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JACOBI'S THEOREM IN LORENTZIAN GEOMETRY

BY

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Abstract. We generalized the Jacobi's theorem of Euclidean space to Minkowski space. Let $c(s)$ be a timelike curve with arclength parameter s in the Minkowski space. Let $\overline{c}$ be the image of the Gauss map of the unit principal vector field of $c(s)$ into the de Sitter space S^2_1. Assume that when the parameter s varies from $s = 0$ to $s = a$, the image $\overline{c}$ is a simple closed curve and not null-homotopic to S^1 and $k(0) = k(a)$ and $w(0) = w(a)$, where $k(t)$ and $w(t)$ be a curvature and torsion at a point $s = t$, respectively. Then $\overline{c}$ divides a "segment" of the de Sitter space into two regions with equal areas.

1. Introduction.

The Jacobi's theorem in 3-dimensional Euclidean space $\mathbb{E}^3$ is followng:

Theorem 1.1.[3]. *Let $\alpha(s) : I \to \mathbb{E}^3$ be a closed, regular, parametrized curve with nonzero curvature. Assume that the Gauss map $\overline{\alpha}$ of the normal vector of $\alpha(s)$ is simple in the unit sphere S^2. Then $\overline{\alpha}$ divides S^2 into two regions with equal areas.*

In the present paper we shall examine this theorem in 3-dimensional Minkowski space $\mathbb{L}^3$. First we recall the Gauss-Bonnet theorem in a 2-

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dimensional Lorentzian manifold, for this is the key theorem to give the Jacobi's theorem. In section 3, we recall the "frame" of a curve in Lorentzian geometry, for the frame formulas in Lorentzian geometry are more complicated than that of Euclidean geometry. In section 4, we give the Jacobi's theorem on a timelike curve (the theorem on a spacelike curve is almost the same as that of Euclidean case). The last section is devoted to give the Jacobi's theorem on a null curve.

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2. Preliminaries. First we recall some definitions and the Gauss-Bonnet theorem for a domain in a 2-dimensional Lorentzian manifold. Since this section is devoted for the preliminary of our new theorems, the statement is abridged slightly. For full explanation about topics of this section, see [1], [2], [5], [7].

Let M be a Lorentzian manifold with the Lorentzian metric g. A vector X at a point of M is called spacelike, timelike or null if $g(X,X) > 0$ or $X = 0$, $g(X,X) < 0$, $g(X,X) = 0$ and $X \neq 0$, respectively. The norm $\|X\|$ of X is defined as $\|X\| := \sqrt{|g(X,X)|}$. The complex-valued norm $\langle X \rangle$ of X is defined as $\langle X \rangle := \sqrt{g(X,X)}$, that is, $\langle X \rangle \in \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^+ i$, where $\mathbb{R}^+$ denotes the set of all positive numbers and $i = \sqrt{-1}$.

On the 3-dimensional Minkowski space $\mathbb{L}^3$, for any two arbitrary vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, $g(X,Y)$ can be written as

$$g(X,Y) = x_1 y_1 + x_2 y_2 - x_3 y_3, \tag{2.1}$$

that is, g is the inner product. The exterior product $X \times Y$ is defined by

$$X \times Y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, -(x_1 y_2 - x_2 y_1)). \tag{2.2}$$

In $\mathbb{L}^3$, the de Sitter space S_1^2 is defined by setting

$$S_1^2 = \{X \mid X \in \mathbb{L}^3, \ \ g(X,X) = 1\}.$$

For two non-null vectors X and Y, non-directed sectional mesure $\varnothing = \varnothing(X,Y)$ is a complex number satisfying the equation

$$\cos\varnothing = \frac{g(X,Y)}{\langle X\rangle \cdot \langle Y\rangle} \tag{2.3}$$

and defined as follows:

(1) If

$$\frac{g(X,Y)}{\langle X\rangle \cdot \langle Y\rangle} \in [-1,1],$$

then $\varnothing \in [0,\pi]$.

(2) If

$$\frac{g(X,Y)}{\langle X\rangle \cdot \langle Y\rangle} > 1,$$

then $\varnothing = \theta i$(when $\|X\| > 0, \|Y\| > 0$) or $\varnothing = \theta/i$(when $\|X\| < 0, \|Y\| < 0$) is uniquely determined by (2.3).

(3) If

$$\frac{g(X,Y)}{\langle X\rangle \cdot \langle Y\rangle} < -1,$$

then $\varnothing = \pi - i\theta$ (when $\|X\| > 0, \|Y\| > 0$) or $\varnothing = \pi - \theta/i$ (when $\|X\| < 0, \|Y\| < 0$), where $\theta(> 0)$ is uniquely determined by (2.3).

(4) If

$$\frac{g(X,Y)}{\langle X\rangle \cdot \langle Y\rangle} \in \mathbb{R}i,$$

then $\varnothing = \frac{\pi}{2} + i\nu$, where ν s uniquely determined by (2.3).

In the Euclidean 2-space $\mathbb{R}^2$, we write a circle S^1 and give 4 arcs $ARC_0 := \overset{\frown}{A_0A_1}$, $ARC_1 := \overset{\frown}{A_1A_2}$, $ARC_2 := \overset{\frown}{A_2A_3}$, $ARC_3 := \overset{\frown}{A_3A_4}$, where $A_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$,

$A_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $A_2 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $A_3 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ (where arcs do not include their end points). In $\mathbb{R}^2$, we define the Lorentzian metric $g = (+, -)$ and make $\mathbb{R}^2$ to $\mathbb{L}^2$.

Then, for $P, Q \in S^1$, fundamental angle $\angle POQ$ is defined as follows.

(1) If $P, Q \in ARC_j (j = \{0, 1, 2, 3\}$, where $\{0, 1, 2, 3\}$ denotes the quotient group modulo 4 of the natural number), then the fundamental angle $\angle POQ$ is non-directed sectional meaure $\varnothing = \varnothing(\overrightarrow{OP}, \overrightarrow{OQ})$.

(2) If $P \in ARC_j$ and $Q \in ARC_{j+1}$, or $Q \in ARC_j$ and $P \in ARC_{j+1}$ and $g(\overrightarrow{OP}, \overrightarrow{OQ}) = 0$, then the fundamental angle $\angle POQ$ is $\frac{\pi}{2}$.

Next, we define the directed sectional measure as follows. When an angle $\angle POQ$ is the fundamental angle, if the varing point moving from the initial point P to the terminal point Q along S^1 in the counterclockwise direction, then the directed sectional measure of the fundamental angle is defined to be the product of the fundamental angle by +1. If the varying point moves in the clockwise direction, product -1. When an angle $\angle POQ$ is not the fundamental angle, we split the angle $\angle POQ$ into successive non-overlapping fundamental angles. Then the directed sectional measure of $\angle POQ$ is the summing up of fundamental angles. (We can easily see that the definitions of the directed sectional measure of "general" angle $\angle POQ$ is independent of the choise of splittings).

Next, we shall define the geodesic curvature. Let M^2 be a 2-dimensional Lorentzian manifold with the Lorentzian metric g. Suppose $c = c(t)$ be a smooth curve on M^2. The length of c with respect $\langle \cdot \rangle$ from $t = a$ to $t = b$ is

$$\alpha = \int_a^b \langle \frac{dc}{dt} \rangle dt.$$

Put

$$U := \frac{\frac{dc}{dt}(a)}{\langle \frac{dc}{dt}(a) \rangle}, \quad V := \frac{\frac{dc}{dt}(b)}{\langle \frac{dc}{dt}(b) \rangle}.$$

By $\overrightarrow{\varnothing}$ we denote the directed sectional measure from U to V. Then the geodesic curvature $k_g(a)$ of the curve c at a is defined as

$$k_g(a) = \lim_{\delta\alpha \to 0} \frac{\delta \overrightarrow{\varnothing}}{\delta\alpha}.$$

Now the Gauss-Bonnet theorem for a domain on an 2-dimensional Lorentzian manifold is stated as follows.

Theorem 2.1. (Gauss-Bonnet Theorem). *Let M^2 be an oriented 2-dimensional Lorentzian manifold and D a simply connected domain on M^2 such that the boundary ∂D consists of finite pieces of either spacelike or timelike curves. Then*

$$\iint_D K dS + \int_{\partial D} k_g d\alpha + \sum \lambda_i = 2\pi$$

where λ_i is the directed sectional measure of the exterior angle at the i-th vertex, K the Gaussian curvature and dS the volume element of M^2.

3. Curves. Let $c = c(t)$ be a curve in the 3-dimensional Minkowski space $\mathbb{L}^3$. If the tangent vector field dc/dt is spacelike, then the curve $c(t)$ is said to be spacelike; similarly for timelike and null.

First we consider spacelike or timelike curve $c(t)$. In this case, we can reparameterize it such that $g(dc/ds, dc/ds) = \varepsilon$ (where $\varepsilon = +1$ if c is spacelike and $\varepsilon = -1$ if c is timelike, respectively). Then this new parameter s is called arclength (or proper time in relativity).

For a timelike curve $c(s)$ with arclength parameter s, the Frenet formula is given as

$$\begin{aligned} \xi_1 &:= \frac{dc}{ds}, \\ \frac{d\xi_1}{ds} &= k\xi_2, \end{aligned}$$

(3.1)
$$\frac{d\xi_2}{ds} = k\xi_1 + w\xi_3,$$
$$\frac{d\xi_3}{ds} = -w\xi_2,$$

where ξ_2 is the unit principal vector field and ξ_3 is the unit binormal vector field, respectively. The scalar function $k = k(s)$ (resp. $w = w(s)$) is called the curvature (resp. torsion) of $c(s)$.

Next, we consider a null curve $c(t)$. In this case, we can not have arclength parameter as spacelike or timelike case. However by a special parameter s, we can have the Cartan frame (cf. [4, 6])

(3.2)
$$\begin{aligned} \eta_1 &:= \frac{dc}{ds}, \\ \frac{d\eta_1}{ds} &= k\xi, \\ \frac{d\eta_2}{ds} &= -w\xi, \\ \frac{d\xi}{ds} &= -w\eta_1 + k\eta_2, \\ g(\eta_i, \eta_i) &= g(\eta_i, \xi) = 0, \quad (i = 1, 2), \\ g(\eta_1, \eta_2) &= -1, \quad g(\xi, \xi) = 1. \end{aligned}$$

The vector field η_1 is called null transversal vector field and ξ is called screen vector field.

4. Jacobi's Theorem of Timelike Curves. In this section, we shall prove the following theorem.

Theorem 4.1. *Let $c(s)$ be a timelike curve with arclength parameter in the 3-dimensional Minkowski space $\mathbb{L}^3$. Let $\overline{c}$ be the image of the Gauss map of the unit principal vector field of $c(s)$ into the de Sitter space S_1^2. Assume that when the arclength parameter s varies from $s = 0$ to $s = a$, the image $\overline{c}$*

is a simple closed curve and not null-homotopic to S^1 *and* $k(0) = k(a)$ *and* $w(0) = w(a)$. *Then* $\overline{c}$ *divides*

$$S_1^2(t_0) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 = 1, |x_3| \le t_0\} \subset S_1^2$$

into two regions with equal areas, where t_0 *is any sufficiently large positive number such that* $\overline{c}$ *is contained in the interior of* $S_1^2(t_0)$.

Proof. Let $\overline{s}$ be the arclength parameter of the curve $\overline{c}$. Since $c(t)$ satisfies (3.2), we have

$$\frac{d\overline{c}}{d\overline{s}} = \frac{d\overline{\xi_2}}{ds}\frac{ds}{d\overline{s}} = (k\xi_1 + w\xi_3)\frac{ds}{d\overline{s}} \tag{4.1}$$

and

$$\begin{aligned} \frac{d^2\overline{c}}{d\overline{s}^2} &= \left(k\frac{d^2 s}{d\overline{s}^2} + k'\left(\frac{ds}{d\overline{s}}\right)^2\right)\xi_1 \\ &\quad +(k^2 - w^2)\left(\frac{ds}{d\overline{s}}\right)^2 \xi_2 + \left(w\frac{d^2 s}{d\overline{s}^2} + w'\left(\frac{ds}{d\overline{s}}\right)^2\right)\xi_3. \end{aligned} \tag{4.2}$$

Since the curve $\overline{c}$ is on the de Sitter space S_1^2, the geodesic curvature $k_g(\overline{c})$ satisfies $k_g(\overline{c}) = g(\frac{d^2\overline{c}}{d\overline{s}^2}, \overline{c} \times \frac{d\overline{c}}{d\overline{s}})$. So, it follows that

$$k_g(\overline{c}) = kw'(\frac{ds}{d\overline{s}})^3 - k'w(\frac{ds}{d\overline{s}})^3 = \frac{kw' - k'w}{w^2 - k^2}(\frac{ds}{d\overline{s}})$$

by virtue of (4.1), (4.3) and the equation

$$\frac{d\overline{s}}{ds} = w^2 - k^2.$$

By assumption $w/k > 1$, we can put $w = k\cosh\theta$, $-b := \cosh^{-1}\frac{w(0)}{k(0)}$,

$b := \cosh^{-1} \frac{w(a)}{k(a)}$. Then, we have

$$\oint_{\overline{c}} k_g(\overline{c})d\overline{s} = \int_{-b}^{b} \frac{1}{\sinh\theta} d\theta = 0. \tag{4.3}$$

Let $S^1(t_0)$ be the circle $x_3 = t_0 (> 0)$. Since the geodesic curvature $k_g(S^1(t_0))$ of $S^1(t_0)$ is equal to $-t_0$, we have

$$\oint_{S^1(t_0)} k_g(S^1(t_0))dt = -2\pi t_0. \tag{4.4}$$

Let L be a timelike curve on S_1^2 and P (resp. Q) the crossing point of L to the circle $S^1(t_0)$ (resp. $\overline{c}$). We consider a simply connected domain D constructed by $[S^1(t_0)] + [\overrightarrow{PQ}(\subset L)] + [\overline{c}] + [\overrightarrow{QP}(\subset L)]$.

Applying the Gauss-Bonnet theorem to D, we obtain

$$\iint_D 1 \cdot dS - 2\pi t_0 + 2\pi = 2\pi,$$

by virtue of (4.3) and (4.4). Therefore

$$[\text{Area}D] = 2\pi t_0 = 2\pi \int_0^{\sinh^{-1} t_0} \cosh t dt = \frac{1}{2}[\text{Area}S_1^2(t_0)].$$

This completes the proof.

5. Jacobi's Theorem of Null Curves. In this section, we shall prove the following Jacobi's theorem of Cartan framed null curves.

Theorem 5.1. *Let $c(s)$ be a Cartan framed null curve in the 3-dimensional Minkowski space $\mathbb{L}^3$. Let $\overline{c}$ be the image of the Gauss map of the screen vector field of $c(s)$ into the de Sitter space S_1^2. Assume that when the parameter s varies from $s = 0$ to $s = a$, the image $\overline{c}$ is a simple closed curve and not null-homotopic to S^1 and $k(0) = k(a)$ and $w(0) = w(a)$. Then $\overline{c}$*

divides

$$S_1^2(t_0) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 = 1, |x_3| \leq t_0\} \subset S_1^2$$

into two regions with equal areas, where t_0 is any sufficiently large positive number such that $\overline{c}$ is contained in the interior of $S_1^2(t_0)$.

Proof. Since $c(s)$ have Cartan frame, we have

$$\frac{d\xi}{d\overline{s}} = (-w\eta_1 + k\eta_2)\frac{ds}{d\overline{s}} \tag{5.1}$$

and

$$\frac{d^2\xi}{d\overline{s}^2} = \left(-w\frac{d^2s}{d\overline{s}} - w'\left(\frac{ds}{d\overline{s}}\right)^2\right)\xi + \left(k\frac{d^2s}{d\overline{s}^2} + k'\left(\frac{ds}{d\overline{s}}\right)^2\right)\xi_2 \tag{5.2}$$

by virtue of (3.3). Hence the Gaussian curvature $k_g(\overline{c})$ satisfies

$$k_g(\overline{c}) = (\frac{k'}{k} - \frac{w'}{w})\frac{ds}{d\overline{s}}$$

so that

$$\oint g_k(\overline{c})d\overline{s} = \int(\frac{k'}{k} - \frac{w'}{w})ds = 0.$$

Therefore, by a similar calculation, as that of Section 3, we obtain the result.

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