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HARMONIC ANALYSIS AND SUBRIEMANNIAN GEOMETRY ON HEISENBERG GROUPS*

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1. Introduction. We shall discuss results concerning the behaviour of differential operators of the form

$$\Delta_X = X_1^2 + \cdots + X_m^2,$$

with a view to their inversion. Here $X_1, \ldots, X_n$ are linearly independent vector fields on an n-dimensional manifold M_n. Δ_X is elliptic when $m = n$. If $m < n$, but $X = \{X_1, \ldots, X_n\}$ and their brackets generate TM_n, we say that Δ_X is subelliptic. An elliptic operator is clearly subelliptic. Δ_X is said to be of step k, if the minimum number of brackets needed to generate TM_n is $k-1$. A subelliptic operator is automatically hypoelliptic by a result of Hörmander, *i.e.*, $\Delta_X u = f \in C^\infty \Rightarrow u \in C^\infty$. Interest in such operators arose in the 1950s from the study of the $\bar{\partial}$-Neumann problem in several complex variables, although analogous operators make their appearance in earlier problems from quantum mechanics.

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The most studied subelliptic operator is constructed from the vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2a_j x_{j+n}\frac{\partial}{\partial t}, \\ X_{j+n} &= \frac{\partial}{\partial x_{j+n}} - 2a_j x_j \frac{\partial}{\partial t}. \end{aligned}$$

where $a_j > 0$, $j = 1, \ldots, n$. Δ_X is step 2 subelliptic in $\mathbf{R}^{2n+1}$. X_j are left-invariant with respect to the Heisenberg translation

$$(x,t)\circ(y,s) = \left(x+y, t+s+2\sum_{j=1}^{n} a_j(x_{j+n}y_j - x_j y_{j+n}) \right),$$

where $(x,t) = (x_1, \ldots, x_{2n}, t)$ and $(y,s) = (y_1, \ldots, y_{2n}, s)$. $\mathbf{R}^{2n+1}$ with this group law is called the Heisenberg group $\mathbf{H}_n$, and

$$\Delta_{\mathbf{H}} = \sum_{j=1}^{n}(X_j^2 + X_{j+n}^2)$$

is the Heisenberg sublaplacian. $\mathbf{H}_n$ and $\Delta_{\mathbf{H}}$ play the role of testing ground for ideas and concepts that appear useful in inverting subelliptic operators on general Heisenberg manifolds, see [6]; this is similar to the role played by the Euclidean Laplacian in the study of the Laplace-Beltrami operator in Riemannian geometry. These concepts may be crudely classified under two separate headings

(1) Harmonic analysis,

(2) Geometry.

Under harmonic analysis we are mainly interested in multiplicative symbolic calculi for left-invariant convolution operators on $\mathbf{H}_n$. Of course they are equivalent and usually amount to the study of the partial Fourier transform along the center of $\mathbf{H}_n$, the group Fourier transform, although different versions are useful in different contexts. One of these versions is called the

Laguerre calculus and uses the full Euclidean Fourier transform on $\mathbf{H}_n$. This is somewhat unusual, since Fourier transforms are useful for differential operators with constant coefficients, which $\Delta_{\mathbf{H}}$ is not, except in the t-variable. In spite of this, the full Fourier tansform $\wedge$ leads to the following curious fact (on $\mathbf{H}_1$ for simplicity):

$$F *_{\mathbf{H}} \phi(x,t) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} e^{ix\cdot\xi + it\tau} \hat{F}(\xi_1 + 2x_2\tau, \xi_2 - 2x_1\tau, \tau)\hat{\phi}(\xi_1, \xi_2, \tau) d\xi d\tau,$$

that is, a convolution operator is a pseudo-differential operator, whose symbol is a function of the Heisenberg vector fields, see [15]. This remark led to a calculus for step 2 pseudo-differential operators on Heisenberg manifolds, see [6].

To understand subelliptic opertors $\Delta_{\mathbf{H}}$ in general we make use of the subriemannian geometry induced by $\Delta_{\mathbf{H}}$: metrics, distances, geodesics, real and complex Hamiltonian mechanics, action integrals, etc. One of the surprises in subriemannian geometry, $\{X\} \subsetneqq TM_n$, is the multitude of geodesic between two points, all of them playing a role in the construction of heat kernels, wave kernels and fundamental solutions for subelliptic operators.

In section 2 we show that the study of differential operators may be reduced to the study of integral operators. In section 3 we discuss the Laguerre calculus and in section 4 we shall study some geometric concepts induced by subelliptic operators.

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We hope this workshop will continue to be held every summer in Taiwan, permitting for the participants to exchange mathematical ideas. In addition, they have very useful discussions concerning research on function theory in several complex variables as well as related topics with mathematicians in Taiwan.

2. Laplace operator and elliptic estimates. Let us start with the Heaviside function:

$$H(x) = 1 \quad \text{when} \quad x > 0 \quad \text{and} \quad 0 \quad \text{when} \quad x < 0.$$

It is obvious that the function H has a jump at $x = 0$ and hence is not differentiable. However, we may define $H'(x)$ by

$$\int_{\mathbf{R}} H'(x)g(x)dx = -\int_{\mathbf{R}} H(x)g'(x)dx = -\int_0^\infty g'(x)dx = g(0)$$

where $g \in C_c^\infty(\mathbf{R})$. Therefore, $H'(x)$ is a "function" defined by

$$\int_{\mathbf{R}} H'(x)g(x)dx = g(0).$$

Let us now consider the convolution operator, for $f, g \in C_c^\infty(\mathbf{R})$,

$$f * g(x) = \int_{\mathbf{R}} f(x-y)g(y)dy.$$

Given f, $f*$ is an operator which acts on functions by convolution

$$([f*]g)(x) = \int_{\mathbf{R}} f(x-y)g(y)dy.$$

Actually, given any "generalized function", we may consider it as a convolution operator. For $g \in C_c^\infty(\mathbf{R})$,

$$g(x) = \int_{\mathbf{R}} \delta(x-y)g(y)dy$$

and

$$\frac{dg}{dx}(x) = \int_{\mathbf{R}} \delta'(x-y)g(y)dy$$

Therefore, we may define the differential operator $\frac{d}{dx}$ as follows:

$$\frac{d}{dx} = (\delta)' *$$

and

$$\int_{\mathbf{R}} \delta'(x)\phi(x)dx = -\int_{\mathbf{R}} \delta(x)\phi'(x)dx = -\phi'(0)$$

for $\phi \in C_c^\infty(\mathbf{R})$.

Example 2.1. Solve $\frac{du}{dx} = f(x)$ where $f \in C_c^\infty(\mathbf{R})$ is given. Assume

$$u(x) = \int_{\mathbf{R}} K(x-y)f(y)dy$$

where K is a kernel to be determined. Since u is a solution, we have

$$\frac{d}{dx}\int_{\mathbf{R}} K(x-y)f(y)dy = f(x).$$

This implies that

$$\int_{\mathbf{R}} K'(x-y)f(y)dy = f(x).$$

We may conclude that $K'(x) = \delta(x)$ or $K(x) = H(x)$. Hence

$$u(x) = \int_{\mathbf{R}} H(x-y)f(y)dy = \int_{-\infty}^{x} f(y)dy.$$

Example 2.2. Solve $\frac{d^2u}{dx^2} = f(x)$ where $f \in C_c^\infty(\mathbf{R})$ is given. Indeed,

$$\frac{d^u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = f(x)$$

Hence, Example 2.1 implies

$$\frac{du}{dx} = \int_{-\infty}^{x} f(y)dy,$$

and we may conclude that

$$u(x) = \int_{-\infty}^{x} dy \int_{-\infty}^{y} f(z)dz.$$

We rewrite this as

$$u(x) = \int_{-\infty}^{x} f(z)dz \int_{z}^{x} dy = \int_{-\infty}^{x} (x-z)f(z)dz.$$

It follows that

$$u = (x_+) * f.$$

Hence

$$\left(\frac{d^2}{dx^2}\right)^{-1} = (x_+) * .$$

Example 2.3. Let us now turn to several variables. Consider the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

It is well known that the Newton potential

$$\frac{C_3}{|x|} = \frac{C_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

is the fundamental solution of the operator Δ. Here C_3 is a constant. Then

$$u(x) = C_3 \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|} dy$$

is a solution of the equation $\Delta u = f$. To see this we write:

$$u(x) = C_3 \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|} dy = C_3 \int_{\mathbf{R}^3} \frac{f(x-y)}{|y|} dy.$$

Then

$$\begin{aligned}
\Delta u(x) &= C_3 \int_{\mathbf{R}^3} \frac{\Delta_x f(x-y)}{(|y|^2)^{1/2}} dy \\
&= \lim_{\varepsilon \to 0} C_3 \int_{\mathbf{R}^3} \frac{\Delta_x f(x-y)}{(|y|^2+\varepsilon^2)^{1/2}} dy \\
&= \lim_{\varepsilon \to 0} C_3 \int_{\mathbf{R}^3} \frac{\Delta_y f(x-y)}{(|y|^2+\varepsilon^2)^{1/2}} dy.
\end{aligned}$$

Now

$$\begin{aligned}
& \sum_{j=1}^{3} \int_{\mathbf{R}^3} \left(\frac{\partial^2 f}{\partial y_j^2} \right) (x-y) \frac{1}{(|y|^2+\varepsilon^2)^{1/2}} dy \\
= & \sum_{j=1}^{3} \int_{\mathbf{R}^3} f(x-y) \frac{\partial^2}{\partial y_j^2} \frac{1}{(|y|^2+\varepsilon^2)^{1/2}} dy \\
= & \int_{\mathbf{R}^3} \frac{-3\varepsilon^2}{(|y|^2+\varepsilon^2)^{5/2}} f(x-y) dy \\
\to & -4\pi f(x) \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

It follows that $C_3 = -\frac{1}{4\pi}$ and

$$\Delta^{-1} f(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{f(y)dy}{|x-y|} = u(x).$$

How do we find $-\frac{1}{4\pi|x|}$? We first consider Δ^{-1} as a convolution operator.

Consider the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbf{R}^3} e^{-ix\cdot\xi} f(x)dx$$

and the inverse Fourier transform

$$f(x) = \int_{\mathbf{R}^3} e^{ix\cdot\xi}\hat{f}(\xi)\frac{d\xi}{(2\pi)^3}.$$

Then

$$\begin{aligned}
\Delta f(x) &= \Delta \int_{\mathbf{R}^3} e^{ix\cdot\xi}\hat{f}(\xi)\frac{d\xi}{(2\pi)^3} \\
&= \sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\int_{\mathbf{R}^3} e^{i(x_1\xi_1+x_2\xi_2+x_3\xi_3)}\hat{f}(\xi)\frac{d\xi}{(2\pi)^3} \\
&= \int_{\mathbf{R}^3} e^{ix\cdot\xi}(-|\xi|^2)\hat{f}(\xi)\frac{d\xi}{(2\pi)^3}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Delta^{-1} f(x) &= \int_{\mathbf{R}^3} e^{ix\cdot\xi}\frac{\hat{f}(\xi)}{-|\xi|^2}\frac{d\xi}{(2\pi)^3} \\
&= \int_{\mathbf{R}^3} e^{ix\cdot\xi}\left(-\frac{1}{|\xi|^2}\right)\int_{\mathbf{R}^3} e^{-y\cdot\xi} f(y)dy\frac{d\xi}{(2\pi)^3} \\
&= \int_{\mathbf{R}^3} f(y)dy\int_{\mathbf{R}^3} e^{i(x-y)\cdot\xi}\left(-\frac{1}{|\xi|^2}\right)\frac{d\xi}{(2\pi)^3}.
\end{aligned}$$

Claim:

$$\frac{1}{4\pi|x|} = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} e^{ix\cdot\xi}\frac{d\xi}{|\xi|^2}.$$

Proof.

$$\begin{aligned}
\int_{\mathbf{R}^3} e^{ix\cdot\xi}\frac{d\xi}{|\xi|^2} &= \int_{\mathbf{R}^3} e^{ix\cdot\xi}d\xi\int_0^\infty e^{-s|\xi|^2}ds \\
&= \int_{\mathbf{R}^3} e^{ix\cdot\xi-s|\xi|^2}d\xi = \prod_{j=1}^{3}\int_{-\infty}^{\infty} e^{-s\xi_j^2+ix_j\xi_j}d\xi_j
\end{aligned}$$

But

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-s\xi_j^2+ix_j\xi_j}d\xi_j &= \int_{-\infty}^{\infty} e^{\left\{-s\left[\xi_j^2-2\xi_j\frac{ix_j}{2s}+\left(\frac{ix_j}{2s}\right)^2\right]-\frac{x_j^2}{4s}\right\}}d\xi_j\\ &= e^{-\frac{x_j^2}{4s}}\int_{-\infty}^{\infty} e^{-s\left(\xi-\frac{ix_j}{2s}\right)^2}d\xi\\ &= e^{-\frac{x_j^2}{4s}}\int_{-\infty}^{\infty} e^{-s\sigma^2}d\sigma = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-\frac{x_j^2}{4s}}.\end{aligned}$$

Hence,

$$\prod_{j=1}^{3}\int_{-\infty}^{\infty} e^{-s\xi_j^2+ix_j\xi_j}d\xi_j = \frac{\pi^{3/2}}{s^{3/2}}e^{-\frac{|x|^2}{4s}}.$$

Finally, we have

$$\begin{aligned}\frac{1}{(2\pi)^3}\int_0^{\infty}\frac{\pi^{3/2}}{s^{3/2}}e^{-\frac{|x|^2}{4s}}ds &= \frac{\pi^{3/2}}{(2\pi)^3}\int_0^{\infty} e^{-\frac{|x|^2u}{4}}\frac{du}{\sqrt{u}}\\ &= 2\frac{\pi^{3/2}}{(2\pi)^3}\int_0^{\infty} e^{-\frac{\sigma^2|x|^2}{4}}d\sigma\\ &= \frac{4}{|x|}\frac{\pi^{3/2}}{(2\pi)^3}\int_0^{\infty} e^{-v^2}dv\\ &= \frac{4}{|x|}\frac{\pi^{3/2}}{(2\pi)^3}\frac{\sqrt{\pi}}{2} = \frac{1}{4\pi|x|}.\end{aligned}$$

Similarly, the inverse of the n-dimensional Laplacian is given by

$$u(x) = \Delta^{-1}f(x) = \frac{1}{(2-n)\Omega_n}\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy, \qquad n > 2.$$

Here Ω_n is the surface area of the n-dimensional unit sphere. Since

$$K(x) = \frac{1}{(2-n)\Omega_n}\frac{1}{|x|^{n-2}}$$

is a locally integrable function, one has

$$\|u\|_{L^p_{loc}(\mathbf{R}^n)} \le C_p\|f\|_{L^p_{loc}(\mathbf{R}^n)}$$

for $1 \le p \le \infty$. Moreover, by the Mikhlin-Calderón-Zygmund theory of singular integrals, one has

$$\left\|\frac{\partial^2 u}{\partial x_j \partial x_\ell}\right\|_{H^p(\mathbf{R}^n)} \le C_p \|f\|_{H^p(\mathbf{R}^n)} \quad \text{and} \quad \left\|\frac{\partial^2 u}{\partial x_j \partial x_\ell}\right\|_{BMO(\mathbf{R}^n)} \le C_p \|f\|_{BMO(\mathbf{R}^n)}$$

for $0 < p < \infty$ and $1 \le j, \ell \le n$. Here $H^p(\mathbf{R}^n)$ is the real Hardy space and $BMO(\mathbf{R}^n)$ is the space of all functions with bounded mean oscillations (see Chapter 3 in Stein [22]). In other words, the solution u for the Laplace operator has "full gain" in all directions. This is an elliptic estimate. We may consider this problem in a more general setting.

Suppose one has n linearly independent vector fields $X_1, \dots, X_n$ on M_n. Then

$$\Delta_X = \sum_{j=1}^n X_j^2$$

is an elliptic differential operator. For $n > 2$,

$$\Delta_X^{-1}(f)(x) = \int_{M_n} \frac{f(y)}{(2-n)\Omega_n d^{n-2}(x,y)} + \cdots$$

where $+\cdots$ stands for negligible error. d stands for the distance and we can calculate it as follows. We define a metric by

$$X_j \perp X_k, \quad 1 \le j, k \le n, \quad j \ne k, \quad \text{and} \quad \|X_j\| = 1.$$

Let $\gamma(s)$ denote a curve connecting x and y in the time τ:

$$\gamma : [0, \tau] \to M_n, \qquad \gamma(0) = x, \quad \gamma(\tau) = y.$$

The length of γ is

$$\ell = \int_0^\tau \sqrt{\sum_{j=1}^n \sigma_j^2(s) ds},$$

where

$$\gamma'(s) = \sum_{j=1}^{n} \sigma_j(s) X_j(s),$$

and the distance between x and y is the minimum of the lengths of such curves.

The Hamiltonian formalism yields a slightly different way of calculating distances. For illustration, we restrict ourselves to $\mathbf{R}^3$ and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

The Hamiltonian function is

$$H = \xi_1^2 + \xi_2^2 + \xi_3^2$$

and Hamilton's equations for the bicharacteristics are

$$\dot{x}_j(s) = \frac{\partial H}{\partial \xi_j} \quad \text{and} \quad \dot{\xi}_j(s) = -\frac{\partial H}{\partial x_j},$$

where $s \in [0, \tau]$, with $x(0) = 0$ and $x(\tau) = x$. Then

$$\dot{x}_j(s) = 2\xi_j \quad \text{and} \quad \dot{\xi}_j(s) = 0$$

This implies that $\xi_j(s) = C_j$, a constant. Hence

$$\begin{aligned} \dot{x}_j(s) = 2C_j \quad &\Rightarrow \quad x_j(s) = 2C_j s + d_j, \\ x_j(0) = 0 \quad &\Rightarrow \quad d_j = 0, \\ x_j(\tau) = x_j \quad &\Rightarrow \quad C_j = \frac{x_j}{2\tau}, \end{aligned}$$

so

$$x_j(s) = \frac{x_j}{\tau} s, \qquad \xi_j(s) = \frac{x_j}{2\tau}.$$

The projection of the bicharacteristic on the base is called the geodesic, *i.e.*, curve with shortest, or extremal, length between x and y. According to our calculations they are straight lines, which we all know. The action integral is

$$\begin{aligned} S &= \int_0^\tau \left[\sum_{j=1}^{3} \xi_j(s)\dot{x}_j(s) - H((x(s),\xi(s)) \right] ds \\ &= \int_0^\tau \left[\frac{|x|^2}{2\tau^2} - \frac{|x|^2}{4\tau^2} \right] ds \\ &= \frac{1}{4}\frac{|x|^2}{\tau}, \end{aligned}$$

which yields the distance and is the solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + H(x, \bigtriangledown S) = 0.$$

Let us look at the heat kernel for the Euclidean Laplacian Δ in $\mathbf{R}^3$. Since Δ is invariant with respect to Euclidean translation, it suffices to construct the heat kernel $P(x,y,t)$ with singularity at the origin, *i.e.*, $P(x,t)$. From our earlier calculation we know that

$$P(x,t) = \frac{1}{(2\sqrt{\pi t})^3} e^{-\frac{|x|^2}{4t}}.$$

Then

$$u(x,t) = P(f)(x,t) = (e^{\Delta t} f)(x,t) = \frac{1}{(2\sqrt{\pi t})^3} \int_{\mathbf{R}^3} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

These results may be generalized to strongly elliptic operators:

$$D = \frac{1}{2}\sum_{j,k=1}^{n} \alpha_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j(x)\frac{\partial}{\partial x_j} + \gamma_0(x),$$

where

- $A = [a_{jk}(x)]$ is symmetric and $A \geq C\mathbf{I}_{n\times n}$ for all $x \in \mathbf{R}^n$. Here C is a positive constant and $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix;
- a_{jk}, b_j, $\gamma_0 \in C_0^\infty(\mathbf{R}^n)$.

Then a heat kernel $P_t(x, y)$ exists, and

$$u(x,t) = \int_{\mathbf{R}^n} P_t(x,y) f(y) dy$$

yields

$$\begin{aligned} &\frac{\partial}{\partial t} u(x,t) = Du(x,t), \qquad t > 0, \quad x \in \mathbf{R}^n, \\ &\lim_{t\to 0^+} u(x,t) = f(x). \end{aligned}$$

The principal part of D yields a Riemannian distance d and one has the following estimates: for any $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$, there exists constants $C_{\mathbf{k}}$ such that

$$\left| \frac{\partial^{|\mathbf{k}|} P_t(x,y)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right| \leq C_{\mathbf{k}} t^{-\frac{n+|\mathbf{k}|}{2}} e^{-\frac{d^2(x,y)}{4t}}, \qquad t > 0.$$

3. The Heisenberg group and the sub-Laplacian. In our talks, we want to generalize the above elliptic case to sub-elliptic case. Let us consider the following two vector fields defined on $\mathbf{R}^3$ with coordinates $(x,t) = (x_1, x_2, t)$:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} + 2ax_2 \frac{\partial}{\partial t} \\ X_2 &= \frac{\partial}{\partial x_2} - 2ax_1 \frac{\partial}{\partial t} \end{aligned}$$

with $a > 0$. It is easy to see that X_1 and X_2 satisfy the Heisenberg uncertainty principle:

$$[X_1, X_2] = -4a \frac{\partial}{\partial t}.$$

Now we consider the following operator:

$$\mathcal{L} = \frac{1}{2}(X_1^2 + X_2^2)$$

Unlike the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial t^2}$ or the Casimir operator

$$L = \mathcal{L} + \frac{1}{2}\frac{\partial^2}{\partial t^2},$$

the operator $\mathcal{L}$ is not elliptic. It is easy to see that the vector fields X_1, X_2 and $T = \frac{\partial}{\partial t}$ and the operator $\mathcal{L}$ are left-invariant with respect to the "Heisenberg translation": for $(x, t) = (x_1, x_2, t)$ and $(y, s) = (y_1, y_2, s) \in \mathbf{R}^3$,

$$(x, t) \circ (y, s) = (x_1 + y_1, x_2 + y_2, t + s + 2a[x_2 y_1 - x_1 y_2]).$$

Actually, the above multiplicative law defines a group sturcture on $\mathbf{R}^3$ which we call the 1-dimensional Heisenberg group with $(x, t)^{-1} = (-x, -t)$. . In general, we may define the n-dimensional Heisenberg group $\mathbf{H}_n$ which is the Lie group whose underlying manifold is $\mathbf{C}^n \times \mathbf{R}$ equipped with the group law:

$$(x, t) \circ (y, s) = \left(x + y, t + s + 2\sum_{j=1}^{n} a_j \left[x_{j+n} y_j - x_j y_{j+n} \right] \right)$$

with $a_j > 0$ for $1 < j < n$. Moreover, the non-isotropic dilaton

$$(x_1, \ldots, x_{2n}, t) \mapsto (\delta x_1, \ldots, \delta x_{2n}, \delta^2 t,), \qquad \delta > 0$$

defines an automorphism on the group $\mathbf{H}_n$. The vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2a_j x_{j+n} \frac{\partial}{\partial t}, \\ X_{j+n} &= \frac{\partial}{\partial x_{j+n}} - 2a_j x_j \frac{\partial}{\partial t}, \end{aligned}$$

and

$$\mathbf{T} = \frac{\partial}{\partial t}$$

form a basis of the Lie algebra of $\mathbf{H}_n$. It is obvious that $\mathbf{H}_n$ is a non-commutative Lie group of step 2, *i.e.,* $X_1, \ldots, X_{2n}$ and their first brackets yield $T\mathbf{H}_n$.

Similarly, we define the sub-Laplacian on $\mathbf{H}_n$ as follows:

$$\mathcal{L} = -\sum_{j=1}^{2n} X_j^2.$$

This operator is a sum of squares of $2n$ "horizontal" vector fields, and it is therefore not elliptic, although it is hypoelliptic, see Hörmander [17], *i.e.,* the solution u of $\mathcal{L}u = f$ is smooth whenever $f \in C^\infty(\mathbf{H}_n)$. (We will see later that the inverse $\mathcal{L}^{-1}$ doesn't have full gain. This is a so-called sub-elliptic estimate.)

Before we go further, let us give some background for the Laguerre calculus on $\mathbf{H}_n$. We first recall a beautiful idea of Mikhlin, contained in his 1936 study of convolution operators on $\mathbf{R}^2$. Let $\mathbf{K}$ denote a principal value convolution operator on $\mathbf{R}^2$:

$$\mathbf{K}(f)(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} K(y) f(x-y) dy,$$

where $f \in C_0^\infty(\mathbf{R}^2)$ and $K \in C^\infty(\mathbf{R}^2 \setminus \{(0,0)\})$ is homogeneous of degree -2 with vanishing mean value, *i.e.,*

$$\int_{|x|=1} K(x) dx = 0.$$

Thus we can write

$$K(x) = \frac{f(\theta)}{r^2}, \qquad x = x_1 + ix_2 = re^{i\theta},$$

where

$$f(\theta) = \sum_{m\in\mathbf{Z},m\neq 0} f_m e^{im\theta}.$$

Suppose that g is another smooth function on $[0, 2\pi]$ with

$$g(\theta) = \sum_{m\in\mathbf{Z},m\neq 0} g_m e^{im\theta}.$$

Then g induces a principal value convolution operator, G, on $\mathbf{R}^2$ with kernel $\frac{g(\theta)}{r^2}$. In [19], Mikhlin found the following identity:

Proposition 3.1.

$$\frac{|m|i^{-|m|}}{2\pi}\frac{e^{im\theta}}{r^2} *_{\mathbf{E}} \frac{|k|i^{-|k|}}{2\pi}\frac{e^{ik\theta}}{r^2} = \frac{|m+k|i^{-|m+k|}}{2\pi}\frac{e^{i(m+k)\theta}}{r^2}.$$

Here "$*_{\mathbf{E}}$" stands for the Euclidean convolution.

Definition 3.2. Let

$$f(\theta) = \sum_{m\in\mathbf{Z},m\neq 0} f_m e^{im\theta}$$

induce a principal value convolution operator K, with kernel $\frac{f(\theta)}{r^2}$, on $C_0^\infty(\mathbf{R}^2)$. Then the symbol $\sigma(K)$ of K is defined by

$$\sigma(K) = \sum_{m\in\mathbf{Z},m\neq 0} \left(\frac{|m|i^{-|m|}}{2\pi}\right)^{-1} f_m e^{im\theta}.$$

With this definition we may rewrite Proposition 3.1 as follows:

Theorem 3.3. *Let K and G be two principal value convolution operators on $C_0^\infty(\mathbf{R}^2)$. Then*

$$\sigma(K *_{\mathbf{E}} G) = \sigma(K)\cdot\sigma(G).$$

Later $\sigma(K)$ turned out to be the Fourier transform of K. One of the purposes of this section is to find a multiplicative calculus for the convolution operator on $\mathbf{H}_n$. Readers may consult the book [8] for detailed discussion on this subject. For $f \in L^1(\mathbf{H}_n)$, denote by

$$\tilde{f}_\lambda(\mathbf{z}) = \hat{f}(\mathbf{z}, \lambda) = \int_{\mathbf{R}} f(\mathbf{z}, t) e^{-i\lambda t} dt.$$

For $f, g \in L^1(\mathbf{H}_n)$, define the left-invariant convolution $f * g$ of f and g by:

$$(f * g)(x, t) = \int_{\mathbf{H}_n} f(y, s) g((y, s)^{-1} \cdot (x, t)) dy ds.$$

For $\lambda \in \mathbf{R}^* = \mathbf{R} \setminus \{0\}$, we define the twisted convolution of f and g by

$$(f *_\lambda g)(\mathbf{z}) = \int_{\mathbf{C}^n} f(\mathbf{z} - \mathbf{w}) g(\mathbf{w}) e^{-2i\lambda \mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}} dm(\mathbf{w}).$$

Here dm is the Lebesgue measure on $\mathbf{C}^n$. Then we have $\widetilde{(f * g)}_\lambda = \tilde{f}_\lambda *_\lambda \tilde{g}_\lambda$. The generalized Laguerre polynomials $L_k^{(p)}(x)$ are defined by the generating function formula:

$$\sum_{k=1}^{\infty} L_k^{(p)}(x) w^k = \frac{1}{(1-w)^{p+1}} \exp\left\{-\frac{xw}{1-w}\right\}$$

for $p \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $x \geq 0$, and $|w| < 1$. We also define the Laguerre functions:

$$\ell_k^{(p)}(x) = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{\frac{1}{2}} x^{\frac{p}{2}} L_k^{(p)}(x) e^{-\frac{x}{2}},$$

where $x \geq 0$ and $p, k \in \mathbb{Z}_+$. It is well known that for each $p = 0, 1, 2, \ldots$, $\{\ell_k^{(p)}(x), k \in \mathbb{Z}_+\}$ form a complete orthonormal basis of the space $L^2([0, \infty))$.

Let $z = |z| e^{i\theta}$ and $k, p \in \mathbb{Z}_+$. Then we define the exponential Laguerre functions as follows:

$$\widetilde{\mathcal{W}}_k^{(\pm p)}(z, \lambda) = (\pm 1)^p \frac{2|\lambda|}{\pi} \ell_k^{(p)}(2|\lambda||z|^2) e^{\pm ip\theta}.$$

Let us first recall a result due to Greiner [15]:

Theorem 3.4. *Let $p,k,q,m = 1,2,\ldots$. Then*

$$\widetilde{\mathcal{W}}^{(p-k)}_{p\wedge k-1} *_{|\lambda|} \widetilde{\mathcal{W}}^{(q-m)}_{q\wedge m-1} = \delta^{(q)}_k \cdot \widetilde{\mathcal{W}}^{(p-m)}_{p\wedge m-1},$$

and

$$\widetilde{\mathcal{W}}^{(p-k)}_{p\wedge k-1} *_{-|\lambda|} \widetilde{\mathcal{W}}^{(q-m)}_{q\wedge m-1} = \delta^{(p)}_m \cdot \widetilde{\mathcal{W}}^{(q-k)}_{q\wedge k-1},$$

where $a\wedge b = \min(a,b)$.

Let $\mathcal{W}^{(p)}_k(z,t)$, $\pm p,k = 0,1,2,\ldots$, denote the inverse Fourier transform of $\widetilde{\mathcal{W}}^{(p)}_k(z,\lambda)$ with respect to λ, *i.e.*,

$$\mathcal{W}^{(p)}_k(z,t) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{i\lambda t}\widetilde{\mathcal{W}}^{(p)}_k(z,\lambda)d\lambda.$$

These are the kernels of the generalized Cauchy-Szegö projection operators on $\mathbf{H}_1$. In particular,

$$\mathcal{W}^{(0)}_0(z,t) = S_+ + S_-$$

where

$$S_\pm = \frac{1}{\pi^2}\frac{1}{(|z|^2 \mp it)^2}.$$

Let K induce a left-invariant convolution operator $\mathbf{K}$ on $\mathbf{H}_n$,

$$\mathbf{K}(\phi)(x,t) = \int_{\mathbf{H}_n} K(y,s)\phi((y,s)^{-1}\cdot(x,t))dyds.$$

Now, $\tilde{K}(z,\lambda)$ has a Laguerre series expansion:

$$\tilde{K}(z,\lambda) = \sum_{|\mathbf{p}|,|\mathbf{k}|=1}^{\infty} K^{(\mathbf{p})}_{\mathbf{k}}(\lambda)\prod_{j=1}^{n}\widetilde{\mathcal{W}}^{(p_j)}_{k_j}(z_j,\lambda).$$

Define the Laguerre tensor $\mathcal{M}(K)$ of K:

$$\mathcal{M}(K) = \mathcal{M}_+(K) \oplus \mathcal{M}_-(K)$$

where

$$\mathcal{M}_+(K) = \left(K_{\mathbf{k}}^{(\mathbf{p})}(\lambda)\right), \quad \lambda > 0$$

and

$$\mathcal{M}_-(K) = \left(K_{\mathbf{k}}^{(\mathbf{p})}(\lambda)\right)^T, \quad \lambda < 0.$$

Here the upper indices represent the diagonal and the lower indices the position in that diagonal. The following theorem is the cornerstone for Laguerre calculus on $\mathbf{H}_n$ which was first proved by Greiner (see [15]) in $\mathbf{H}_1$ and later generalized by Beals, Gaveau, Greiner and Vauthier (see [5]) to $\mathbf{H}_n$:

Theorem 3.5. *Let F and G induce two convolution operators on $\mathbf{H}_n$. $\mathcal{M}(F)$ and $\mathcal{M}(G)$ denote the Laguerre tensors of F and G respectively. Then*

$$\mathcal{M}(F * G) = \mathcal{M}_+(F) \cdot \mathcal{M}_+(G) \oplus \mathcal{M}_-(F) \cdot \mathcal{M}_-(G).$$

Corollary 3.6. *The identity operator $\mathbf{I}$ on $C_0^\infty(\mathbf{H}_n)$ is induced by the identity Laguerre tensor*

$$\mathcal{M}_\pm(\mathbf{I}) = (\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)}).$$

Let $f \in L^p(\mathbf{H}_n)$, $1 < p < \infty$. Then

$$\lim_{r \to 1^-} \sum_{|\mathbf{k}|=0}^{\infty} r^{|\mathbf{k}|} f * \mathcal{W}_{\mathbf{k}}^{(\mathbf{0})} = f$$

in L^p-norm (see [5], [9] and [24]).

A left-invariant differential operator $\mathcal{P}$ on $\mathbf{H}_n$ is a polynomial $\mathcal{P}(\mathbf{Z}, \bar{\mathbf{Z}}, \mathbf{T})$ with constant coefficents. Then

$$\mathcal{P} = \mathcal{P}\mathbf{I} = \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{P}\mathcal{W}_{k_1,\dots,k_n}^{(0,\dots,0)} *$$

where $\mathbf{I} = \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{W}_{\mathbf{k}}^{(\mathbf{0})} *$ is the identity operator on $C_0^\infty(\mathbf{H}_n)$.

In particular, we have the following:

- $\mathcal{M}(\mathbf{T}) = i\tau(\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)})$.
- $\mathcal{M}(\mathbf{Z}_j) = \mathcal{M}_+(\mathbf{Z}_j) \oplus \mathcal{M}_-(\mathbf{Z}_j)$ where

$$\mathcal{M}_-(\mathbf{Z}_j)_{k_1,\dots,k_n}^{(p_1,\dots,p_n)} = \sqrt{2|\lambda|p_j}\,\delta_{k_1}^{(p_1)} \cdots \delta_{k_j}^{(p_j+1)} \cdots \delta_{k_n}^{(p_n)}$$

  and $\mathcal{M}_+(\mathbf{Z}_j) = \mathcal{M}_-(\mathbf{Z}_j)^T$.
- $\mathcal{M}(\bar{\mathbf{Z}}_j) = -\mathcal{M}(\mathbf{Z}_j)^T$.

Here $\mathbf{Z}_j = \frac{1}{2}(X_j + iX_{n+j})$ and $\bar{\mathbf{Z}}_j = \frac{1}{2}(X_j - iX_{n+j})$, $j = 1, \dots, n$.

Example 3.7. When $n = 1$ and $a = 1$, we have

$$\mathcal{M}_+(\mathbf{Z}_1) = \sqrt{2|\tau|} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1}$$

and

$$\mathcal{M}_-(\mathbf{Z}_1) = [\mathcal{M}_+(\mathbf{Z}_1)]^t.$$

Now we may set

$$\mathcal{M}_+(K) = \frac{1}{\sqrt{2|\tau|}} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{\sqrt{1}} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2}$$

and

$$\mathcal{M}_-(K) = [\mathcal{M}_+(K)]^t.$$

Thus

$$\tilde{K}_\pm(z,\tau) = \frac{1}{\sqrt{2|\tau|}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\tau).$$

Using the defintion of $\widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\tau)$, we sum the series

$$\tilde{K}(z,\tau) = \frac{2|\tau| z e^{-|\tau||z|^2}}{\pi} \int_0^1 \sum_{k=0}^{\infty} r^k L_k^{(1)}(2|\tau||z|^2)\,dr.$$

We have

$$\sum_{k=0}^{\infty} r^k L_k^{(1)}(x) = \frac{e^x}{(1-r)^2} e^{-x/(1-r)},$$

so

$$\tilde{K}(z,\tau) = \frac{1}{\pi} \frac{e^{-|\tau||z|^2}}{\bar{z}},$$

and

$$K(z,t) = \frac{1}{2\pi^2 \bar{z}} \int_{\mathbf{R}} e^{it\tau - |\tau||z|^2} d\tau = \frac{z}{\pi^2(|z|^4 + t^2)}.$$

This recovers the Greiner, Kohn, Stein Theorem [16] on the Heisenberg group:

$$\begin{aligned} \mathbf{Z}_1 K &= \mathbf{I} - \mathcal{W}^{(0)}_{-,0} = \mathbf{I} - S_-, \\ K\mathbf{Z}_1 &= \mathbf{I} - \mathcal{W}^{(0)}_{+,0} = \mathbf{I} - S_+. \end{aligned}$$

The Heisenberg group and its sub-Laplacian are at the cross-roads of many domains of analysis and geometry: nilpotent Lie group theory, hypoelliptic second order partial differential equations, strongly pseudoconvex domains in complex analysis, probability theory of degenerate diffusion process, subriemannian geometry, control theory and semiclassical analysis of quantum mechanics, see *e.g.,* [1], [2], [3], [4], [5], [6], [9], [11], [13], [14], [21], [24]; we note that $\mathcal{M}_\pm$ are the annihilation and creation operators in quantum mechanics.

Now let us turn to the sub-Laplacian:

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n}(\mathbf{Z}_j\bar{\mathbf{Z}}_j + \bar{\mathbf{Z}}_j\mathbf{Z}_j) = -\frac{1}{4}\sum_{j=1}^{2n} X_j^2.$$

In fact, we may handle problem in a much more general setting. Let us consider the following operator:

$$\mathcal{L}_\alpha^m = \left[\frac{1}{2}\sum_{j=1}^{n}(\mathbf{Z}_j\bar{\mathbf{Z}}_j + \bar{\mathbf{Z}}_j\mathbf{Z}_j) - i\alpha\mathbf{T}\right]^m,$$

with $1 \le m < n$. First we will find the Laguerre tensor of the operator $\mathcal{L}_\alpha^m$. We can take the Fourier transform with respect to t, and write $\widetilde{\mathcal{L}}_\alpha^m$ as a twisted convolution:

$$\widetilde{\mathcal{L}}_\alpha^m = \widetilde{\mathcal{L}}_\alpha^m \widetilde{\mathbf{I}} = \sum_{|\mathbf{k}|=0}^{\infty}\left[-\frac{1}{2}\sum_{j=1}^{n}(\widetilde{\mathbf{Z}}_j\widetilde{\bar{\mathbf{Z}}}_j + \widetilde{\bar{\mathbf{Z}}}_j\widetilde{\mathbf{Z}}_j) - \alpha\tau\right]^m \prod_{j=1}^{n} a_j \widetilde{\mathcal{W}}_{k_j}^{(0)}(\sqrt{a_j}z_j,\tau)*_\tau$$

Then, the Laguerre tensors of $\mathbf{Z}_j$, $\bar{\mathbf{Z}}_j$ and $\mathbf{T}$ yield

$$\widetilde{\mathcal{L}}_\alpha^m = \sum_{|\mathbf{k}|=0}^{\infty}\left(\sum_{j=1}^{n}(2k_j+1)|\tau|a_j - \alpha\tau\right)^m \prod_{j=1}^{n} a_j \widetilde{\mathcal{W}}_{k_j}^{(0)}(\sqrt{a_j}z_j,\tau) *_\tau .$$

Consequently the Laguerre tensor of the convolution operator induced by

$\mathcal{L}_\alpha^m$ is

$$\mathcal{M}(\widetilde{\mathcal{L}}_\alpha^m) = |\tau|^m \left(\left[\sum_{j=1}^n (2k_j+1)a_j - \alpha \mathrm{sgn}(\tau) \right]^m \delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)} \right), \tag{3}$$

which is invertible as long as α does not belong to the exceptional set $\mathcal{E}_\alpha$, where

$$\mathcal{E}_\alpha = \left\{ \pm \sum_{j=1}^n (2k_j+1)a_j : \quad \mathbf{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}_+^n \right\}.$$

According to Theorem 3.5, the inverse Laguerre tensor of (3) is

$$\mathcal{M}(\widetilde{\mathcal{L}}_\alpha^{-m}) = |\tau|^{-m} \left(\left[\sum_{j=1}^n (2k_j+1)a_j - \alpha \mathrm{sgn}(\tau) \right]^{-m} \delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)} \right), \tag{4}$$

and we write its kernel $\widetilde{\Psi}_m(\mathbf{z}, \tau)$ in the Laguerre series expansion:

$$\widetilde{\Psi}_m(\mathbf{z},\tau) = |\tau|^{-m} \sum_{|\mathbf{k}|=0}^{\infty} \left(\sum_{j=1}^n (2k_j+1)a_j - \alpha \mathrm{sgn}(\tau) \right)^{-m} \prod_{j=1}^n a_j \widetilde{\mathcal{W}}_{k_j}^{(0)}(\sqrt{a_j} z_j, \tau). \tag{5}$$

To find the fundamental solution of $\mathcal{L}_\alpha^m$, we can sum this series and take the inverse Fourier transform with respect to τ. First we introduce the following integral representation of A^{-m}:

$$\frac{1}{A^m} = \frac{1}{\Gamma(m)} \int_0^\infty s^{m-1} e^{-As} ds \quad \text{for} \quad \mathrm{Re}\,(A) > 0.$$

Then we can write (5) in the following form:

$$\begin{aligned} \widetilde{\Psi}_m(\mathbf{z},\tau) &= \frac{|\tau|^{-m}}{\Gamma(m)} \sum_{|\mathbf{k}|=0}^{\infty} \int_0^\infty s^{m-1} e^{-\left(\sum_{j=1}^n (2k_j+1)a_j - \alpha \mathrm{sgn}(\tau)\right)s} ds \\ &\quad \cdot \prod_{j=1}^n a_j \widetilde{\mathcal{W}}_{k_j}^{(0)}(\sqrt{a_j} z_j, \tau). \end{aligned} \tag{6}$$

Next we interchange the summation and integration, and use the definitions of $\widetilde{\mathcal{W}}^{(0)}_{k_j}$,

$$\begin{aligned}\widetilde{\Psi}_m(\mathbf{z},\tau) &= \frac{|\tau|^{-m}}{\Gamma(m)}\int_0^\infty s^{m-1}\sum_{|\mathbf{k}|=0}^\infty e^{-\left(\sum_{j=1}^n (2k_j+1)a_j-\alpha\operatorname{sgn}(\tau)\right)s}ds\\ &\quad\cdot\prod_{j=1}^n a_j\widetilde{\mathcal{W}}^{(0)}_{k_j}(\sqrt{a_j}z_j,\tau)\\ &= \frac{|\tau|^{n-m}}{\pi^n\Gamma(m)}\int_0^\infty s^{m-1}\sum_{|\mathbf{k}|=0}^\infty e^{-\left(\sum_{j=1}^n (2k_j+1)a_j-\alpha\operatorname{sgn}(\tau)\right)s}ds\\ &\quad\cdot\prod_{j=1}^n 2a_je^{-a_j|\tau||z_j|^2}L^{(0)}_{k_j}(2a_j|\tau||z_j|^2)\\ &= \frac{|\tau|^{n-m}}{\pi^n\Gamma(m)}\int_0^\infty s^{m-1}e^{\alpha\operatorname{sgn}(\tau)s}\prod_{j=1}^n 2a_je^{-a_js-a_j|\tau||z_j|^2}\\ &\quad\cdot\sum_{k_j=0}^\infty (e^{-2a_js})^{k_j}L^{(0)}_{k_j}(2a_j|\tau||z_j|^2)ds\end{aligned}$$

Apply the generating formula for the Laguerre polynomials

$$\sum_{k=0}^\infty L_k^{(p)}(x)z^k=\frac{1}{(1-z)^{p+1}}\exp\left\{-\frac{xz}{1-z}\right\}$$

to the last formula for $\widetilde{\Psi}_m(\mathbf{z},\tau)$, we obtain

$$\begin{aligned}\widetilde{\Psi}_m(\mathbf{z},\tau) &= \frac{|\tau|^{n-m}}{\pi^n\Gamma(m)}\int_0^\infty s^{m-1}e^{\alpha\operatorname{sgn}(\tau)s}\prod_{j=1}^n\frac{2a_je^{-a_js}}{1-e^{-2a_js}}\\ &\quad\cdot\exp\left\{-a_j|\tau||z_j|^2\left[1+\frac{2e^{-2a_js}}{1-e^{-2a_js}}\right]\right\}ds\\ &= \frac{|\tau|^{n-m}}{\pi^n\Gamma(m)}\int_0^\infty s^{m-1}e^{\alpha\operatorname{sgn}(\tau)s}\left[\prod_{j=1}^n\frac{a_j}{\sinh(a_js)}\right]\\ &\quad\cdot\exp\left\{-|\tau|\sum_{j=1}^n a_j|z_j|^2\coth(a_js)\right\}ds\end{aligned}$$

Set $\gamma(s;\mathbf{z})=\sum_{j=1}^n a_j|z_j|^2\coth(a_js)$ and take the inverse Fourier transform with respect to τ,

$$\Psi_m(\mathbf{z},t) = \frac{\Gamma(n-m+1)}{2\pi^{n+1}\Gamma(m)}\left[\int_0^\infty s^{m-1}e^{\alpha s}\left(\prod_{j=1}^n \frac{a_j}{\sinh(a_j s)}\right)\frac{ds}{[\gamma(s;\mathbf{z})-it]^{n-m+1}}\right.$$

$$\left. + \int_0^\infty s^{m-1}e^{-\alpha s}\left(\prod_{j=1}^n \frac{a_j}{\sinh(a_j s)}\right)\frac{ds}{[\gamma(s;\mathbf{z})+it]^{n-m+1}}\right], \tag{7}$$

$$= \frac{\Gamma(n-m)}{2\pi^{n+1}\Gamma(m)}\int_{-\infty}^\infty \left(\prod_{j=1}^n \frac{a_j}{\sinh(a_j s)}\right)\frac{e^{\alpha s}s^{m-1}ds}{[\gamma(s;\mathbf{z})-it]^{n-m+1}}, \quad 1 \le m < n. \tag{8}$$

Here we used the following identity:

$$\int_{-\infty}^\infty |\tau|^{n-m}e^{\alpha \operatorname{sgn}(\tau)s+it\tau-|\tau|\gamma(s;\mathbf{z})}d\tau = \frac{\Gamma(n-m+1)e^{\alpha s}}{[\gamma(s;\mathbf{z})-it]^{n-m+1}} + \frac{\Gamma(n-m+1)e^{-\alpha s}}{[\gamma(s;\mathbf{z})+it]^{n-m+1}}$$

to obtain (7), then switched the sign of s in the second integral of (7) to get (8), since $\gamma(-s;\mathbf{z}) = -\gamma(s;\mathbf{z})$.

Following the notation in [2] (see also Section 4 below), we introduce the complex distance and volume element on the Heisenberg group:

$$g(s;\mathbf{z},t) = \gamma(2s;\mathbf{z}) - it \quad \text{and} \quad v(s) = \prod_{j=1}^n \frac{2a_j}{\sinh(2a_j s)}. \tag{9}$$

With this notation we have

$$\Psi_m(\mathbf{z},t) = \frac{2^m(n-m)!}{(2\pi)^{n+1}\Gamma(m)}\int_{-\infty}^\infty e^{2\alpha s}s^{m-1}\frac{v(s)ds}{[g(s;\mathbf{z},t)]^{n-m+1}} \tag{10}$$

When $|\mathbf{z}| = 0$, $t \neq 0$, then $g(s;\mathbf{z},t) = -it$. The integrand of (10) is not integrable at $s = 0$. To regularize the integral we deform its path from

$(-\infty, \infty)$ to

$$(-\infty + i\varepsilon\text{sgn}t, \infty + i\varepsilon\text{sgn}t), \quad \text{where} \quad 0 < \varepsilon < \min_{1\le j\le n} \frac{\pi}{2a_j},$$

see [6]. Then

$$\Psi_m(\mathbf{z}, t) = \frac{2^m(n-m)!}{(2\pi)^{n+1}\Gamma(m)} \int_{-\infty+i\varepsilon\text{sgn}t}^{\infty+i\varepsilon\text{sgn}t} e^{2\alpha s} s^{m-1} \frac{v(s)ds}{[g(s;\mathbf{z},t)]^{n-m+1}}.$$

Remark. When $a_j = a$, $j = 1, \dots, n$, the unitary group $\mathcal{U}(n)$ acts on $\mathbf{H}_n$ via

$$U(\mathbf{z}, t) = (U(\mathbf{z}), t) \quad \text{for} \quad U \in \mathcal{U}(n), \quad (\mathbf{z}, t) \in \mathbf{H}_n.$$

The operator $\mathcal{L}_\alpha^m$ is invariant under the $\mathcal{U}(n)$-action. In this case,

$$\begin{aligned}
\mathcal{L}_0^{-m} = \mathcal{L}^{-m} &= \frac{(n-m)!}{2\pi^{n+1}\Gamma(m)} \int_{-\infty}^{\infty} \left(\frac{a}{\sinh(a\tau)}\right)^n \frac{\tau^{m-1}d\tau}{[a|\mathbf{z}|^2\coth(a\tau) - it]^{n-m+1}} \\
&= \frac{a^{n-m+1}(n-m)!}{2\pi^{n+1}\Gamma(m)} \int_{-\infty}^{\infty} \left(\frac{a\tau}{\sinh(a\tau)}\right)^{m-1} \\
&\quad \cdot \left[a|\mathbf{z}|^2\cosh(a\tau) - it\sinh(a\tau)\right]^{n-m+1} d\tau.
\end{aligned}$$

This coincides with a result of Benson, Dolley and Ratcliff [7], $a = \frac{1}{4}$. In particular, when $m = 1$,

$$\mathcal{L}^{-1} = \frac{2^{2n-2}P^2\left(\frac{n}{2}\right)}{\pi^{n+1}} \frac{1}{(|\mathbf{z}|^4 + t^2)^{\frac{n}{2}}}.$$

This recovers a result of Folland [12].

It can be shown that the operator

$$T_m(f)(\mathbf{z}, t) = f * K_m(\mathbf{z}, t) = f * \mathcal{P}_{2m}(\mathbf{Z}, \bar{\mathbf{Z}})\Psi_m(\mathbf{z}, t)$$

orginally defined on $C_0^\infty(\mathbf{H}_n)$ can be extended as a bounded operator from $\mathcal{H}^p(\mathbf{H}_n)$ into itself for $0 < p < \infty$. Here, $\mathcal{P}_{2m}(\mathbf{Z}, \bar{\mathbf{Z}})$ is any monomial of degree $2m$ in the "good" vector fields $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$, $\bar{\mathbf{Z}}_1, \ldots, \bar{\mathbf{Z}}_n$ and $\mathcal{H}^p$ is the atomic Hardy space defined on $\mathbf{H}_n$. In particular, when $m = 1$,

$$T(f)(\mathbf{z}, t) = f * K_1(\mathbf{z}, t) = f * \mathcal{P}_2(\mathbf{Z}, \bar{\mathbf{Z}})\Psi_1(\mathbf{z}, t)$$

orginally defined on $C_0^\infty(\mathbf{H}_n)$ can be extended as bounded operator from $\mathcal{H}^p(\mathbf{H}_n)$ into itself for $0 < p < \infty$. Since $\mathbf{T} = \frac{1}{2ia_j}[\bar{\mathbf{Z}}_j, \mathbf{Z}_j]$, it is easy to see that

$$\|\mathcal{L}^{-1}(f)\|_{\mathcal{H}^p_{k+1,loc}(\mathbf{H}_n)} \leq C_{n,p}\|f\|_{\mathcal{H}^p_{k,loc}(\mathbf{H}_n)}$$

for $0 < p < \infty$ and $k \in \mathbb{Z}_+$. Here $\mathcal{H}^p_k(\mathbf{H}_n)$ is the local version Hardy-Sobolev space of order k (see Chang and Tie [10], Folland and Stein [13]). The inverse $\mathcal{L}_\alpha^{-1}$ of the sub-Laplacian gains two in all "good" directions but only gains one in the "bad" direction. This is the so-called sub-elliptic estimate.

Next, we will compute the heat kernel $h_u(\mathbf{z}, t) = \exp\{-u\mathcal{L}_\alpha\}\delta_0$. Here u is the time variable. In the isotropic case, the heat kernel was independently studied by Gaveau [14] and Hulanicki [18] via probability. Later, Beals and Greiner [6] solved the general case by a different method. We will see that $h_u(\mathbf{z}, t)$ can be obtained easily via the Laguerre calculus.

We first take the Fourier transform with respect to the t-variable and write the heat kernel $\widetilde{h}_u(\mathbf{z}, t)$ as a twisted convolution operator.

$$\begin{aligned} \widetilde{h}_u(\mathbf{z}, \tau) &= \exp\{-s\widetilde{\mathcal{L}}_\alpha\}\widetilde{\mathbf{I}} = \sum_{|\mathbf{k}|=0}^{\infty} \exp\{-u\widetilde{\mathcal{L}}_\alpha\}\left[\prod_{j=1}^{n} a_j \widetilde{\mathcal{W}}^{(0)}_{k_j}(\sqrt{a_j}z_j, \tau)\right] \\ &= \sum_{|\mathbf{k}|=0}^{\infty} e^{-u\sum_{j=1}^{n} a_j|\tau|(2k_j+1)+u\alpha\tau} \prod_{j=1}^{n} a_j \widetilde{\mathcal{W}}^{(0)}_{k_j}(\sqrt{a_j}z_j, \tau). \end{aligned}$$

Next, a computation similar to those above leads to

$$(11)\ \widetilde{h}_u(\mathbf{z}, \tau) = \frac{e^{\alpha\tau u}}{\pi^n}\left[\prod_{j=1}^{n} \frac{a_j|\tau|}{\sinh(a_j|\tau|u)}\right]\exp\left\{-|\tau|\sum_{j=1}^{n} a_j|z_j|^2\coth(a_j|\tau|u)\right\}.$$

Since

$$\prod_{j=1}^{n} \frac{a_j|\tau|}{\sinh(a_j|\tau|u)} = \prod_{j=1}^{n} \frac{a_j\tau}{\sinh(a_j\tau u)} \quad \text{and} \quad |\tau|\coth(a_j|\tau|u) = \tau\coth(a_j\tau u),$$

we can simplify (11),

$$\widetilde{h}_u(\mathbf{z},\tau) = \frac{e^{\alpha\tau u}}{\pi^n}\left[\prod_{j=1}^{n} \frac{a_j\tau}{\sinh(a_j\tau u)}\right] e^{-\tau\gamma(\tau u;\mathbf{z})},$$

where, again $\gamma(\tau u;\mathbf{z}) = \sum_{j=1}^{n} a_j|z_j|^2\coth(a_j\tau u)$.

Then the inverse Fourier transform with respect to the τ-variable yields the heat kernel:

$$\begin{aligned} h_u(\mathbf{z},t) &= \frac{1}{2\pi^{n+1}}\int_{-\infty}^{+\infty}\left[\prod_{j=1}^{n} \frac{a_j\tau}{\sinh(a_j\tau u)}\right] e^{\alpha\tau u+it\tau-\tau\gamma(\tau u;\mathbf{z})}d\tau \\ &= \frac{1}{2(\pi u)^{n+1}}\int_{-\infty}^{+\infty}\left[\prod_{j=1}^{n} \frac{a_j\tau}{\sinh(a_j\tau)}\right] e^{\alpha\tau+i\frac{\tau}{u}t-\frac{\tau}{u}\gamma(\mathbf{z},\tau)}d\tau, \end{aligned}$$

and replacing τ by 2τ one has

$$h_u(\mathbf{z},t) = \frac{1}{(\pi u)^{n+1}}\int_{-\infty}^{+\infty} v(\tau)e^{2\alpha\tau-\frac{2\tau}{u}g(\tau;\mathbf{z},t)}d\tau. \tag{12}$$

4. The Hamilton-Jacobi equation and the heat kernel. The heat kernel is closely associated to Hamiltonian mechanics on $\mathbf{H}_n$. In fact, the heat kernel $h_u(\mathbf{z},t)$ we just calculated can be interpreted in terms of an action function associated to complex Hamiltonian mechanics. In this case, the Hamiltonian function is the symbol of $\mathcal{L}$ (see [4]). To simplify the calculations, we shall concentrate on the 1-dimensional Heisenberg group:

$$H_{\mathcal{L}}(x,\xi,\theta) = \frac{1}{2}(\xi_1+2ax_2\theta)^2 + \frac{1}{2}(\xi_2-2ax_1\theta)^2 = \frac{1}{2}(\zeta_1^2+\zeta_2^2),$$

where $\zeta_1 = \xi_1 + 2ax_2\theta$ and $\zeta_2 = \xi_2 - 2ax_1\theta$. In this notation Hamilton-Jacobi equations for a curve $(x_1(s), x_2(s), t(s), \xi_1(s), \xi_2(s), \theta(s))$ take the form:

$$
\begin{aligned}
(13)\quad \dot{x}_1(s) &= \frac{\partial H_{\mathcal{L}}}{\partial \xi_1} = \xi_1 + 2ax_2\theta = \zeta_1(s), \quad \dot{x}_2(s) = \frac{\partial H_{\mathcal{L}}}{\partial \xi_2} = \xi_2 - 2ax_1\theta = \zeta_2(s), \\
(14)\quad \dot{t}(s) &= \frac{\partial H_{\mathcal{L}}}{\partial \theta} = (\xi_1 + 2ax_2\theta)(2ax_2) - (\xi_2 - 2ax_1\theta)(2ax_1) \\
&= 2a(\zeta_1 x_1 - \zeta_2 x_2), \\
(15)\quad \dot{\xi}_1(s) &= -\frac{\partial H_{\mathcal{L}}}{\partial x_1} = (2a\theta)(\xi_2 - 2ax_1\theta) = (2a\theta)\zeta_2, \\
(16)\quad \dot{\xi}_2(s) &= -\frac{\partial H_{\mathcal{L}}}{\partial x_2} = -(2a\theta)(\xi_1 + 2ax_2\theta) = -(2a\theta)\zeta_1, \\
(17)\quad \dot{\theta}(s) &= -\frac{\partial H_{\mathcal{L}}}{\partial t} = 0,
\end{aligned}
$$

where the dot denotes $\frac{d}{ds}$. We let s run along the ray from 0 to a point $\tau \in \mathbf{C}$. Because of group invariance we need to consider paths relative to the origin and a point $(x, t) = (x_1, x_2, t)$only, and assume boundary conditions

$$(18)\quad x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(\tau) = x_1, \quad x_2(\tau) = x_2, \quad t(\tau) = t.$$

From (13), the Hamiltonian,

$$\frac{1}{2}\dot{x}_1^2(s) + \frac{1}{2}\dot{x}_2^2(s) = H_{\mathcal{L}}(x, \xi, \theta) = H_0 \equiv \frac{1}{2}(\zeta_1(0)\zeta_1(0) + \zeta_2(0)\zeta_2(0)).$$

is constant along a given bicharacteristic. Form (17), we know that $\theta(s) = \theta(0) = \theta$ and we may take it to be the free parameter. The equations (13), (15) and (16) imply that

$$
\begin{aligned}
\dot{\zeta}_1 &= \dot{\xi}_1 + 2a\theta\dot{x}_2 = 2a\theta\zeta_2 + 2a\theta\zeta_2 = 4a\theta\zeta_2, \\
\dot{\zeta}_2 &= \dot{\xi}_2 - 2a\theta\dot{x}_1 = -2a\theta\zeta_1 - 2a\theta\zeta_1 = -4a\theta\zeta_1.
\end{aligned}
$$

Hence,

$$\zeta_1(s) = \cos(4a\theta s)\zeta_1(0) + \sin(4a\theta s)\zeta_2(0),$$

$$\zeta_2(s) = -\sin(4a\theta s)\zeta_1(0) + \cos(4a\theta s)\zeta_2(0).$$

Therefore, we may solve for $x(s)$ as a function of x, τ and θ, and then solve for $t(s)$ as a function of x, t, τ and θ. Here are the calculations.

$$\begin{aligned} x_1(s) &= \int_0^s \zeta_1(\rho)d\rho \\ &= \left\{\int_0^s \cos(4a\theta\rho)d\rho\right\}\zeta_1(0) + \left\{\int_0^s \sin(4a\theta\rho)d\rho\right\}\zeta_2(0) \\ &= -\frac{1}{4a\theta}\{-\sin(4a\theta s)\zeta_1(0) + \cos(4a\theta s)\zeta_2(0) - \zeta_2(0)\} \\ &= -\frac{1}{4a\theta}\{\zeta_2(s) - \zeta_2(0)\} \\ &= -\frac{1}{4a\theta}\left\{-\sin(4a\theta s)\zeta_1(0) + [\cos(4a\theta s) - 1]\zeta_2(0)\right\}. \end{aligned}$$

and

$$\begin{aligned} x_2(s) &= \frac{1}{4a\theta}\{\zeta_1(s) - \zeta_1(0)\} \\ &= \frac{1}{4a\theta}\left\{[\cos(4a\theta s) - 1]\zeta_1(0) + \sin(4a\theta s)\zeta_2(0)\right\}, \end{aligned}$$

Therefore,

$$(19)\quad \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \end{bmatrix} = \frac{4a\theta}{\sin^2(2a\theta\tau)} \begin{bmatrix} \sin(4a\theta\tau) & \cos(4a\theta\tau) - 1 \\ -\cos(4a\theta\tau) + 1 & \sin(4a\theta\tau) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It follows that

$$H_0 = \frac{1}{2}\left(\zeta_1(0)\zeta_1(0) + \zeta_2(0)\zeta_2(0)\right) = \frac{(2a\theta)^2}{2\sin^2(2a\theta\tau)}(x_1^2 + x_2^2) = \frac{(2a\theta)^2}{2\sin^2(2a\theta\tau)}\|x\|^2.$$

When $\theta = 0$, we have $\zeta(s) = \zeta(0)$, $x(s) = \zeta(0)s$ and $t(s) = t(0)$. Substituting these calculations into (14), we have

$$\begin{aligned} t - t(s) &= 2a\int_s^\tau [\zeta_1(\rho)x_2(\rho) - \zeta_2(\rho)x_1(\rho)]\,d\rho \\ &= \frac{2a}{4a\theta}\int_s^\tau \{\cos(4a\theta\rho)\zeta_1(0) + \sin(4a\theta\rho)\zeta_2(0)\} \end{aligned}$$

$$
\begin{aligned}
& \times \{[\cos(4a\theta\rho) - 1]\zeta_1(0) + \sin(4a\theta\rho)\zeta_2(0)\}\, d\rho \\
& + \frac{2a}{4a\theta}\int_s^\tau \{-\sin(4a\theta\rho)\zeta_1(0) + \cos(4a\theta\rho)\zeta_2(0)\} \\
& \qquad \times \{-\sin(4a\theta\rho)\zeta_1(0) + [\cos(4a\theta\rho) - 1]\zeta_2(0)\}\, d\rho \qquad (20) \\
= {} & \frac{1}{2\theta}\int_s^\tau [1 - \cos(4a\theta\rho)]d\rho \cdot \left[\zeta_1^2(0) + \zeta_2^2(0)\right] \\
= {} & \frac{(\tau - s)}{2\theta}\left[\zeta_1^2(0) + \zeta_2^2(0)\right] + \frac{1}{2\theta}\cdot\frac{1}{4a\theta}[\sin(4a\theta\tau) - \sin(4a\theta s)] \\
& \quad \times \left[\zeta_1^2(0) + \zeta_2^2(0)\right] \\
= {} & (\tau - s)\frac{2a^2\theta}{\sin^2(2a\theta\tau)}\|x\|^2 - \frac{a}{2}\cdot\frac{\sin(4a\theta\tau) - \sin(4a\theta s)}{\sin^2(2a\theta\tau)}\|x\|^2.
\end{aligned}
$$

Theorem 4.1. *The solution of equations* (13) *and* (14) *with boundary conditions* (18) *is*

$$x_1(s) = \frac{\sin(2a\theta s)}{\sin(2a\theta\tau)}\{\cos[2a\theta(s-\tau)]x_1 + \sin[2a\theta(s-\tau)]x_2\}, \tag{21}$$

$$x_2(s) = \frac{\sin(2a\theta s)}{\sin(2a\theta\tau)}\{-\sin[2a\theta(s-\tau)]x_1 + \cos[2a\theta(s-\tau)]x_2\} \tag{22}$$

$$t(s) = \left[\frac{a}{2}\frac{\sin(4a\theta\tau)-\sin(4a\theta s)}{\sin^2(2a\theta\tau)} - (\tau-s)\frac{2a^2\theta}{\sin^2(2a\theta\tau)}\right](x_1^2+x_2^2) - t. \tag{23}$$

The value of the Hamiltonian $H_{\mathcal{L}}$ on this path is

$$H_0 = \frac{2a^2\theta^2}{\sin^2(2a\theta\tau)}(x_1^2 + x_2^2).$$

Next (20) yields

$$t - t(0) = a\mu(2a\theta\tau)\|x\|^2,$$

where we set

$$\mu(z) = \frac{z}{\sin^2 z} - \cot z.$$

The action integral associated to the Hamiltonian curve is

$$S(x,t,\tau,\theta)=\int_0^\tau\left\{\sum_{j=1}^2\xi_j(s)\dot{x}_j(s)+\theta\dot{t}(s)-H_{\mathcal{L}}(x(s),\xi(s),\theta)\right\}ds.$$

H is homogeneous of degree 2 with respect to (ξ_1,ξ_2,θ), so

$$(24)\quad S=\int_0^\tau\left\{\sum_{j=1}^2\xi_j\frac{\partial H_{\mathcal{L}}}{\partial\xi_j}+\theta\frac{\partial H_{\mathcal{L}}}{\partial\theta}-H_{\mathcal{L}}\right\}ds=\int_0^\tau(2H_{\mathcal{L}}-H_{\mathcal{L}})ds=\tau H_0.$$

From (21)-(24), we have the following theorem:

Theorem 4.2. *The action integral $S(x,t,\tau,\theta)$ is given by*

$$\begin{aligned}S(x,t,\tau,\theta)&=\frac{\tau(2a\theta)^2}{2\sin^2(2a\theta\tau)}\|x\|^2,\\&=[t-t(0)]\theta+a\theta\cot(2a\theta\tau)(x_1^2+x_2^2),\quad\theta\in\left[0,\frac{\pi}{a}\right).\end{aligned}$$

It is convenient to fix τ, $\tau=1$. Then the Hamiltonian paths are determined entirely by the parameter θ. We may take the end points to be $(0,0)$ and (x,t). Then θ must satisfy

$$t=a\mu(2a\theta)(x_1^2+x_2^2)=a\mu(2a\theta)\|x\|^2.$$

It can be shown that μ is a monotone increasing diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbf{R}$. On each interval $(m\pi,(m+1)\pi)$, $m=1,2,\ldots$, μ has a unique critical point z_m. On this interval μ decreases strictly from $+\infty$ to $\mu(z_m)$ and then increases strictly from $\mu(z_m)$ to $+\infty$. Now the complete picture of the geodesics is given in the following two theorems.

Theorem 4.3. *There are finitely many geodesics that join the origin to (x,t) if and only if $x\neq 0$. These geodesics are parametrized by the solutions*

θ of

$$a\mu(2a\theta)\|x\|^2 = |t|, \tag{25}$$

and their lengths increase strictly with θ. There is exactly one such geodesic if and only if

$$|t| < a\mu(z_1)\|x\|^2,$$

and the number of geodesics increases without bound as $\frac{|t|}{a\|x\|^2} \to \infty$.

The square of the length of the geodesic associated to a solution θ of (25) is

$$\begin{aligned} 2S(x,|t|,1,\theta) &= \frac{(2a\theta)^2}{\sin^2(2a\theta)}(x_1^2+x_2^2) \\ &= \frac{(2a\theta)^2}{\sin^2(2a\theta)}\frac{(x_1^2+x_2^2)}{(x_1^2+x_2^2)+|t|/a}\left[\frac{|t|}{a}+(x_1^2+x_2^2)\right] \\ &= \frac{(2a\theta)^2}{\sin^2(2a\theta)}\frac{1}{1+\mu(2a\theta)}\left[\frac{|t|}{a}+(x_1^2+x_2^2)\right] \\ &= \nu(2a\theta)\left(\frac{|t|}{a}+\|x\|^2\right), \end{aligned}$$

where $\nu(0) = 2$ and otherwise

$$\nu(z) = \frac{z^2}{z+\sin^2 z - \sin z\cos z}.$$

Consequently, if $2a\theta \in (m\pi, (m+1)\pi)$ the length d_θ of the geodesic satisfies

$$\frac{m^2\pi^2}{(m+1)\pi+2}\left(\frac{|t|}{a}+\|x\|^2\right) < (d_\theta)^2 < \frac{(m+1)^2\pi^2}{m\pi}\left(\frac{|t|}{a}+\|x\|^2\right).$$

When $x = 0$, we need to find the Hamiltonian paths connecting the origin to $(0,t)$, *i.e.*, $x_1(1) = 0$, $x_2(1) = 0$, $t(1) = t$. This implies that $\zeta_1(1) = \zeta_1(0)$ and $\zeta_2(1) = \zeta_2(0)$. It follows that

$$\zeta_1(1) = \cos(4a\theta)\zeta_1(0) + \sin(4a\theta)\zeta_2(0) = \zeta_1(0),$$

$$\zeta_2(1) = -\sin(4a\theta)\zeta_1(0) + \cos(4a\theta)\zeta_2(0) = \zeta_2(0)$$

This implies that

$$\sin(4a\theta) = 0, \quad \text{and} \quad \cos(4a\theta) = 1$$

i.e.,

$$2a\theta = m\pi, \qquad \text{with} \quad m = 1, 2, 3, \dots. \tag{26}$$

In this case,

$$t = \frac{1}{2\theta}(\zeta_1^2(0) + \zeta_2^2(0)),$$

therefore, $\theta \neq 0$ and $m \neq 0$ in (26). We also know that

$$d_m^2 = \frac{m\pi|t|}{a}.$$

Summaring, we have the following theorem.

Theorem 4.4. *The geodesics that join the origin to a point* $(0, t)$ *have lengths* d_1, d_2, $d_3, \dots$, *where*

$$d_m^2 = \frac{m\pi|t|}{a}.$$

Since $x_1(1) = x_2(1) = 0$, we may use (26) and $(\zeta_1(0), \zeta_2(0))$ to obtain the geodesics as follows:

$$\begin{aligned} x_1^{(m)}(s) &= -\frac{1}{2m\pi}\left\{-\sin(2m\pi s)\zeta_1(0) + [\cos(2m\pi s) - 1]\zeta_2(0)\right\} \\ &= \left(\frac{t}{4am\pi}\right)^{\frac{1}{2}}\left\{\sin(2m\pi s)\frac{\zeta_1(0)}{\|\zeta(0)\|} + [1 - \cos(2m\pi s)]\frac{\zeta_2(0)}{\|\zeta(0)\|}\right\}, \end{aligned}$$

where $\|\zeta(0)\| = \sqrt{\zeta_1^2(0)+\zeta_2^2(0)}$. Similarly, we have

$$\begin{aligned} x_2^{(m)}(s) &= \frac{1}{2m\pi}\left\{[\cos(2m\pi s)-1]\zeta_1(0)+\sin(2m\pi s)\zeta_2(0)\right\} \\ &= \left(\frac{t}{4am\pi}\right)^{\frac{1}{2}}\left\{[\cos(2m\pi s)-1]\frac{\zeta_1(0)}{\|\zeta(0)\|}+\sin(2m\pi s)\frac{\zeta_2(0)}{\|\zeta(0)\|}\right\}, \end{aligned}$$

and

$$t^{(m)}(s) = [2m\pi s - \sin(2m\pi s)]\,\frac{t}{2m\pi}.$$

This shows that for each fixed m, $m = 1, 2, \ldots$, the geodesics $(x_1^{(m)}(s), x_2^{(m)}(s), t^{(m)}(s))$ can be parametrized by a unit vector $\zeta(0)/\|\zeta(0)\|$ on the unit circle. These curves lie in a cylinder around the t-axis whose radius is $\mathcal{O}(1/\sqrt{m})$. To consider the complex action, we fix θ,

$$\theta = -i.$$

The complex action integral is defined by:

$$\begin{aligned} g(x,t,\tau) &= -it + \int_0^1 [\dot{x}_1\xi_1 + \dot{x}_2\xi_2 - H_{\mathcal{L}}]\,ds \\ &= -it + a\coth(2a\tau)(x_1^2+x_2^2). \end{aligned}$$

Moreover, g satisfies the Hamilton-Jacobi equation

$$\frac{\partial g}{\partial\tau} + \frac{1}{2}(X_1 g)^2 + \frac{1}{2}(X_2 g)^2 = \frac{\partial g}{\partial\tau} + 2(\mathbf{Z}g)(\bar{\mathbf{Z}}g) = 0.$$

From classical calculations one has

$$\frac{\partial g}{\partial x_1} = \xi_1, \quad \frac{\partial g}{\partial x_2} = \xi_2, \quad \text{and} \quad \frac{\partial g}{\partial t} = \theta.$$

Therefore,

$$H_{\mathcal{L}}\left(x_1, x_2, \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right) = \frac{1}{2}(\zeta_1^2(0)+\zeta_2^2(0)) = H_0,$$

and the Hamilton-Jacobi equation yields

$$0 = \frac{\partial g}{\partial \tau} + H_0 = \frac{\partial g}{\partial \tau} + \frac{S}{\tau} = \frac{\partial g}{\partial \tau} + \frac{1}{\tau}[g(x,t,\tau) + it(0)].$$

Theorem 4.5. *Suppose $x \neq 0$. Then the unique critical point with respect to τ of the modified complex action function*

$$f(x,t,\tau) = \tau g(x,t,\tau) = -i\tau t + a\tau \coth(2a\tau)(x_1^2 + x_2^2)$$

in the strip $\{|\mathrm{Im}\,(\tau)| < \pi/2a\}$ is the point $\tau_c(x,t) = i\theta_c(x,t)$, where θ_c is the solution of the equation (25) *in this interval. At the critical point*

$$f(x,t,\tau_c(x,t)) = S(x,t,1;\theta_c) = \frac{1}{2}d^2(x,t). \tag{27}$$

The identity (27) *is also valid at points* $(0,t)$, $t \neq 0$.

Comparing the above calculations with (12), we can rewrite the heat kernel associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_0$ as follows:

$$P_{\mathcal{L}} f(x,t,u) = e^{-\mathcal{L}u} f(x,t,u) = \int_{\mathbf{H}_n} h_u((y,s)^{-1} \circ (x,t)) f(y,s)dyds,$$

where

$$h_u(x,t) = \frac{1}{(2\pi u)^{n+1}} \int_{\mathbf{R}} e^{-\frac{f(x,t,\tau)}{u}} V(r)d\tau.$$

Here

$$f(x,t,\tau) = \tau g(x,t,\tau) = -i\tau u + a\tau(x_1^2 + x_2^2)\coth(2a\tau),$$

and

$$V(r) = \frac{2a\tau}{\sinh(2a\tau)}.$$

Using the complex action function $f(x,t,\tau)$, we may discuss the small time behavior of the heat kernel. To simplify the problem, we assume that $a_j = a$ for all $j = 1, \ldots, n$ (see [4]).

Theorem 4.6. *Given a fixed point (x,t), $x \neq \mathbf{0}$, let θ_c denote the solution of equation (25) in the interval $[0,\pi/2a)$. Then the heat kernel for isotropic Heisenberg groups has the following small time behavior:*

$$h_u(x,t) = \frac{1}{(2\pi u)^{n+1}} e^{-\frac{d^2(x,t)}{2u}} \left\{ \Theta(x,t)\sqrt{2\pi u} + \mathcal{O}(u) \right\}, \qquad u \to 0^+,$$

where

$$\Theta(x,t) = \Big(\frac{1}{f''(i\theta_c)}\Big)^{\frac{1}{2}} \nu(i\theta_c) = \frac{\theta_c}{\sqrt{[1-2a\theta_c \cot(2a\theta_c)](x_1^2+x_2^2)}} \Big\{\frac{2a\theta_c}{\sin(2a\theta_c)}\Big\}^{n-1},$$

with $f''(\tau) = \frac{d^2 f}{d\tau^2}$.

Theorem 4.7. *At point $(\mathbf{0},t)$, $t \neq 0$, we have*

$$h_u(x,t) = \frac{t^{n-1}}{(n-1)!(2\sqrt{a}u)^{2n}} e^{-\frac{d^2(\mathbf{0},t)}{2u}} \left\{1 + \mathcal{O}(u)\right\}, \qquad u \to 0^+.$$

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