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EMBEDDINGS OF UNREDUCED CURVES IN PROJECTIVE SPACES: POSTULATION AND MINIMAL FREE RESOLUTION

BY

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Abstract. Here we study the minimal free resolution of certain locally Cohen-Macaulay linearly embedded multiple curves $X \subset \boldsymbol{P}^n$ with $C := X_{\text{red}}$ irreducible. $\deg(C) \geq 2p_a(C) + 2$ and such the conormal sheaf of C in X is "positive". In particular such curves X have maximal rank and their homogeneous ideal is generated by quadrics.

Introduction. Let $X \subset \boldsymbol{P}^n$ be a locally Cohen-Macaulay curve such that $C := X_{\text{red}}$ is irreducible. The projective geometry and the cohomological properties of the embedded scheme X (e.g. its minimal free resolution and its Hilbert function, i.e. its postulation) depend very much on the nilpotent structure of X. For instance there are double lines with arbitrarily negative arithmetic genus. Here we study a class of such unreduced curves for which the good properties of the embedding of C into its linear span $< C >$ may be extended to good properties of X; for instance X and C have the same index of regularity in the sense of Castelnuovo-Mumford's

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theory. A key example of such X (as abstract scheme) is given by a multiple of a contractible (or with negative normal bundle) curve C in a smooth surface or an infinitesimal neighborhood with very negative normal bundle of a smooth curve C embedded in a smooth variety Z. For a bound on the negativity required when the normal bundle is semistable, see Section 3. We allow also the case in which the normal bundle $N_{C/Z}$ of C in Z is a direct sum of degree 0 line bundles (e.g. if $N_{C/Z}$ is trivial). Let I be the ideal sheaf of C in X. Hence I/I^2 is the conormal sheaf of C in X. We consider the following properties which the curve X may have.

Condition (S). For every integer $t > 0$ the O_C-sheaf I^t/I^{t+1} is locally free and it has an increasing filtration by subbundles, say $\{F_j\}_{j\geq 0}$, such that every F_{j+1}/F_j is a line bundle of non-negative degree.

Condition (L). I/I^2 is locally free and for every integer $t > 0$ the natural map $S^t(I/I^2) \to I^t/I^{t+1}$ is an isomorphism.

Condition (L) is satisfied if C is a locally complete intersection curve contained in a smooth variety Z and X is the s-th infinitesimal neighborhood of C in Z. If Condition (L) is satisfied, then Condition (S) is equivalent to the fact that I/I^2 has a filtration with non-negative degree line bundles as graded subquotients.

Now we may state our main result.

Theorem 0.1. *Assume that the ground field has characteristic zero. C smooth and Conditions* (S) *and* (L). *Set $g := p_a(C)$. Fix an integer $y > 0$. Fix $L \in Pic(X)$ such that $\deg(L|C) \geq 2g + y + 1$. Then $H^1(X, L^{\otimes a}) = 0$ for every $a > 0$, L is very ample and for the corresponding embedding $X \subset \prod := \boldsymbol{P}(H^0(X, L))$ the restriction maps $H^0(\prod, O_{\Pi}(k)) \to H^0(X, L^{\otimes k})$ are surjective for every integer $k \geq 1$. Furthermore, L satisfies Property (N_y), i.e. for every integer b with $0 \leq b \leq y$ the b-th term of the minimal* *free resolution of X in $\prod$ is generated in degree $\leq b+2$. In particular the homogeneous ideal of X is generated by quadrics.*

We cannot say that X is arithmetically normal because in general we have $h^0(X, O_X) > 1$; here the situation in negative degrees seems to be quite wild.

To prove Theorem 0.1 we will study the general hyperplane section of X (see Proposition 2.1). We think that such study may have an independent interest. Apart from this study we will use standard techniques of Koszul cohomology for the linearly normal embedding of X in Π (see [5] or [7] for background, notations and results on the minimal free resolution of smooth curves and finite sets). In the last section we give a few remarks on the existence of rank r vector bundles, E. On a smooth curve C such that E is equipped with a filtration $\{E_i\}_{0\leq i\leq r}$ with $E_0 = \{0\}$. $E_r = E$, $E_i/E_{i-1} \in Pic(C)$ for $1 \leq i \leq r$ and $\deg(E_i/E_{i-1}) \geq 0$. We may take such rank r vector bundles as conormal bundles for multiple structures on C with Conditions (S) and (L).

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1. Working with X. In this section we work with a non-degenerate $X \subset \mathbf{P}^n$ and prove Theorem 0.1 modulo a result (Theorem 2.2) on the postulation of X which will be stated and proved in the next section because it uses a result on the general hyperplane section of X.

For every integer $i \geq 0$ set $C(i) := X/I^{i+1}$ and let $\Pi(i) :=< C(i) >$ be the linear span of $C(i)$. Let s be the first integer with $C(s) = X$, i.e. with $I^{s+1} = \{0\}$. Set $\Pi := \Pi(s)$. Set $n(i) := \dim(\Pi(i))$; hence $n = n(i)$ for every $i \geq s$. We are interested in the case $n(0) < n$. For every integer $t \geq 0$ we have the following exact sequence

$$0 \to I^t/I^{t+1} \to O_{C(t+1)} \to O_{C(t)} \to 0 \tag{1}$$

Fix $L \in Pic(X)$: we are interested in the case in which L is very ample. Set $L(i) := L|C(i)$ and $V := H^0(X, L)$. Let $V(t) \subseteq H^0(C(t), L(t))$ be the image of V.

Lemma 1.1. *If* $\deg(L(0)) > 2p_a(C) - 2$ *and Conditions* (L) *and* (S) *are satisfied, then* $H^1(C(t), L(t)) = 0$ *and* $V(t) = H^0(C(t), L(t))$.

Proof. For degree reasons. Conditions (S) and (L) imply the vanishing of $H^1(C(0), L(0) \otimes I^i/I^{i+1})$ for every integer $i > 0$. Hence the result follows tensoring (1) with L and using induction on the integer t.

We recall the following well-known result.

Lemma 1.2. *Let* C *be an integral projective curve. Set* $g := p_a(C)$. *Every* $L \in Pic^d(C)$ *with* $d \geq 2g + 1$ *is very ample.*

Proof. By [4], Remark 2.1, we need to check that for every zero-dimensional subscheme, Z, of C with length$(Z) = 2$ the restriction map $H^0(C, L) \to L|Z$ is surjective. Hence it is sufficient to check that $h^1(C, L \otimes \boldsymbol{I}_Z) = 0$. By Grothendieck duality for locally Cohen-Maculay schemes ([1]) $H^1(C, L \otimes \boldsymbol{I}_Z)$ is dual to $H^0(C, Hom(L \otimes \boldsymbol{I}_Z, \omega_C))$, which is zero because $\deg(L \otimes \boldsymbol{I}_Z) = d - 2 > 2g - 2 = \deg(\omega_C)$.

Lemma 1.3. *If* $\deg(L(0)) \geq 2p_a(C) + 1$ *and Conditions* (L) *and* (S) *are satisfied, then* L *is very ample.*

Proof. By [4], Remark 2.1, we need to check that for every zero-dimensional subscheme, Z, of X with length $(Z) = 2$ the restriction map $H^0(X, L) \to L|Z$ is surjective. Let i be the first integer $\leq s$ such that

$Z \subseteq C(i)$. Since the restriction map $H^0(X,L) \to H^0(C(i),L(i))$ is surjective (Lemma 1.1), it is sufficient to prove the surjectivity of the restriction map $H^0(C(i),L(i)) \to L(i)|Z$. Since $L(0)$ is ver ample (Lemma 1.2), we have won if $i=0$. Assume $i>0$ and set $P := Z_{\rm red}$. We have $Z \cap C(i-1) = \{P\}$ (as schemes). It suffices to check that $h^1(C(i),L(i)\otimes \boldsymbol{I}_{Z,C(i)}) = 0$. Tensor the exact sequence (1) for $t = i-1$ by $L \otimes \boldsymbol{I}_{Z,C(i)}$. The quotient of $L(i-1)\otimes \boldsymbol{I}_{Z,C(i)}$ by its torsion, T, as $\boldsymbol{O}_{C(i-1)}$-module is just $L(i-1)\otimes \boldsymbol{I}_{\{P\},C(i-1)}$. We have $h^1(C(i-1),T) = 0$ because T is supported on a zero-dimensional scheme, P. Hence, as in the proof of Lemma 1.1 we obtain $h^1(C(i-1),L(i-1)\otimes \boldsymbol{I}_{Z,C(i)}) = 0$. For the same reason using C instead of $C(i-1)$ we have $h^1(C,L(0)\otimes I_{i-1}/I^i \otimes \boldsymbol{I}_{Z,C(i-1)}) = 0$ and we conclude.

Assume that L is very ample and set $V := H^0(X,L)$. We are interested in the linearly normal embedding of X into $\mathbf{P}(V)$ associated to L. Hence we take $\mathbf{P}^n = \Pi(s) = \mathbf{P}(V)$ and we will see X as a subscheme of $\mathbf{P}(V)$. The restriction to X of the dual of the Euler sequence of $T\mathbf{P}(V)$ induces an exact sequence of vector bundles on X:

$$0 \to M_L \to V \otimes O_X \to L \to 0 \tag{2}$$

with $M_L \cong \Omega_{\mathbf{P}(V)}(1)|X$. For the relations between M_L and the minimal free resolution of X in $\mathbf{P}(V)$, see for instance [7]. For every integer i with $0 \le i \le s$ we have $M_L|C(i) \cong M_{L(i)} \oplus \boldsymbol{O}_{C(i)^{\oplus(n-n(i))}}$. Applying the functor Λ^X to these isomorphisms and then using the s exact sequences (1) and Lemma 1.1 we obtain the following result.

Proposition 1.4. *Assume Conditions* (S) *and* (L)*. Fix an integer x such that $H^1(C,\Lambda^j(M_{L(0)})\otimes L(0)) = 0$ for every integer j with $0 \le j \le x$. Then $H^1(X,\Lambda^x(M_L)\otimes L) = 0$.*

Proposition 1.5. *Assume Conditions* (S) *and* (L) *and* $\deg(L(0)) \geq 2p_a(C) + 2$. *Let* $X \subset \Pi := P(H^0(X, L))$ *be the complete embedding. Then for all integers* $k \geq 1$ *we have* $H^1(\Pi, \boldsymbol{I}_X(k)) = 0$ *and the restriction map* $\rho_{X,k} : H^0(\Pi, \boldsymbol{O}_\Pi(k)) \to H^0(X, O_X(k))$ *is surjective.*

Proof. Since $H^1(\Pi, O_\Pi(k)) = 0$ for every integer $k \geq 1$, the two assertions are equivalent. The map $\rho_{X,1}$ is bijective because we are considering a linearly normal embedding. Fix $k \geq 2$ and assume the surjectivity of $\rho_{X,k-1}$, i.e. $H^1(\Pi \boldsymbol{I}_X(k-1)) = 0$. Fix a general hyperplance H of Π. We have the exact sequence

$$0 \to \boldsymbol{I}_X(k-1) \to \boldsymbol{I}_X(k) \to \boldsymbol{I}_{X\cap H,H}(K) \to 0 \tag{3}$$

By Theorem 2.2 for every integer $k \geq 2$ the restriction map $\rho_{X\cap H,k,H}$: $H^0(H, O_H(k)) \to H^0(X \cap H, \boldsymbol{O}_{X\cap H}(k))$, i.e. $H^1(H, \boldsymbol{I}_{X\cap H,H}(k)) = 0$. By the exact sequence (3) we obtain the vanishing of $H^1(\Pi, \boldsymbol{I}_X(k))$ and the surjectivity of $\rho_{X,k}$.

Proof of 0.1. Use Lemma 1.4, Proposition 1.5, [7], Lemma 1.10, and a theorem of M. Green ([5]. 4.a.1 or [7], Prop 3.2).

Remark 1.6. It is easy to give some generalizations of Theorem 0.1, i.e. weaken the condition on $\deg(L|C)$ just making other very weak assumptions. One need just to check the literature to handle the embedding of C giving by $L(0)$ (for instance by [7], Th. 3.3, if C is smooth and $\deg(L|C) = 2g + y$ one have just to exclude the case in which C is a hyperelliptic or a "very multisecant" linear subspace of dimension less than y) and then work with the general hyperplane section. If one is interested only in results on the surjectivity of the maps $\rho_{X,u}$ with $u \geq 2$ this is easy and for the part on the curve one can use a huge literature (see [6]). To extend Propositions 1.4 and 1.5 to these borderline cases it is easier to assume instead of Condition (S) the following stronger condition which is satisfied in most of our motivating examples, but not when X is an infinitesimal neighborhood of the smooth curve C in the smooth variety Z with trivial normal bundle.

Condition (S+): For every integer $t > 0$ the $\boldsymbol{O}_C$-sheaf I^t/I^{t+1} is locally free and it has an increasing filtration by subbundles, say $\{F_j\}_{j\geq 0}$, such that every F_{j+1}/F_j is a line bundle of degree > 0.

The surjectivity of the restriction map $Pic(X) \to Pic(C)$ was checked in [8], proof of Step 3 of the proof of Th. IV. 3.1 at p.179. Thus from the existence for a fixed integer d of general $R \in Pic^d(C)$ with good properties we may obtain the existence of "general" $L \in Pic(X)$ with $L|C \cong R$ and hence (by our machinery) with certain good properties.

2. General hyperplane section and postulation. In this section we study the postulation of a general hyperplane section, say $X \cap H$, of a curve satisfying Conditions (L) and (S). We always assume that the monodromy group of the general hyperplane section of C (as non-degenerate integral curve of $< C >$ is either the full symmetric group or the alternating group): we will call Condition (H) this assumption. Condition (H) is satisfied if char $(\mathbf{K}) = 0$ ([15], first line of p.571 and Cor. 2.2) or if C is reflexive ([15], Cor. 2.2) or if C is smooth and $n(0) \geq 4$ ([15], Cor. 2.7) or if C is not strange and $n(0) \geq 6$ ([15], Th. 2.5) or if C is not strange, $\deg(C) \geq 24$ and $n(0) \geq 4$ ([15]. Th. 2.5). Set $x := \deg(X)/\deg(C)$. By Conditions (L) and (S) there is a decreasing filtration $\{J_i\}_{0\leq i\leq x-1}$ of I with rank $(J_{i+1}/J_i) = 1$ for every $i, J_{x-1} = \{0\}$, $J_0 = I$ and $\deg(J_{i+1}/J_i) \geq 0$ (as line bundle on C). Set $D(i) := X/J_i$. Let H be a general hyperplane. Set $M(i) := H\cap < D(i) >$, $m(i) := \dim(M(i))$ and $S(i) := M(i) \cap X$. Hence length $(S(i)) = (i + 1)\deg(C)$. For every integer i the zero-dimensioinal scheme $S(i)$ spans $M(i)$ and $S(0)$ is in linearly general position inside $S(0)$. The following result is the best approximation to the linearly general position we can conceive for a scheme like $X \cap H$ with a filtration by subschemes of length $> \dim(H)$

and contained in proper linear subspaces. We feel that the following result 2.1 and the proof of 2.2 given below have an intrinsic interest and that they are very useful to attack this kind of question. Furthermore, we have obtained (thanks to Condition (S)) a class of zero-dimensional schemes, say $X \cap H$ for general H, such that their cohomological properties are governed by their "core" $(X \cap H)_{red}$ which is contained in a linear subspace with high codimension.

Proposition 2.1. *Assume Conditioin* (L), (S) *and* (H). *For every integer* $i > 0$ *and every subscheme* T *of* $S(i)$ *containing* $S(i-1)$ *we have* $\dim(< T >) = \min\{m(i), m(i-1) + \mathit{length}(T) - i(\deg(C)).\}$

Proof. By Conditon (H) if the statement of 2.1 is false for one such subscheme T, then it is false for all such subschemes. Since by definition $S(i)$ spans $M(i)$, we conclude.

For any zero-dimensional subscheme Z of a projective space A, let $\rho_{Z,k,A}$: $H^0(A, \boldsymbol{O}_A(k)) \to H^0(Z, \boldsymbol{O}_Z(k))$ be the restriction map. If $A \subseteq H$ we have $Im(\rho_{Z,k,A}) = Im(\rho_{Z.k.H})$.

Theorem 2.2. *Assume Conditions* (L), (S) *and* (H). *Assume* $\deg(C) \geq 2p_z(C) + 2$. *Then for every integer* $k \geq 2$ *the restriction map* $\rho_{S,k,H}$ *is surjective.*

Proof. By Lemma 1.1 we have $m(i) \geq m(i-1) + m(0)$ for every integer i with $0 < i \leq x - 1$. By Castelnuove-Mumford's lemma it is sufficient to prove the surjectivity of $\rho_{S,2,H}$. We will check by induction on i that for every integer i with $0 \leq i \leq x - 1$ the restriction map $\rho_{S(i),k,M(i)}$ is surjective. For $i = 0$ this is true becasue $S(0)$ is a reduced set of points in linearly general position in $M(0)$ and card $(S(0)) \leq 2m(0)$ (see e.g [7]). Assume the result for an integer $i - 1$ with $0 < i \leq x - 1$. To prove the result for the integer i it is sufficient to check that for every $P \in S(0)$ there is a quadrie hypersurface of $M(i)$ containing $S(i-1)$, all the connected components of $S(i)$ not containing P but not $S(i)$. Assume that this is false for one $P \in S(0)$. Since $S(0)$ is in linear general position in $S(0)$ and card $(S(0)) \leq 2m(0)$ we may find two hyperplanes. A and A', of $M(0)$ with card $(A \cap S(0)) = m(0)$, $A' \cap S(0) = S(0) \setminus (A \cap S(0))$, $P \in A'$ and a hyperplane A'' and A' containing all the points of $A' \cap S(0)$ except P. By Lemma 1.1 and the inequalities $m(j) \geq m(j-1) + m(0)$ for every $j > 0$ we may extend A and A' to hyperplances B, B' of $M(i)$ with as $S(i) \cap B$ (resp. $S(i) \cap B'$) the connected components of the scheme $S(i)$ with support $S(0) \cap A$ (resp. $S(0) \cap A'$). Hence length $(S(i) \cap B) = (i+1)m(0)$ and length $(S(i) \cap A') = (i+1)(\text{length}(S(i) \cap A'))$. Furthermore, we may extend A'' to a codimension $i+1$ subspace B'' of A'' such that $B'' \cap A''$ contains all the connected components of $S(i) \cap A'$ not supported by P. Take an increasing flag of linear subspaces of B', say $\{B_u\}_{0 \leq u \leq i+1}$, $B_0 = B''$, $B_{i+1} = B'$, $\dim(B_u) = \dim(B_0) + u$ for every i. To check the existence of the flag $\{B_u\}_{0 \leq u \leq i+1}$, just use again Proposition 2.1. Using reducible quadrics with B as one component, while the other contains B_u, we see that the connected component, Z, of $S(i)$ supported by P give $i+1$ independent conditions for quadrics with respect to $S(i) \setminus Z$: in particular we have the result concerning the scheme T.

3. Good filtrations of vector bundles on C. Let C be a smooth connected projective curve of genus g. Motivated by Condition (S) used in the first part of the paper we consider the following definition.

Definition 3.1. Let E be a rank r vector bundle on C. An increasing filtration $\{E_i\}_{0 \leq i \leq r}$ of E by subbundles with $E_0 = \{0\}$, $E_r = E$ and $E_{i+1}/E_i \in Pic(C)$ for every integer i with $0 \leq i < r$ will be called *good* if $\deg(E_{i+1}/E_i) \geq 0$ for every integer i with $0 \leq i < r$.

Fixed an integer $r \geq 2$. Motivated by the first part of this paper (Condition (S) and its use) we consider the following questions.

Question 3.2. (i) For which integers d there exists a stable (or semistable) degree d rank r vector bundle on C with a good filtration?

(ii) For which integers d every stable (or semistable) degree d rank r vector bundle on C has a good filtration?

(iii) Assume $g \geq 2$. For which integers d a general stable vector bundle with degree d and rank r on C has a good filtration?

Obviously we need $d \geq 0$. For semistability Queston 3.2 (i) has obviously a trivial affirmative answer for $d = 0$: just take the bundle $\boldsymbol{O}_{C^{\oplus r}}$.

Remark 3.3. These questions are trivial for $g = 0$. Here we will check that Atiyah's classification of vector bundles on an elliptic curve ([2], Part II) gives an answer if $g = 1$. The question for stable bundles is solved if the question for semistable bundles is solved because a semistable bundle on an elliptic curve is stable if and only if its rank and degree are coprime. An indecomposable vector bundle is semistable and every semistable bundle is a direct sum of indecomposable bundles with the same slope. If $0 < d < r$ every indecomposable bundle (and hence every semistable bundle) has a good filtration ([2], Lemma 6 and Lemma 15 (i)). The same is true if $d = r$ ([2], Lemma 11). If $d > r$, set $x := [(d-1)/r]$ and fix $P \in C$. By the cases with $0 < d \leq r$ for every semistable bundle E with $\mathrm{rank}(E) = r$ and $\deg(E) = d$ the bundle $E(-xP)$ has a good filtration $\{F_i\}_{0 \leq i \leq r}$. Hence the filtration $\{F(xP)_i\}_{0 \leq i \leq r}$ is a good filtration of E.

Remark 3.4. If $r = 2$ and $g \geq 2$ the answer to these questions are known (see the introduction of [13]. For every integer $d > 0$ there exists a stable vector bundle with a good filtration. For every integer $d \geq g - 1$ every rank 2 stable vector bundle of degree d has a good filtration. For every integer $d \leq g - 2$ a general rank 2 stable vector bundle of degree d does not admit a good filtration.

Remark 3.5. Assume $g \geq 2$. Fix an integer $r \geq 2$. As in the last part of Remark 3.3, Question 3.2 (i) is true for every integer $d > r$ if it is true for every integer d with $0 < d \leq r$. By [3], Prop, 1.7, for every integer d with $0 < d \leq r$ there exists a stable vector bundle on C with these numerical invariants and with a good filtration.

Remark 3.6. Assume $g \geq 2$ and that the ground field has characteristic zero. Fix integers r, d with $r \geq 2$ and a vector bundle E on C with these numerical invariants. If $d \geq (r-1)g$ the bundle E has a rank 1 subbundle with non-negative degree ([11]) or [14]). Furthermore for a general stable bundle with degree d and rank r the condition $d \geq (r-1)g$ is necessary for the existence of such line subbundle ([11] or [9] or [10] or [12]). Now assume $r \geq 3$ and $d \geq (r-1)g$. Take a line subbundle R of E of maximal degree and set $d' := \deg(R)$. Then continue looking for a good filtration of E/R. If E is a general stable bundle with degree d and rank r we have $d - rd' = (r-1)(g-1) + \varepsilon$, where ε is the unique integer with $0 \leq \varepsilon \leq r-1$ and $\varepsilon + (r-1)(g-1) \equiv d \bmod (r)$ ([12], Remark 3.14, or [9] or [10]). If E is general , then E/R is a general stable bundle with rank $r-1$ and degree $d - d'$ ([11]). Hence we may continue. In this way we obtain that Question 3.2 (ii) and 3.2 (iii) have the same answers.

References

1. A. Altman, S. L. Kleiman, *Introduction to Grothendieck Duality Theory*, Lect. Notes in Math., **146**, Springer-Verlag, Heidelberg, 1970.

2. M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc., **7**(3)(1957), 414-452. reprinted in : Michael Atiyah Collected Works, Oxford Science Publications, Claredon Press, Oxford, **1**(1988), 105-143.

3. E. Ballico, *Stability of bundles with a given filtration*, Proceedings of the Royal Society of Edinburgh (Mathematics, A), **129**(1999), 229-234.

4. F. Catanese, M. Franciosi, K. Hulek and M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J., **154**(1999), 185-220.

5. M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom., **19**(1984), 125-171.

6. M. Green, and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math., **83**(1986), 73-90.

7. M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Composition Math. **67**(1988), 301-314.

8. R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Lect. Notes in Math. **156**, Springer-Verlag, 1970.

9. A. Hirschowitz, *Problémes de Brill-Noether en rang supérieur*, Unpublished preprint, partially published as the next reference, 1984.

10. A. Hirschowitz, *Problémes de Brill-Noether en rang supérieur*, Comptes Rendus Acad. Sci. Paris, Série I, **307**(1988), 153-156.

11. H. Lange, *Zur Klassifikation von Regelmannigfaltigkeite*, Math. Ann., **262**(1983), 447-459.

12. H. Lange, *Some geometrical aspects of vector bundles on curves*, Aportaciones Matematicas Notas de Investigation, **5**(1992), 53-74.

13. H. Lange and M. S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Math. Ann., **266**(1983), 55-72.

14. S. Mukai and F. Sakai, *Maximal subbundles of vector bundles on a curve*, Manuscripta Math., **52**(1985), 251-256.

15. J. Rathmann, *The uniform position principle for curves in characteristic p*, Math. Ann., **276**(1987), 565-579.

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