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HOMOGENEOUS MAXIMAL ESTIMATES FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

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Abstract. Homogeneous maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation. Also more general oscillatory integrals are studied.

1. Introduction. Let f belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}(\xi) d\xi, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

where $a > 1$. Here $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x) dx.$$

It then follows that $u(x,0) = f(x)$ and in the case $a = 2$ u is a solution to the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We shall here consider maximal functions

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \qquad x \in \mathbb{R}^n,$$

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and

$$S^{**}f(x) = \sup_{t>0} |S_t f(x)|, \qquad x \in \mathbb{R}^n.$$

We also introduce Sobolev spaces H_s by setting

$$H_s = \{f \in \mathcal{S}';\ \|f\|_{H_s} < \infty\}, \qquad s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

We shall also consider homogeneous Sobolev spaces $\dot{H}_s$ defined by

$$\dot{H}_s = \{f \in \mathcal{S}';\ \|f\|_{\dot{H}_s} < \infty\}, \qquad s \in \mathbb{R},$$

where

$$\|f\|_{\dot{H}_s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

It is of interest to study local estimates

$$\|S^* f\|_{L^q(B)} \leq C_B \|f\|_H$$

and

$$\|S^{**} f\|_{L^q(B)} \leq C_B \|f\|_H$$

where B is an arbitrary ball in $\mathbb{R}^n$ and H denotes H_s or $\dot{H}_s$, and global estimates

$$\|S^* f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_H$$

and

$$\|S^{**} f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_H.$$

Estimates of this type have been considered in P. Sjölin [6], [7], [8], [9], [10], and F. Gülkan [2], and in several other papers. We do not give a complete list of references but refer to the references in the mentioned papers. We shall here concentrate on the estimate

$$\|S^{**}f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{\dot{H}_s}. \tag{1}$$

We shall always assume $1 \le q \le \infty$ and $s \in \mathbb{R}$. We shall obtain the following theorem as a consequence of results and methods in the papers [6], [7], [9], [10], and [2].

Theorem 1. *In the case* $n = 1$ *the estimate* (1) *holds if and only if* $1/4 \le s < 1/2$ *and* $q = 2/(1-2s)$. *In the case* $n \ge 2$ *and* f *radial the estimate* (1) *holds if and only if* $1/4 \le s < n/2$ *and* $q = 2n/(n-2s)$.

Remark 1. In the case $n \ge 2$ and f general (not necessarily radial) we have only the following partial results. If the estimate (1) holds then $n/2(n+1) \le s < n/2$ and $q = 2n/(n-2s)$. If $n/4 \le s < n/2$ and $q = 2n/(n-2s)$ then the estimate (1) holds.

Remark 2. Special cases of the above results are contained in C. E. Kenig, G. Ponce, and L. Vega [3], [4], [5], but we have not been able to find the complete results in the literature.

2. Proofs.

Proof of the Theorem. We first observe that a necessary condition for (1) to hold is $s \ge 1/4$. This follows from counter-examples in [6], p. 712–713, and [9], p. 55–58. These counter-examples are originally constructed in the case of inhomogeneous Sobolev spaces H_s but they also work for homogeneous spaces $\dot{H}_s$.

We shall then prove that $q = 2n/(n-2s)$ is a necessary condition for (1). For $f \in \mathcal{S}$ we set $f_R(x) = f(Rx)$, $R > 0$, and then have $\hat{f}_R(\xi) = R^{-n}\hat{f}(\xi/R)$.

It is easy to see that $S_t f_R(x) = S_{tR^a}(Rx)$ and we therefore have

$$S^{**} f_R(x) = S^{**} f(Rx).$$

Now assume that (1) holds. Then

$$\|S^{**} f_R\|_q \leq C \|f_R\|_{\dot{H}_s},$$

where we have written $\| \quad \|_q$ instead of $\| \quad \|_{L^q(\mathbb{R}^n)}$. The left hand side above equals $R^{-n/q}\|S^{**}f\|_q$ and the right hand side is equal to $R^{-n/2+s}\|f\|_{\dot{H}_s}$. It follows that

$$R^{-n/q}\|S^{**}f\|_q \leq C R^{-n/2+s}\|f\|_{\dot{H}_s}.$$

This can hold for all $R > 0$ only if

$$-\frac{n}{q} = -\frac{n}{2} + s,$$

which gives the equality $q = 2n/(n-2s)$.

Since $1 \leq q \leq \infty$ it follows that $s \leq n/2$ is a necessary condition for (1). However, it is easy to see that the case $s = n/2$, $q = \infty$ is impossible (cf. [9]), and we have therefore proved that $1/4 \leq s < n/2$ and $q = 2n/(n-2s)$ is a necessary condition for (1) (also in the case of radial functions).

By use of an extension of a method in [6], it is proved in [2] that (1) holds if $n/4 \leq s < n/2$ and $q = 2n/(n-2s)$ (see Theorems 2.5, 2.6 and 2.12 in [2]). The proof of the Theorem in the case $n = 1$ is therefore complete.

We now turn to the case $n \geq 2$ and f radial. We shall prove that

$$\|S^{**} f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{H_s} \tag{2}$$

in the case $s = 1/4$ and $q = 4n/(2n-1)$. The sufficiency part of the result in the Theorem will then follow from interpolation between this result and the case $s = n/4$, $q = 4$.

To study the case $s = 1/4$, $q = 4n/(2n-1)$ we shall extend a method in [7]. It is proved in [7] that the estimate

$$\|S^* f\|_{L^q(B)} \leq C_B \|f\|_{H_{1/4}}$$

holds for the above value of q and f radial. It is observed in [2] that one also has

$$\|S^{**} f\|_{L^q(B)} \leq C_B \|f\|_{H_{1/4}}$$

and we shall prove that also

$$\|S^{**} f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{H_{1/4}}. \tag{3}$$

Following [7] we set

$$P^* g(r) = (1+r^2)^{-1/8} r^{1/2} \int_0^\infty J_{n/2-1}(rs) e^{-it(s)r^a} s^{(n+1)/4n} g(s) ds$$

for $0 < r < \infty$, where $g \in L^1(0,\infty)$ and has compact support. Here $J_{n/2-1}$ denotes a Bessel function and $t(s)$ is a positive measurable function on $(0,\infty)$. For $q = 4n/(2n-1)$ we define p by the formula $1/p + 1/q = 1$, so that $4/3 < p < 2$. Arguing as in [7] we then conclude that to prove (3) it is sufficient to prove that

$$\Big(\int_0^\infty |P^* g(r)|^2 dr \Big)^{1/2} \leq C \Big(\int_0^\infty |g(s)|^p ds \Big)^{1/p} \tag{4}$$

for all $g \in L^p(0,\infty)$ with compact support.

Following [7] we write

$$P^* g(r) = b_1 A(r) + b_2 B(r) + Q(r),$$

where

$$A(r) = (1+r^2)^{-1/8}\int_0^\infty e^{irs}e^{-it(s)r^a}s^{-\gamma}g(s)ds,$$
$$B(r) = (1+r^2)^{-1/8}\int_0^\infty e^{-irs}e^{-it(s)r^a}s^{-\gamma}g(s)ds$$

and

$$|Q(r)| \leq C(1+r^2)^{-1/8}\int_0^\infty \min\Big(1,\,\frac{1}{rs}\Big)s^{-\gamma}|g(s)|ds.$$

Here $\gamma = 1/q - 1/4 = (n-1)/4n$ and b_1 and b_2 denote constants.

A and B can then be estimated as in [7]. The proof in [7] treats the case $0 < t(s) < 1$ but the same proof works in our case $t(s) > 0$. Also the estimate

$$\Big(\int_1^\infty |Q(r)|^2 dr\Big)^{1/2} \leq C\|g\|_p$$

can be proved by use of the method in [7]. To prove that also

$$\Big(\int_0^1 |Q(r)|^2 dr\Big)^{1/2} \leq C\|g\|_p \tag{5}$$

we can argue in the following way. For $0 < r < 1$ we have

$$|Q(r)| \leq C\int_0^{1/r} s^{-\gamma}|g(s)|ds + C\int_{1/r}^\infty \frac{1}{rs}\,s^{-\gamma}|g(s)|ds.$$

Observing that $\gamma q = 1 - q/4$ and $(-1-\gamma)q = -1 - 3q/4$ we obtain

$$\begin{aligned} |Q(r)| &\leq C\Big(\int_0^{1/r} s^{-\gamma q}ds\Big)^{1/q}\|g\|_p + C\,\frac{1}{r}\Big(\int_{1/r}^\infty s^{(-1-\gamma)q}ds\Big)^{1/q}\|g\|_p \\ &= C_1\Big(\frac{1}{r}\Big)^{(1-\gamma q)/q}\|g\|_p + C_2\,\frac{1}{r}\Big(\frac{1}{r}\Big)^{[(-1-\gamma)q+1]/q}\|g\|_p \\ &= C_1 r^{-1/4}\|g\|_p + C_2 r^{-1+3/4}\|g\|_p = Cr^{-1/4}\|g\|_p \end{aligned}$$

for $0 < r < 1$, and (5) follows. Thus (4) and (3) are proved.

Interpolating between (3) and the corresponding estimate in the case $s = n/4$, $q = 4$, we obtain (2) for $1/4 \leq s \leq n/4$ and $q = 2n/(n-2s)$ (see

J. Bergh and J. Löfström [1], p. 120). Using the proof of Theorem 2.6 in [2] we can then conclude that we also have the homogeneous estimate (1) for the same values of s and q. Thus we have proved that the homogeneous estimate (1) for radial functions holds for $1/4 \leq s < n/2$ and $q = 2n/(n-2s)$. The proof of the Theorem is complete.

The sufficiency condition in Remark 1 follows from the proof of the Theorem. To obtain the necessary condition in Remark 1 we argue as follows. We know from the proof of the Theorem that $1/4 \leq s < n/2$ and $q = 2n/(n-2s)$ is a necessary condition for the homogeneous estimate (1). To obtain also the necessary condition $s \geq n/2(n+1)$ we invoke a result from Sjölin [10], which states that if

$$\left(\int_B |S^* f|^q dx\right)^{1/q} \leq C_B \|f\|_{H_s} \tag{6}$$

for every ball B, then

$$s + \frac{n-1}{2q} \geq \frac{n}{4}. \tag{7}$$

Since the homogeneous estimate (1) is stronger than (6) we conclude that (1) implies (7). Combining (7) with the equality $q = 2n/(n-2s)$, we obtain

$$s + \frac{(n-1)(n-2s)}{4n} \quad \geq \quad \frac{n}{4}$$

and

$$4ns + (n-1)(n-2s) \quad \geq \quad n^2.$$

Simplifying this inequality we obtain $s \geq n/2(n+1)$ and hence the results in Remark 1 are proved.

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