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ON THE ISOMETRY GROUP OF A NON-EUCLIDEAN PLANE

BY

C. THAS

Abstract. Any isometry of the hyperbolic plane **H** determines uniquely a special conic in or outside this plane. Conversely, any conic of this kind correspond with an isometry and its inverse. In this paper we give a representation of the isometry group of **H** wherein the elements of the group are these conics.

1. Introduction. We use the Cayley-Klein model in the Euclidean plane (also called the Beltrami model) for the hyperbolic plane **H**, i.e. we consider in the Euclidean plane Π the real unit circle $X^2 + Y^2 = 1$ or in homogeneous coordinates $x^2 + y^2 - z^2 = 0$, which is the absolute conic Γ of **H** and the points of the hyperbolic plane are the points of Π inside Γ (a point p is *inside* Γ if there exist two conjugate imaginary tangents to the real circle Γ through p). Points of Γ are absolute points and points outside Γ are ultra points (a point p is *outside* Γ if there exist two real tangents to Γ through p). Straight lines of **H** are the parts inside Γ of straight lines of Π, which have two real common points with Γ. In the following, we use the same notation for an **H**-line and for the corresponding (extended) line in the Euclidean plane. The **H**-distance $h(p,q)$ between two **H**-points p and q is given by: $h(p,q) = \frac{1}{2} ln(p\ q\ i_1\ i_2)$, where $i_1,\ i_2$ are the intersection points of the line pq with Γ and where ln is the natural logarythm of the cross-ratio of the four points $p,\ q,\ i_1,\ i_2$.

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Next, we recall the classification of the isometries of **H**. An isometry of **H** is the restriction to the inside of Γ of a projectivity of Π which leaves the absolute circle Γ invariant.

1. The first kind is the **H**-symmetry with respect to an **H**-line or the **H**-reflection in an **H**-line. Suppose that L is a **H**-line and that the ultra point l is its pole with regard to Γ. The **H**-reflection in L is the restriction to the inside of Γ of the harmonical homology with center l and axis L. It is well known that any isometry of **H** is the product of two or three **H**-reflections. We have the following possibilities.
2. For the second kind, we consider two **H**-lines L_1 and L_2, which have a common **H**-point p. The product of the **H**-reflections in L_1 and L_2 is an **H**-rotation with center (fixed point) p. We find a special case if L_1 and L_2 are **H**-orthogonal (conjugate lines with regard to Γ): the **H**-reflection in the point p or the **H**-symmetry with respect to p, i.e. the (restriction of the) harmonical homology with center p and axis the polar line of p with respect to Γ.
3. The third kind is the **H**-translation, which is the product of two **H**-reflections in **H**-lines L_1 and L_2 with a common ultra point p.
4. Next, we have the **H**-glide reflection. Consider again two **H**-lines L_1 and L_2 with a common ultra point p. The product of the **H**-reflections in L_1, L_2 and the **H**-reflection in the common **H**-perpendicular P of L_1, L_2 (this is the polar line of p with regard to Γ), is an **H**-glide reflection, called parallel with the **H**-line P.
5. Finally, if L_1 and L_2 are **H**-lines with a common absolute point p, the product of the **H**-reflections in L_1 and L_2 is an **H**-parallel displacement.
6. Of course, we have also the identity transformation of **H**.

2. The Conic Associated with an H-Isometry. First, consider in the hyperbolic plane an **H**-rotation with fixed **H**-point p, and suppose that it is not the **H**-reflection in p, i.e. it is not a rotation over an **H**-angle

π. In the Euclidean plane Π this rotation extends to a projectivity which leaves Γ invariant and the three fixed points of this projectivity are p and the two imaginary intersection points i_1, i_2 of Γ with the polar line of p with regard to Γ. The restriction of this projectivity to Γ gives a projectivity on Γ with fixed points i_1, i_2 and it is well known that the lines connecting corresponding points of this projectivity are tangent lines of a conic $\mathcal{C}$ which is double tangent at i_1 and i_2 with Γ (e.g. see [4], page 112, theorem 7.61). It is easy to see that this conic $\mathcal{C}$ lies entirely inside Γ and thus it is an **H**-circle. Of course the **H**-center of $\mathcal{C}$ is p and $\mathcal{C}$ is invariant for the rotation. So, any **H**-rotation, which is not a point reflection, determines a unique **H**-circle. Remark that, calling two **H**-lines parallel if they intersect on the absolute Γ, the tangent lines of this **H**-circle are the **H**-lines which have a parallel image under the rotation.

Conversely, with a given **H**-circle correspond in this way clearly two **H**-rotations: a rotation and its inverse. But an oriented **H**-circle $\mathcal{C}$ determines in an obvious way just one **H**-isometry: an orientation on $\mathcal{C}$ generates an orientation on each tangent line T at $\mathcal{C}$ and if $T \cap \Gamma = \{a, b\}$, the **H**-rotation corresponding with the oriented **H**-circle $\mathcal{C}$ induces on Γ the projectivity which transforms a into b if the orientation of the vector $\vec{ab}$ coincides with the orientation on T.

Next, consider the **H**-reflection in an **H**-point p. The lines connecting corresponding points of the induced involution on Γ are of course the lines through p and are tangent lines of the degenerate **H**-circle $\mathcal{DC}$ with center p and components the two imaginary tangent lines through p at Γ. In this case $\mathcal{DC}$ can be considered as an **H**-circle with radius 0.

If the **H**-isometry is an **H**-translation, the product of the **H**-reflections in **H**-lines L_1 and L_2, where $L_1 \cap L_2$ is an ultra point p, we have, for the induced projectivity on Γ, two different real fixed points q_1, q_2 on Γ (the intersection points of Γ with the polar line P of p with respect to Γ) and the lines connecting corresponding points on Γ are the tangent lines of a conic

$\mathcal{E}$, inside Γ, which is double tangent with Γ at the points $q_1,\ q_2$. This means that $\mathcal{E}$ is an equidistant curve, i.e. it is the locus of the **H**-points which are at a constant **H**-distance from the **H**-line q_1q_2.

So, any **H**-translation determines a unique equidistant curve $\mathcal{E}$ of the hyperbolic plane and conversely, an equidistant curve $\mathcal{E}$ with an (opposite) orientation on both parts of $\mathcal{E}$ separated by the line connecting the contact points of $\mathcal{E}$ and Γ, determines a unique **H**-translation (see the example in section 5). Let us call such equidistant curve with two opposite orientations on both parts, an "oriented equidistant curve".

Next, consider a glide reflection, the product of the **H**-reflections in two **H**-lines L_1 and L_2, where $L_1 \cap L_2$ is an ultra point p, and of the **H**-reflection in the common **H**-perpendicular P of L_1 and L_2. On Γ, we have again two fixed real points $q_1,\ q_2$ (the intersection points of P with Γ) and the lines connecting corresponding points on Γ are the tangent lines of a conic $\mathcal{UE}$ which lies outside Γ and is double tangent at Γ at $q_1,\ q_2$. Why is $\mathcal{UE}$ a conic outside Γ (an ultra conic)? It is easy to see that in the case of a translation, a line connecting corresponding points on Γ has a common ultra point with the line q_1q_2, while for a glide reflection such line intersects q_1q_2 in an **H**-point and $\mathcal{UE}$ is a conic outside Γ. We call $\mathcal{UE}$ an *ultra equidistant curve.* Remark that Γ lies outside any $\mathcal{UE}$: through each real point of Γ, different from the contact points q_1 and q_2, there are two real tangent lines at the $\mathcal{UE}$.

Any **H**-glide reflection determines a unique ultra equidistant curve $\mathcal{UE}$ and conversely, in the same way as for an **H**-translation, an ultra equidistant curve $\mathcal{UE}$ with two opposite orientations on both parts of it, determines a unique glide reflection (see the example in section 5). We call it an "oriented ultra equidistant curve".

If the **H**-isometry is a parallel translation, we have one fixed real point q on Γ and the **H**-lines which have a parallel image are the tangent lines of a conic $\mathcal{H}$, inside Γ, which has only q common with Γ (four common points with Γ at q or hyper osculating with Γ at q), i.e. $\mathcal{H}$ is an horocycle of **H**. So, any **H**-parallel translation determines a unique horocycle $\mathcal{H}$ and conversely, as for an **H**-rotation, any oriented horocycle determines a unique parallel translation.

Next, consider an **H**-reflection in an **H**-line L. Any line connecting corresponding points on Γ is a line through the polar ultra point l of L with regard to Γ and is a tangent line of what we call a *degenerate ultra equidistant curve*, i.e. a degenerate conic $\mathcal{DUE}$ with components the two real tangents through the ultra point l at Γ.

Finally, with the identity transformation of **H** corresponds the identity transformation on Γ and the corresponding conic is Γ.

Definition. We call each kind of conic corresponding with an **H**-isometry a *generalized* **H**-*circle*, a $\mathcal{GC}$ for short.

This means that a $\mathcal{GC}$ is a non degenerate real conic of the Euclidean plane which is double tangent or hyper osculating at Γ, such that Γ lies outside the conic; or the $\mathcal{GC}$ is a degenerate conic consisting in two real or two conjugate imaginary tangent lines of Γ. Moreover Γ itself is also a $\mathcal{GC}$.

We have:

Theorem. *With any* **H**-*isometry corresponds a unique* (*oriented if non degenerate and different from* Γ) $\mathcal{GC}$ *in the following way:*

Identical transformation of **H** $\leftrightarrow$ *the absolute* Γ.
H-*rotation* (*not over* π) $\leftrightarrow$ *oriented* **H**-*circle* $\mathcal{C}$.
H-*point reflection* $\leftrightarrow$ *degenerate* **H**-*circle* $\mathcal{DC}$.
H-*translation* $\leftrightarrow$ *oriented equidistant curve* $\mathcal{E}$.
H-*glide reflection* $\leftrightarrow$ *oriented ultra equidistant curve* $\mathcal{UE}$.
H-*line reflection* $\leftrightarrow$ *degenerate ultra equidistant curve* $\mathcal{DUE}$.
H-*parallel translation* $\leftrightarrow$ *oriented horocycle* $\mathcal{H}$.

Conversely, with any oriented (if non degenerate and different from Γ*)* $\mathcal{GC}$ *corresponds a unique* **H***-isometry.*

Because of this theorem we have a bijection between the **H**-isometries and the elements of the set of the generalized circles with an orientation (except for Γ and for the degenerate $\mathcal{GC}$'s):

$\mathcal{GC} = \{\Gamma,\ \mathcal{C},\ \mathcal{DC},\ \mathcal{E},\ \mathcal{UE},\ \mathcal{DUE},\ \mathcal{H},\ \|$ where $\Gamma,\ \mathcal{C},\ \mathcal{DC},\ \mathcal{E},\ \mathcal{UE},\ \mathcal{DUE},\ \mathcal{H}$ are the generalized circles and where the conics $\mathcal{C},\ \mathcal{E},\ \mathcal{UE},\ \mathcal{H}$ are oriented $\}$.

The group structure of the **H**-isometry group $\mathbf{I_H}$ induces a group structure on this set $\mathcal{GC}$ and so we have:

$$\mathbf{I_H} \cong \mathcal{GC}.$$

This means that the isometry group $\mathbf{I_H}$ of the hyperbolic plane is isomorphic with a group whose elements are (oriented) conics.

3. The Equation of a Generalized Circle of H. Consider again the model of the hyperbolic plane inside the circle Γ with equation $x^2+y^2-z^2=0$ of the Euclidean plane. A generalized circle is double tangent or hyper osculating at Γ and has an equation of the form

$$k(x^2+y^2-z^2)+(xx_0+yy_0-zz_0)^2=0,\quad k, x_0, y_0, z_0 \in \mathbf{R} \tag{3.1}$$

Of course, the value of k is not arbitrary, because not every conic of this kind is a generalized circle.

For $k=\infty$, we find Γ. If $k \neq \infty$, we get a conic $\mathcal{K}$ for which the common points with Γ are given by $x^2+y^2-z^2=0,\ (xx_0+yy_0-zz_0)^2=0$ and we find two real different common points if $p(x_0,y_0,z_0)$ is an ultra point or $x_0^2+y_0^2-z_0^2>0$, we have two conjugate imaginary points if $p(x_0,y_0,z_0)$ is an **H**-point or $x_0^2+y_0^2-z_0^2<0$, we find one real point if $p(x_0,y_0,z_0)$ is an absolute point or $x_0^2+y_0^2-z_0^2=0$.

A straightforward calculation shows that the conic is degenerate iff $k = z_0^2 - x_0^2 - y_0^2$ or $k = 0$. Suppose that $p(x_0, y_0, z_0)$ is an **H**-point ($x_0^2 + y_0^2 - z_0^2 < 0$). Then the conic (3.1) is a degenerate **H**-circle $\mathcal{DC}$ if $k = z_0^2 - x_0^2 - y_0^2$ and it will be a non degenerate **H**-circle $\mathcal{C}$ if it is a real non degenerate conic (an ellipse in the Euclidean plane) which lies inside Γ, i.e. if $k > z_0^2 - x_0^2 - y_0^2$ ($0 < k < z_0^2 - x_0^2 - y_0^2$ gives an imaginary conic (an ellipse in the Euclidean plane) and with $k < 0$ corresponds a real conic (ellipse, parabola or hyperbola) outside Γ).

Next, assume that $p(x_0, y_0, z_0)$ is an ultra point ($x_0^2 + y_0^2 - z_0^2 > 0$). Then the conic (3.1) is a degenerate ultra equidistant curve $\mathcal{DUE}$ if $k = z_0^2 - x_0^2 - y_0^2$. It is an equidistant curve $\mathcal{E}$ if $k > 0$ (a real ellipse in the Euclidean plane) and it will be an ultra equidistant curve $\mathcal{UE}$ if $z_0^2 - x_0^2 - y_0^2 < k < 0$ (a hyperbola in the Euclidean plane). Remark that $k < z_0^2 - x_0^2 - y_0^2 (< 0)$ gives a real conic $\mathcal{K}$ (ellipse, hyperbola or parabola) such that Γ lies inside $\mathcal{K}$, and this is not a generalized circle.

If $p(x_0, y_0, z_0)$ is an absolute point, then the conic (3.1) is non degenerate and real (if $k \neq 0$). With $k > 0$ corresponds an horocycle $\mathcal{H}$, while $k < 0$ gives a real conic which is outside Γ and is hyperosculating at Γ, thus not a generalized circle.

We get the following list:

$$k(x^2 + y^2 - z^2) + (xx_0 + yy_0 - zz_0)^2 = 0. \tag{3.2}$$

$k = \infty \leftrightarrow$ the absolute conic $\Gamma \leftrightarrow$ identical transformation of **H**. $x_0^2 + y_0^2 - z_0^2 < 0$ and $k > z_0^2 - x_0^2 - y_0^2 \leftrightarrow$ **H**-circle $\mathcal{C} \leftarrow$ **H**-rotation. $x_0^2 + y_0^2 - z_0^2 < 0$ and $k = z_0^2 - x_0^2 - y_0^2 \leftrightarrow$ degenerate **H**-circle $\mathcal{DC} \leftrightarrow$ **H**-point reflection. $x_0^2 + y_0^2 - z_0^2 > 0$ and $k > 0 \leftrightarrow$ equidistant curve $\mathcal{E} \leftarrow$ **H**-translation. $x_0^2 + y_0^2 - z_0^2 > 0$ and $z_0^2 - x_0^2 - y_0^2 < k < 0 \leftrightarrow$ ultra equidistant curve $\mathcal{UE} \leftarrow$ **H**-glide reflection. $x_0^2 + y_0^2 - z_0^2 > 0$ and $k = z_0^2 - x_0^2 - y_0^2 \leftrightarrow$ degenerate ultra equidistant curve

$\mathcal{DUE} \leftrightarrow$ **H**-line reflection. $x_0^2 + y_0^2 - z_0^2 = 0$ and $k > 0 \leftrightarrow$ horocycle $\mathcal{H} \leftarrow$ **H**-parallel translation.

4. A Representation of an H-Isometry and the Corresponding $\mathcal{GC}$. A matrix representation of an **H**-isometry is not so easy to find, and at first sight it seems that the equation of the corresponding generalized circle will be extremely complicated. Moreover, trying to work with these oriented $\mathcal{GC}$'s as elements of a group, i.e. the problem of calculating the equation of the composition of two such generalized circles looks almost impossible. But there is a way of doing it, starting with a method which can be found in Coolidge ([2], page 95).

A parametric representation of the absolute conic $\Gamma, x^2 + y^2 - z^2 = 0$, is given by: $x = t^2 - 1, y = 2t, z = t^2 + 1$ (in Coolidge there is slight difference : Γ has the equation $-x^2 + y^2 + z^2 = 0$).

A non singular projectivity which transforms the absolute Γ into itself is given by :

$$t' = \frac{a_{11}t + a_{12}}{a_{21}t + a_{22}}, \text{ with } a_{ij} \in \mathbf{R} \text{ and } \Delta = a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{4.1}$$

This projectivity determines the following **H**-isometry (the calculation was worked out with MACSYMA):

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{22}^2 - a_{21}^2 - a_{12}^2 + a_{11}^2 & 2(a_{11}a_{12} - a_{21}a_{22}) & -a_{22}^2 - a_{21}^2 + a_{12}^2 + a_{11}^2 \\ 2(a_{11}a_{21} - a_{12}a_{22}) & 2(a_{11}a_{22} + a_{12}a_{21}) & 2(a_{12}a_{22} + a_{11}a_{21}) \\ -a_{22}^2 + a_{21}^2 - a_{12}^2 + a_{11}^2 & 2(a_{21}a_{22} + a_{11}a_{12}) & a_{22}^2 + a_{21}^2 + a_{12}^2 + a_{11}^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.2}$$

The determinant of the matrix of this isometry is :

$$8(a_{11}a_{22} - a_{12}a_{21})^3 = 8\Delta^3 \neq 0. \tag{4.3}$$

Next, in order to find the equation of the corresponding $\mathcal{GC}$, we work as follows: the variable point $p(t^2 - 1, 2t, t^2 + 1)$ of the absolute conic Γ is transformed by the projectivity (4.1) into the point $p'((a_{11}^2 - a_{21}^2)t^2 + 2t(a_{11}a_{12} - a_{22}a_{21})t + a_{12}^2 - a_{22}^2, 2a_{11}a_{21}t^2 + 2t(a_{12}a_{21} + a_{11}a_{22}) + 2a_{12}a_{22}, (a_{11}^2 + a_{21}^2)t^2 + 2t(a_{11}a_{12} + a_{21}a_{22}) + a_{12}^2 + a_{22}^2)$.

The equation of the line pp' is given by :

$$(-a_{11}t^2 + (a_{21} - a_{12})t + a_{22})x - (a_{21}t^2 + (a_{11} + a_{22})t - a_{12})y + (a_{11}t^2 + (a_{12} + a_{21})t + a_{22})z = 0. \tag{4.4}$$

The lines (4.4) are the tangents of the generalized circle which corresponds with the isometry (4.2) and we find its equation by eliminating t out of (4.4) and of the differentiation of (4.4) with respect to t:

$$(-2a_{11}t+a_{21}-a_{12})x-(2a_{21}t+a_{11}+a_{22})y+(2a_{11}t+a_{21}+a_{12})z = 0. \tag{4.5}$$

The result of this elimination is (again done with and factored by MACSYMA):

$$(-a_{11}x - a_{21}y + a_{11}z)((-(a_{12} - a_{21})^2 - 4a_{11}a_{22})x^2 + (4a_{12}a_{21} - (a_{11} + a_{22})^2)y^2 + (4a_{11}a_{22} - (a_{12} + a_{21})^2)z^2 + 2(a_{11} - a_{22})(a_{12} + a_{21})xy + 2(a_{12}^2 - a_{21}^2)xz + 2(a_{11} - a_{22})(a_{21} - a_{12})yz) = 0. \tag{4.6}$$

Remark that the line $-a_{11}x - a_{21}y + a_{11}z = 0$ is a singular part which corresponds with the value $t = \infty$ in (4.4) and in (4.5).

The determinant of the matrix of the conic (second factor in (4.6)) is given by :

$$16(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})^2 = 16(a_{11} + a_{22})\Delta^2,$$

and this conic is degenerate iff (recall that $\Delta \neq 0$) $a_{11} + a_{22} = 0$, i.e. iff the projectivity (4.1) is an involution and the corresponding **H**-isometry is a point reflection or a line reflection.

Next, this conic has to be a generalized circle or the equation of this conic must be of the form (3.2). A straightforward calculation shows that the second factor of (4.6) can be written as follows (up to the sign):

$$\begin{aligned}(4.7)\qquad & 4(a_{11}a_{22}-a_{12}a_{21})(x^2+y^2-z^2)+(x(a_{12}+a_{21})\\ &\qquad +y(a_{22}-a_{11})+z(a_{21}-a_{12}))^2=0.\end{aligned}$$

We retake the classification of the generalized circles (at the end of section 3), but now with the equation (4.7) or $l(x^2+y^2-z^2)+(xx_0+yy_0-zz_0)^2=0$ with $l=4\Delta$, $x_0=a_{12}+a_{21}$, $y_0=a_{22}-a_{11}$, and $z_0=a_{12}-a_{21}$.

We have: $x_0^2+y_0^2-z_0^2=4a_{12}a_{21}+(a_{22}-a_{11})^2=(a_{11}+a_{22})^2-4\Delta$, which is the discriminant of the equation $a_{21}t^2+(a_{22}-a_{11})t-a_{12}=0$ which determines the t-values corresponding with the fixed points of the projectivity (4.1) on the absolute circle Γ.

We get: $x_0^2+y_0^2-z_0^2<0 \longleftrightarrow (a_{11}+a_{22})^2-4\Delta<0$: the conic (4.7) has two conjugate imaginary common points with Γ.

If $a_{11}+a_{22}\neq 0$, the conic is non degenerate and $l=4\Delta>4\Delta-(a_{11}+a_{22})^2=z_0^2-x_0^2-y_0^2$: the conic (4.7) is an **H**-circle $\mathcal{C}$. If $a_{11}+a_{22}=0$, the conic is degenerate and $l=4\Delta=z_0^2-x_0^2-y_0^2$: we have a degenerate **H**-circle $\mathcal{DC}$. Remark that it is impossible that $0<l<z_0^2-x_0^2-y_0^2$ (or $0<4\Delta<4\Delta-(a_{11}+a_{22})^2$) and that $l<0$ (or $4\Delta=(a_{11}+a_{22})^2+z_0^2-x_0^2-y_0^2<0$).
$x_0^2+y_0^2-z_0^2>0 \longleftrightarrow (a_{11}+a_{22})^2-4\Delta>0$: the conic (4.7) has two real different common points with Γ.

We have three possibilities in this case:

$a_{11}+a_{22}\neq 0$ and $l=4\Delta>0$, which gives an equidistant curve $\mathcal{E}$.

$a_{11}+a_{22}\neq 0$ and $4\Delta-(a_{11}+a_{22})^2<4\Delta<0$ (or $z_0^2-x_0^2-y_0^2<l<0$), which corresponds with an ultra equidistant curve $\mathcal{UE}$.

$a_{11}+a_{22}=0$ and $l=4\Delta=z_0^2-x_0^2-y_0^2$, which gives a degenerate ultra equidistant curve $\mathcal{DUE}$.

Remark that it is impossible that $l < z_0^2 - x_0^2 - y_0^2$ (or $4\Delta < 4\Delta - (a_{11} + a_{22})^2$). $x_0^2 + y_0^2 - z_0^2 = 0 \longleftrightarrow (a_{11} + a_{22})^2 = 4\Delta$.

If $a_{11} + a_{22} = 0$ then $\Delta = 0$ and the projectivity (4.1) on Γ is singular. So $a_{11} + a_{22} \neq 0$ and $l = 4\Delta > 0$: the conic (4.7) is a horocycle $\mathcal{H}$.

The next problem is : given a generalized circle with equation $l(x^2 + y^2 - z^2) + (xx_0 + yy_0 - zz_0)^2 = 0$, find the matrices of the two **H**-isometries corresponding with it.

First, the inverse of the **H**-isometry (4.2) has the following matrix (again with MACSYMA):

$$(4.8) \quad \begin{pmatrix} a_{22}^2 - a_{21}^2 - a_{12}^2 + a_{11}^2 & -2(a_{12}a_{22} - a_{11}a_{21}) & a_{22}^2 - a_{21}^2 + a_{12}^2 - a_{11}^2 \\ -2(a_{21}a_{22} - a_{11}a_{12}) & 2(a_{11}a_{22} + a_{12}a_{21}) & -2(a_{21}a_{22} + a_{11}a_{12}) \\ a_{22}^2 + a_{21}^2 - a_{12}^2 - a_{11}^2 & -2(a_{12}a_{22} + a_{11}a_{21}) & a_{22}^2 + a_{21}^2 + a_{12}^2 + a_{11}^2 \end{pmatrix}.$$

So, putting $B = (b_{ij})_{i,j=1,2,3}$ for the matrix in (4.2) and $B' = (b'_{ij})_{i,j=1,2,3}$ for the matrix (4.8), we have:

$b_{ii} = b'_{ii}, \quad i = 1, 2, 3$; $b_{21} = b'_{12}$, $b_{12} = b'_{21}$, $b_{13} = -b'_{31}$, $b_{31} = -b'_{13}$, $b_{23} = -b'_{32}$ and $b_{32} = -b'_{23}$.

Working with $x_0 = a_{12} + a_{21}$, $y_0 = a_{22} - a_{11}$, $z_0 = a_{12} - a_{21}$ and $l = 4(a_{11}a_{22} - a_{12}a_{21})$ we find :

$$\begin{aligned} b_{11} &= a_{22}^2 - a_{21}^2 - a_{12}^2 + a_{11}^2 &&= \frac{l}{2} + y_0^2 - z_0^2 \\ b_{22} &= 2(a_{11}a_{22} + a_{12}a_{21}) &&= \frac{l}{2} + x_0^2 - z_0^2 \\ b_{33} &= a_{22}^2 + a_{21}^2 + a_{12}^2 + a_{11}^2 &&= \frac{l}{2} + x_0^2 + y_0^2; \end{aligned}$$

$(a_{11} + a_{22})^2 = l + x_0^2 + y_0^2 - z_0^2$;

$\frac{b_{12}+b_{21}}{2} = (a_{11} - a_{22})(a_{12} + a_{21}) = -x_0y_0$; $\frac{b_{12}-b_{21}}{2} = (a_{11} + a_{22})(a_{12} - a_{21}) = \pm z_0\sqrt{l + x_0^2 + y_0^2 - z_0^2}$;

$\frac{b_{23}-b_{32}}{2} = (a_{22} - a_{11})(a_{12} - a_{21}) = y_0 z_0$; $\frac{b_{23}+b_{32}}{2} = (a_{11} + a_{22})(a_{12} + a_{21}) = \pm x_0\sqrt{l + x_0^2 + y_0^2 - z_0^2}$;

$\frac{b_{13}-b_{31}}{2} = -a_{21}^2 + a_{12}^2 = x_0 z_0$; $\frac{b_{13}+b_{31}}{2} = -a_{22}^2 + a_{11}^2 = (a_{11} + a_{22})(a_{11} - a_{22}) = \mp y_0\sqrt{l + x_0^2 + y_0^2 - z_0^2}$.

From all this, we get the two matrices of the **H**-isometries which correspond with the generalized circle $l(x^2 + y^2 - z^2) + (xx_0 + yy_0 - zz_0)^2 = 0$:

$$\begin{pmatrix} \frac{l}{2}+y_0^2-z_0^2 & -x_0y_0+z_0\sqrt{l+x_0^2+y_0^2-z_0^2} & x_0z_0-y_0\sqrt{l+x_0^2+y_0^2-z_0^2} \\ -x_0y_0-z_0\sqrt{l+x_0^2+y_0^2-z_0^2} & \frac{l}{2}+x_0^2-z_0^2 & y_0z_0+x_0\sqrt{l+x_0^2+y_0^2-z_0^2} \\ -x_0z_0-y_0\sqrt{l+x_0^2+y_0^2-z_0^2} & -y_0z_0+x_0\sqrt{l+x_0^2+y_0^2-z_0^2} & \frac{l}{2}+x_0^2+y_0^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{l}{2}+y_0^2-z_0^2 & -x_0y_0-z_0\sqrt{l+x_0^2+y_0^2-z_0^2} & x_0z_0+y_0\sqrt{l+x_0^2+y_0^2-z_0^2} \\ -x_0y_0+z_0\sqrt{l+x_0^2+y_0^2-z_0^2} & \frac{l}{2}+x_0^2-z_0^2 & y_0z_0-x_0\sqrt{l+x_0^2+y_0^2-z_0^2} \\ -x_0z_0+y_0\sqrt{l+x_0^2+y_0^2-z_0^2} & -y_0z_0-x_0\sqrt{l+x_0^2+y_0^2-z_0^2} & \frac{l}{2}+x_0^2+y_0^2 \end{pmatrix}$$

Next, let us again consider a generalized circle with equation

$$l(x^2 + y^2 - z^2) + (xx_0 + yy_0 - zz_0)^2 = 0. \tag{4.9}$$

The coefficients a_{ij} of the projectivities (a projectivity and its inverse) induced on the absolute conic Γ by the **H**-isometries which correspond with the $\mathcal{GC}$ (4.9) are calculated out of:

$$\begin{cases} 4(a_{11}a_{22} - a_{12}a_{21}) = l \\ a_{12} + a_{21} = x_0 \\ a_{22} - a_{11} = y_0 \\ a_{12} - a_{21} = z_0. \end{cases}$$

We find

$$a_{12} = \frac{x_0 + z_0}{2}, \quad a_{21} = \frac{x_0 - z_0}{2}$$

$$a_{11}+a_{22}=\pm\sqrt{l+x_0^2+y_0^2-z_0^2} \quad \text{and}$$
$$a_{11}=(-y_0\pm\sqrt{l+x_0^2+y_0^2-z_0^2})/2,\ a_{22}=(y_0\pm\sqrt{l+x_0^2+y_0^2-z_0^2})/2.$$

So, the matrices of the induced projectivities on Γ are (up to the factor $\frac{1}{2}$)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -y_0\pm\sqrt{l+x_0^2+y_0^2-z_0^2} & x_0+z_0 \\ x_0-z_0 & y_0\pm\sqrt{l+x_0^2+y_0^2-z_0^2} \end{pmatrix}. \tag{4.10}$$

Remark that the product of these two matrices is equal to $\begin{pmatrix} -l & 0 \\ 0 & -l \end{pmatrix}$.
Now we give a second generalized circle :

$$k(x^2+y^2-z^2)+(xx_1+yy_1-zz_1)^2=0. \tag{4.11}$$

Working with the $\mathcal{GC}$'s as elements of a group, we can find the equation of the composition of the generalized circles (4.9) and (4.11) (first (4.9) and then (4.11)) as follows.

Let us take the + sign in the matrix (4.10) corresponding with the $\mathcal{GC}$ (4.9) and also the + sign in the analogous matrix (4.10) corresponding with the $\mathcal{GC}$ (4.11). Actually, because of this choice the two $\mathcal{GC}$'s are now oriented if they are non degenerate and different from Γ (see section 2 and also section 5).

The product of the two matrices is denoted by

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= (-y_1+\sqrt{l+x_1^2+y_1^2-z_1^2})(-y_0+\sqrt{k+x_0^2+y_0^2-z_0^2})+(x_1+z_1)(x_0-z_0) \\ A_{12} &= (-y_1+\sqrt{l+x_1^2+y_1^2-z_1^2})(x_0+z_0)+(x_1+z_1)(y_0+\sqrt{k+x_0^2+y_0^2-z_0^2}) \end{aligned}$$

$$A_{21} = (x_1-z_1)(-y_0+\sqrt{k+x_0^2+y_0^2-z_0^2})+(x_0-z_0)(y_1+\sqrt{l+x_1^2+y_1^2-z_1^2})$$
$$A_{22} = (x_1-z_1)(x_0+z_0)+(y_1+\sqrt{l+x_1^2+y_1^2-z_1^2})(y_0+\sqrt{k+x_0^2+y_0^2-z_0^2}).$$

It follows from the foregoing that the composition of the (oriented) generalized circles (4.9) and (4.11) has the equation

$$L_0(x^2+y^2-z^2)+(xX_0+yY_0-zZ_0)^2=0,$$

where

$$\begin{aligned}
X_0 &= (A_{12}+A_{21})/2 = z_1y_0-y_1z_0+x_1\sqrt{k+x_0^2+y_0^2-z_0^2}+x_0\sqrt{l+x_1^2+y_1^2-z_1^2}\\
Y_0 &= (A_{22}-A_{11})/2 = x_1z_0-z_1x_0+y_1\sqrt{k+x_0^2+y_0^2-z_0^2}+y_0\sqrt{l+x_1^2+y_1^2-z_1^2}\\
Z_0 &= (A_{12}-A_{21})/2 = x_1y_0-y_1x_0+z_1\sqrt{k+x_0^2+y_0^2-z_0^2}+z_0\sqrt{l+x_1^2+y_1^2-z_1^2}\\
L_0 &= A_{11}A_{22}-A_{12}A_{21} = kl.
\end{aligned}$$

5. Examples.

5.1. An oriented H-circle $\mathcal{C}$ and the corresponding H-rotation. Let us consider, in the general equation $l(x^2+y^2-z^2)+(xx_0+yy_0-zz_0)^2=0$, the special case where $(x_0,y_0,z_0)=(0,0,z_0)$ with $z_0>0$ and $l>z_0^2$. So we may put $(x_0,y_0,z_0)=(0,0,1)$ and $l>1$: we get an **H**-circle with **H**-center $(0,0,1)$ and it is well known that in the Beltrami model, this is a circle in the Euclidean plane with equation $l(x^2+y^2-z^2)+z^2=0$ or $l(x^2+y^2)-z^2(l^2-1)=0$.

Choosing the + sign for the square root $\sqrt{l+x_0^2+y_0^2-z_0^2}$ we find the following matrix for the corresponding **H**-rotation:

$$\begin{pmatrix} \frac{l}{2}-1 & \sqrt{l-1} & 0\\ -\sqrt{l-1} & \frac{l}{2}-1 & 0\\ 0 & 0 & \frac{l}{2}\end{pmatrix}.$$

In non homogeneous coordinates (X, Y) the representation becomes:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1-\frac{2}{l} & \frac{2}{l}\sqrt{l-1} \\ -\frac{2}{l}\sqrt{l-1} & 1-\frac{2}{l} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

which, for each $l > 1$, clearly is a rotation in the oriented Euclidean plane over the oriented angle Φ determined by $cos\Phi = 1-\frac{2}{l}$ and $sin\Phi = -\frac{2}{l}(\sqrt{l-1})$. A straightforward calculation shows that $\pi < \Phi < 2\pi \pmod{2\pi}$ and we find an **H**-circle $\mathcal{C}$ with a counterclockwise orientation on it (for instance: the image of the point $(1, 0)$ is the point $(1-\frac{2}{l}, -\frac{2}{l}\sqrt{l-1})$, which is always a point "under" the X-axis).

5.2. An oriented horocycle $\mathcal{H}$ and the corresponding H-parallel translation. Put $(x_0, y_0, z_0) = (-1, 0, 1)$ (or $(-z_0, 0, z_0)$ with $z_0 > 0$) and $l > 0$: we find the horocycle $\mathcal{H}$ with equation $l(x^2+y^2-z^2)+(-x-z)^2 = 0$ or $x^2(l+1)+ly^2+z^2(1-l)+2xz = 0$, which is an ellipse in the Euclidean plane which has only the point $(-1, 0, 1)$ common with Γ.

With the $+$ sign for the square root $\sqrt{l+x_0^2+y_0^2-z_0^2}$, the matrix of the **H**-parallel translation becomes:

$$\begin{pmatrix} \frac{l}{2} & \sqrt{l} & -1 \\ -\sqrt{l} & \frac{l}{2} & -\sqrt{l} \\ 1 & -\sqrt{l} & \frac{l}{2}+1 \end{pmatrix}.$$

It is easy to see that this **H**-isometry corresponds with the counterclockwise orientation on the horocycle $\mathcal{H}$: the image of $(1, 0, 1)$ is the point $(\frac{l}{2} - 2, -2\sqrt{l}, \frac{l}{2}+2)$ which is a point under the X-axis.

5.3. An oriented equidistant curve $\mathcal{E}$ and the corresponding H-translation. Consider for instance $(x_0, y_0, z_0) = (1, 0, 0)$ (or $(x_0, 0, 0)$ with $x_0 > 0$) and $l > 0$: we find the equidistant curve $\mathcal{E}$ with equation

$l(x^2+y^2-z^2)+x^2=0$ or $x^2(l+1)+ly^2-lz^2=0$, which is an ellipse in the Euclidean plane which is double tangent with Γ at the points $(0,1,1)$ and $(0,-1,1)$ on the Y-axis. The matrix of the **H**-translation becomes (again with the + sign for the square root $\sqrt{l+x_0^2+y_0^2-z_0^2}$):

$$\begin{pmatrix} \frac{l}{2} & 0 & 0 \\ 0 & \frac{l}{2}+1 & \sqrt{l+1} \\ 0 & \sqrt{l+1} & \frac{l}{2}+1 \end{pmatrix}. \tag{5.3.1}$$

Since $(1,0,1)$ is transformed in $(\frac{l}{2},\sqrt{l+1},\frac{l}{2}+1)$, which is a point in the first quadrant and since the image of $(-1,0,1)$ is $(-\frac{l}{2},\sqrt{l+1},\frac{l}{2}+1)$, which is a point in the second quadrant, the **H**-translation corresponds with the clockwise orientation on the "right side" of $\mathcal{E}$ (part of $\mathcal{E}$ at the right side of the Y-axis) and the counterclockwise orientation on the "left side" of $\mathcal{E}$.

5.4. An oriented ultra equidistant curve $\mathcal{UE}$ and the corresponding H-glide reflection. Put $(x_0,y_0,z_0)=(1,0,0)$ (or $(x_0,0,0)$ with $x_0>0$) and $-1<l<0$: we find the ultra equidistant curve $\mathcal{UE}$ with equation $l(x^2+y^2-z^2)+x^2=0$ or $x^2(l+1)+ly^2-lz^2=0$, which is in the Euclidean plane an hyperbola with axes $X=0$ and $Y=0$ and with real tops $(0,\pm1,1)$ on the Y-axis (where the $\mathcal{UE}$ is double tangent with the absolute circle Γ).

The matrix of the **H**-glide reflection is (again with the + sign for the square root $\sqrt{l+x_0^2+y_0^2-z_0^2}$) exactly the same as (5.3.1) and the images of $(1,0,1)$ and $(-1,0,1)$ have the same coordinates as in 5.3. But now $(\frac{l}{2},\sqrt{l+1},\frac{l}{2}+1)$ is a point in the second quadrant while $(-\frac{l}{2},\sqrt{l+1},\frac{l}{2}+1)$ is a point of the first quadrant and it is easy to see that the **H**-glide reflection induces on the $\mathcal{UE}$ the clockwise orientation on the "right side" of the $\mathcal{UE}$ (part at the right side of the Y-axis) and the counterclockwise orientation on the "left side" of the $\mathcal{UE}$, just as in 5.3 for $\mathcal{E}$.

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University of Ghent, Department of Pure Mathematics and Computeralgebra, Krijgslaan, 278, B-9000 Gent, Belgium.

E-mail: pt@cage.rug.ac.be