,q}(r)$ then $\rho_{p,q}(r) + c/\log^{[q]} r$ is also a proximate order satifying (i) & (ii) and consequently, the generalized (p, q)-type turns to be $e^c T^*(p, q)$.

An entire function f with index-pair(p, q) is said to be of minimal, normal and maximal (p, q)-type with respect to a proximate order according as $T^*(p, q)$ is zero, positive finite and infinite respectively.

Fugard [4] connected classcal order and type of an entire harmonic function h in terms of its derivatives at the origin in R^N, $N \geq 2$. These results obviously leave a big class of entire harmonic functions, such as entire harmonic functions of slow growth or of fast growth etc. Also it has been noticed that these results fail to compare the derivatives of those h which have same positive finite order but their types are infinity. In order to include this important class of functions we shall utilize the concept of proximate order due to Levin [10]. In this paper we extend the results of Fugard [4] in terms of $H_m(e^*)$ and derivatives of h in R^N, $N \geq 2$. Moreover, for the inclusion of those h of slow growth and fast growth these results will also be extended to (p, q)-scale introduced by Juneja et al. [6,7]. It is significant to mention that Shah [13] and Kapoor and Nautiyal [8] have studied the results for (α, β)-growth. However, for entire functions of slow growth Shah's results fail to exist and for such functions separate studies has been done. That's why in our studies the (p, q)-growth has been preferred to (α, β)-growth. Our method is different from those of Fugard [4].

The text has been devided in three parts, Section 1, contains the introductory exposition of the topic and in Section 2, the results have been obtained in terms of $H_m(e^*)$. A saturation theorem has been proved for (p, q)-order 0 or 1 and of minimal generalized (p, q)-type. Finally, in Section 3, we obtained the results in terms of the derivatives of h in R^N.

We shall use the following notations throughout the paper.

Notations:

$$\exp^{[m]} x = \log^{[-m]} x = \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x), m = \pm 1, \pm 2, \cdots,$$

$$\Delta_{[r]}x = \prod_{i=0}^{r} \log^{[i]} x \qquad \text{for } r = 0, 1, \cdots,$$

$$P(L(p,q)) = \begin{cases} L(p,q) & \text{if } q < p < \infty, \\ 1 + L(p,q) & \text{if } p = q = 2, \\ \max(1, L(p,q)) & \text{if } 3 \le p = q, \\ \infty & \text{if } p = q = \infty, \end{cases}$$

It is more convenient to measure growth in terms of L^2-norm defined by

$$M_2(f,r) = \Big(\int_S |f(rx)|^2 d\sigma(x) \Big)^{1/2},$$

where S is the unit sphere in R^N (or the unit circle in C) and σ is $(N-1)$-dimensional surface measure (or length measure) normalized so that $\sigma(s) = 1$. The value of order and type are uneffected if $M_\infty(f,r)$ is replaced by $M_2(f,r)$ in their definitions ([4, Lemma 2.2] for harmonic function h).

2. In this section, we have generalized the concepts of index-pair (p,q), $p \ge q \ge 1$ to entire harmonic functions on R^N (instead of entire holomorphic functions on C), and the results were characterized in terms of $H_m(e^*)$.

Theorem 1. *If $h \in \aleph_N$ can be extended to an entire function with index - pair (p,q), (p,q)-order $\rho(p,q)(b < \rho(p,q) < \infty)$, generalized (p,q)-type $T^*(p,q)$, then for every $H_m(e^*)$, there exists a unique entire function*

$$f(u) = h(ue^*) = \sum_{m=0}^{\infty} H_m(e^*) u^m \qquad \textit{such that} \tag{2.1}$$

$$\rho(p,q,h) = \rho(p,q,f) \qquad \textit{and} \quad T^*(p,q,h) = \beta T^*(p,q,f),$$

where $\beta = 2^{-\rho(p,1)}$ *for* $q = 1$ *and* $\beta = 1$ *for* $q > 1$.

Proof. Let

$$f(z)=\sum_{m=0}^{\infty}H_m(e^*)z^m. \tag{2.2}$$

The power series converges for all real numbers and hence all complex z, so $f\in\mathcal{E}$ and clearly (2,1) holds for all real u, The uniqueness of f is also clear, since entire functions that agree on the real line are identical.

We have by Brelot and Choquet [2, Prop. 4]

$$|H_m(e^*)|\le\sqrt{d_m}r^{-m}M_2(H,r)\qquad(H\in\aleph_{m,N},r>0), \tag{2.3}$$

where $d_m=\dim\aleph_{m,N}$. The spaces $\aleph_{m,N}$ are mutualy orthogonal in the sense

$$\begin{aligned}&\int_S H_m(rx)H_n(rx)d\sigma(x)\\ =&\left.\begin{cases}0 & \text{for } m\neq n\\ \sqrt{d_m}|H_m(e^*)| & \text{for } m=n\end{cases}, r>0, H_m\in\aleph_{m,N}, H_n\in\aleph_{n,N}\right\}\end{aligned}$$

[1, pp.75]. Since the series $h=\sum_{j=0}^{\infty}H_j$ converges uniformly on every sphere, we have

$$M_2^2(h,r)=\sum_{j=0}^{\infty}M_2^2(H_{j,r})\quad(r>0). \tag{2.4}$$

Using (2.3) and (2.4), we get

$$|H_m(e^*)|\le\sqrt{d_m}r^{-m}M_2(h,r). \tag{2.5}$$

Using (2.5) in the power series expansion of $f(z)$, we obtain

$$M_\infty(f,r)=\sum_{m=0}^{\infty}|H_m(e^*)|r^m\le M_2(h,r)\sum_{m=0}^{\infty}\sqrt{d_m} \tag{2.6}$$

Since $d_m\to\frac{2m^{(N-2)}}{(N-2)!}$ as $m\to\infty$, it follows that there is a constant $K_0=$

$K_0 (h \in N)$ such that

$$M_\infty(f,r) \leq M_2(h,r) \sum_{m=0}^{\infty} K_0 m^{(N-2)/2} \quad (m \geq 1, r > 0)$$

or

$$\log M_\infty(f,r) \leq o(1) + \log M_2(h,r). \tag{2.7}$$

Thus in view of above inequality with definition of (p,q)-order and generalized (p,q)-type for $p \geq 2$ and $q = 1$, we get

$$\text{and} \quad \begin{array}{ccc} \rho(p,1,f) \leq \rho(p,1,h) & \text{and} & T^*(p,1,f) \leq 2^{\rho(p,1)} T^*(p,1,h) \\ \text{for } p \geq 2 \text{ and } q > 1, & & \\ \rho(p,q,f) \leq \rho(p,q,h) & \text{and} & T^*(p,q,f) \leq T^*(p,q,h). \end{array} .$$

Combining both inequalities for all (p,q), we have

$$\rho(p,q,f) \leq \rho(p,q,h) \qquad \text{and } \beta T^*(p,q,f) \leq T^*(p,q,h). \tag{2.8}$$

Now we have

$$\begin{aligned} h(xe^*) &= \sum_{m=0}^{\infty} H_m(xe^*), \\ M(h,r) &= \max_{|x|=r} |h(xe^*)| \leq \sum_{m=0}^{\infty} \|H_m(e^*)\|_\infty r^m \leq \sum_{m=0}^{\infty} \sqrt{d_m} \|H_m(e^*)\|_2 r^m \\ &\leq \sum_{m=0}^{\infty} |H_m(e^*)| r^m = M_\infty(f,r). \end{aligned}$$

Thus

$$M(h,r) \leq M_\infty(f,r). \tag{2.9}$$

We observe that $q = 1$,

$$\rho(p,1,h) \leq \rho(p,1,f) \qquad \text{and } T^*(p,1,h) \leq 2^{-\rho(p,1)} T^*(p,1,f)$$

and for $q > 1$,

$$\rho(p,q,h) \le \rho(p,q,f) \qquad \text{and } T^*(p,q,h) \le T^*(p,q,f).$$

Hence for all index-pair (p,q),

$$\rho(p,q,h) \le \rho(p,q,f) \qquad \text{and } T^*(p,q,h) \le \beta T^*(p,1,f). \tag{2.10}$$

Combining (2.8) and (2.10), we have

$$\rho(p,q,h) = \rho(p,q,f) \qquad \text{and } T^*(p,q,h) = T^*(p,q,f).$$

Theorem 2. *Let $h \in \aleph_N$ and satisfies* (2.1)*. Then h can be extended to an entire harmonic function of (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ if and only if*

$$\rho(p,q) = P(L(p,q))$$

where

$$L(p,q) = \limsup_{m\to\infty} \frac{\log^{[p-1]} m}{\log^{[q-1]} |H_m(e^*)|^{-1/m}}.$$

Proof. We have seen that if $h \in \aleph_N$ and satifies (2.1), then f is an entire function. Moreover, by Theorem 1, h and f have the same (p,q)-order, Applying Corollary 1 to the function $f(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$ and by Juneja et al. [6, pp. 62], Theorem 2 is established.

Theorem 3. *If $h \in \aleph_N$ and satisfies* (2.1)*. Then h can be extended to an entire harmonic function of (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ and generalized (p,q)-type $T^*(p,q)$ $(0 < T^*(p,q) < \infty)$ if and only if*

$$\frac{T^*(p,q)}{\beta M(p,q)} = \limsup_{m\to\infty} \Big[\frac{\phi(\log^{[p-2]} m)}{\log^{[q-1]} |H_m(e^*)|^{-1/m}}\Big]^{\rho(p,q)-A},$$

where β is defined as in Theorem 1 *and*

$$M(p,q)=\begin{cases}(\rho(2,2)-1)^{\rho(2.2)-1}/(\rho(2.2))^{\rho(2.2)} & \textit{if } (p,q)=(2,2),\\ 1/e\rho(2,1) & \textit{if } (p,q)=(2,1),\\ 1 & \textit{otherwise.}\end{cases}$$

Proof. This Theorem follows by Kasana [9] and applying Theorem 1 to the function $f(z)$ and the resulting characterization of $T^*(p,q,f)$ in terms of $H_m(e^*)$ and the relation $T^*(p,q,h)=\beta T^*(p,q,f)$.

If we take $\rho_{p,q}(r)=\rho(p,q)r>r_0$ and $\phi(x)=x^{\Gamma/\rho(p,q)-A}$, the following corollary for (p,q)-type $T(p,q)$ in terms of $H_m(e^*)$ can be obtained.

Corollary. *Let $h\in\aleph_N$ and satifies $(2,1)$. The h can be extended to an entire harmonic function having (p,q)-order $\rho(p,q)$ ($b<\rho(p,q)<\infty$) and (p,q)-type $T(p,q)$ $(0<T(p,q)<\infty)$ if and only if*

$$\frac{T(p,q)}{\beta M(p,q)}=\limsup_{m\to\infty}\frac{\log^{[p-2]}m}{[\log^{[q-1]}|H_m(e^*)|^{-1/m}]^{\rho(p,q)-A}}.$$

Saturation Theorem. *If $h\in\aleph_N$ can be extended to an entire harmonic function of (p,q)-order $\rho(p,q)$ such that $\rho(p,q)=b$, then for every $\delta>0$,*

$$\limsup_{m\to\infty}\frac{\phi(\log^{[p-2]}m)^{\delta}}{[\log^{[q-1]}|H_m(e^*)|^{-1/m}]}=0$$

Further, if $\rho(p,q)>b$ and h is of minimal generalized (p,q)-type, then

$$\limsup_{m\to\infty}\frac{\phi(\log^{[p-2]}m)}{[\log^{[q-1]}|H_m(e^*)|^{-1/m}]}=0$$

Proof. Since $\rho(p,q)=b$, it follows from Theorem 1, by definition of

(p,q) order of $h \in \aleph_N$ for given $\varepsilon > 0$ and $r > r_0$.

$$\log M(h,r) < \exp^{[p-2]}(\log^{[q-1]} r)^{b+\varepsilon}.$$

Using (2.5) and the fact that $M(h,r)$ can be replaced by $M_2(h,r)$ [4, Lemma 2.2] in above, we get,

$$\log |H_m(e^*)|^{-1/m} < \frac{1}{2m}\log d_m - \log r + \frac{\exp^{[p-2]}(\log^{[q-1]} r)^{b+\varepsilon}}{m}. \tag{2.11}$$

Choose the value of r satisfying

$$r = \exp^{[q-1]}\Big(\log^{[p-2]}\frac{m}{b+\varepsilon}\Big)^{1/b+\varepsilon}. \tag{2.12}$$

For $(p,q) = (2,1)$, (2.11) yields $r = (m/\varepsilon)^{1/\varepsilon}$, and by using (2.10), we obtain

$$|H_m(e^*)| < \sqrt{d_m}\Big(\frac{e\varepsilon}{m}\Big)^{m/\varepsilon},$$

$$|H_m(e^*)|^{1/m} < (d_m)^{1/2m}\Big(\frac{e\varepsilon}{m}\Big)^{1/\varepsilon},$$

$$\limsup_{m\to\infty} m^{1/\varepsilon}|H_m(e^*)|^{1/m} < \infty. \tag{2.13}$$

For $(p,q) = (2,2)$ we observe that $\log r = \big(\frac{m}{1+\varepsilon}\big)^{1/(1+\varepsilon)}$ which reduces to (2.4) & (2.5) to the expression

$$\log |H_m(e^*)| < m\log\sqrt{d_m} - m\Big(\frac{m}{1+\varepsilon}\Big)^{1/(1+\varepsilon)} + \frac{m}{1+\varepsilon}$$

or

$$\begin{aligned}\log |H_m(e^*)|^{-1/m} &> \Big(\frac{m}{1+\varepsilon}\Big)^{\frac{1}{(1+\varepsilon)}}\Big[1 - \frac{m^{-1(1+\varepsilon)}}{(1+\varepsilon)^{\varepsilon/(1+\varepsilon)}} - \Big(\frac{1+\varepsilon}{m}\Big)^{\frac{1}{(1+\varepsilon)}}\log\sqrt{d_m}\Big]\\ &= \Big(1 + o(1)\Big(\frac{m}{1+\varepsilon}\Big)^{1/(1+\varepsilon)}\Big).\end{aligned}$$

Thus

$$\limsup_{m\to\infty}\frac{m^{1/(1+\varepsilon)}}{\log|H_m(e^*)|^{-1/m}}\le 1. \tag{2.14}$$

Finally, for $(p,q)\neq(2,1)$ and (2,2), (2.11) and (2.12) is reduced to

$$\log^{[q-1]}|H_m(e^*)|^{-1/m}>(1+o(1))\Big(\log^{[p-2]}\frac{m}{\varepsilon}\Big)^{1/\varepsilon},\ p>q,$$

and

$$\log^{[q-1]}|H_m(e^*)|^{-1/m}>(1+o(1))\Big(\log^{[p-2]}\frac{m}{1+\varepsilon}\Big)^{1/(1+\varepsilon)},\ p=q.$$

Thus, for all $p\ge q\ge 3$, we have

$$\limsup_{m\to\infty}\frac{(\log^{[p-2]}m)^{1/(b+\varepsilon)}}{\log^{[q-1]}|H_m(e^*)|^{-1/m}}\le 1. \tag{2.15}$$

Clearly, combining (2.13), (2.14) and (2.15) we have.

$$\limsup_{m\to\infty}\frac{(\log^{[p-2]}m)^{\delta}}{\log^{[q-1]}|H_m(e^*)|^{-1/m}}\le\infty. \tag{2.16}$$

for every $\delta>0$. If limit superior in (2.16) is finite positive for some $\delta>0$, then for every $\alpha>0$, we get

$$\limsup_{m\to\infty}\frac{(\log^{[p-2]}m)^{\delta+\alpha}}{\log^{[q-1]}|H_m(e^*)|^{-1/m}}=\infty. \tag{2.17}$$

This is a contradiction to (2.16) and hence the first part is proved.

For the second part, by taking $T^*(p,q)=0$ in Theorem 2, we obtain the required result.

3. In this section we obtain the results in terms of derivatives of h. But first, we define the m-gradient of $h(ue^*)$ which is similar to that of Fugard [4],

$$|\nabla_m h(ue^*)| = \Big[m! \sum_{|a|=m} (D^a h(ue^*))^2/a!\Big]^{1/2},$$

where for each n-tuple $a = (a_1, a_2, \cdots, a_n)$ of non-negative integers, let

$$|a| = a_1 + a_2 + \cdots + a_n,\ a! = a_1!a_2!\cdots a_n! \qquad \text{and } D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1}\partial x_2^{a_2}\cdots\partial x_n^{a_n}}.$$

Now we prove

Lemma 1. *Let $h \in \aleph_N$ and satisfies* (2.1). *Then for all $r < R$*

$$\begin{aligned}\Big[\Gamma\Big(\frac{N}{2}\Big)\sum_{m=0}^{\infty}\frac{|\nabla_m(h(ue^*)|^2}{m!\Gamma\Big(m+\frac{N}{2}\Big)}r^{2m}\Big]^{1/2} &\leq M_\infty(h,r)\\ &\leq \Big[\Gamma\Big(\frac{N}{2}\Big)\sum_{m=0}^{\infty}d_m\frac{|\nabla_m(h(ue^*)|^2}{m!\Gamma\Big(m+\frac{N}{2}\Big)}r^{2m}\Big]^{1/2}.\end{aligned}$$

Proof. By inequalities (2.6) and (2.9), we have

$$M(h,r) \leq M_\infty(f,r) \leq M_2(h,r)\sum_{m=0}^{\infty}\sqrt{d_m}.$$

This lemma follows by applying [3, Lemma 2] and by (2.1). We see that

$$\begin{aligned}b_m &= \frac{\Gamma\Big(\frac{N}{2}\Big)|\nabla_m h(ue^*)|^2}{m!\Gamma\Big(m+\frac{N}{2}\Big)}\\ \log b_m^{-1} &\sim \log\frac{m!\Gamma\Big(m+\frac{N}{2}\Big)}{|\nabla_m h(ue^*)|^2}.\end{aligned}$$

It follows by using Stiriling formula,

$$\text{(3.1)} \qquad \sim \log\Big\{\frac{m!}{|\nabla_m h(ue^*)|}\Big\}^2 \qquad \text{as } m \to \infty.$$

Also from

$$a_m = \frac{\Gamma\Big(\frac{N}{2}\Big) d_m |\nabla_m h(ue^*)|^2}{m!\Gamma\Big(m+\frac{N}{2}\Big)},$$

we have

$$\log a_m^{-1} \sim \log \frac{m!\Gamma\Big(m+\frac{N}{2}\Big)}{|\nabla_m h(ue^*)|^2}.$$

and hence

$$\sim \log \left\{ \frac{m!}{|\nabla_m h(ue^*)|} \right\}^2 \qquad \text{as } m \to \infty. \tag{3.2}$$

Remark. Theorem 1, 2, 3 and Saturation theorem can be characterized in terms of $\frac{|\nabla_m h(ue^*)|}{m!}$ in place of $H_m(e^*)$ using Lemma 1 (3.1), (3.2) and note that the growth of $M_\infty(h, r)$ is equal to that of $h(ue^*)$.

Acknowledgement. The author is extremely thankful to the referee for valuable suggestions.

References

1. S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer-Verlag, New York, 1992.

2. M. Brelot and G. Choquet, *Polynomes Harmoniques et Polyharmoniques*, Seconde Colloque sur les equations aux derives partiells, Bruxelles, 1954.

3. A. Fryant and H. Shankar, *Growth of harmonic functions in hyperspheres*, J. Math. Anal. Appl., **122**(1987),453-462.

4. T. B. Fugard, *Growth of entire harmonic functions in $R^N, n \geq 2$*, J. Math. Anal. Appl., **74**(1980), 286-291.

5. B. Grigorjan, *Uniqueness theorems for harmonic functions of three variables in a domain of rotaition*, Izv. Akad. Nauk Armjan SSR Ser. Mat., **7**(1972), 81-89 [In Russian] English Translation, Sov. Mat., **7**(1972), 81-89.

6. O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p, q)-order and lower (p, q)-order of an entire function*, J. reine Angew. Math., **282**(1976), 53-67.

7. ——, *On the (p, q)-type and lower (p, q)-type of an entire function*, J. Reine Angew. Math., **290**(1977), 180-190.

8. G. P. Kapoor and A. Nautiyal, *Polynomial approximation of an entire function of slow growth*, J. Approx. Theory, **32**(1981), 64-75.

9. H. S. Kasana, *The generalized type of entire functions with index-pair* (p, q), Comment. Math. Prac. Mat., (2)**29**(1990), 215-222.

10. B. Ja Levin, *Distribution of Zeros of Entire Functions*, Translation of Mathematical Monograph **5**, Amer. Math. Soc. Providence, R. I. 1964, Revised edition 1980.

11. K. Nandan, R. P. Doherey and R. S. L. Srivastava, *Proximate order of an entire function with index-pair* (p, q), Indian J. Pure Appl. Math., **11**(1980), 33-39.

12. N. V. Rao, *Carlson theorem for harmonic function in* R^N, J. Approx. Theory, **12**(1974), 309-314.

13. S. M. Shah, *Polynomial approximation of an entire function and generalized orders*, J. Approx. Theory, **32**(1977), 64-75.

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