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AN EQUIVALENT DENJOY TYPE DEFINITION OF THE GENERALIZED HENSTOCK STIELTJES INTEGRAL

BY

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Abstract. The present work is to characterise the HS_k-integral as the $[AC_{gk}G^*]$ function, produced here, leading to the introduction of the equivalent Denjoy Stieltjes type integral, the DS_k^*-integral.

1. Introduction. Das, Nath and Sahu [3] introduce a definition of a generalized Henstock-Stieltjes integral, which they call the HS_k-integral. It is shown in [3] that the HS_k-integral includes the LS_k-integral of Bhattacharyya and Das [1] and generalizes the integrals of Pfeffer [6] in some sense or other.

The purpose of the present paper is to introduce Denjoy type integral, the DS_k^*-integral, as a countable extension of the LS_k-integral of Bhattacharyya and Das [1]. To this end the notion of the concepts of AC_{gk}^*, BV_{gk}^* functions is introduced in Section 2 along with their generalized versions $[AC_{gk}G^*]$, $[BV_{gk}G^*]$ etc functions.

It is shown in Section 3 that the HS_k-integral includes the DS_k^*-integral.

In Section 4 we obtain the gk-differential property of the indefinite HS_k-integral which is shown to be $[AC_{gk}G^*]$. This leads to the fact that the HS_k-

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integral is included in the DS_k^*-integral. Thus the DS_k^*-integral becomes the descriptive definition of the HS_k-integral and the generalization actually resembles the classical relations.

For notations and definitions not produced here we refer to Russell [8], Bhattacharyya and Das [1, 2], Ray and Das [7] and Das, Nath and Sahu [3]. However, we recall the definition of the HS_k-integral of Das, Nath and Sahu [3].

Let g be k-convex on $[a,b]$ and let $g_+^{(k-1)}(a)$ and $g_\pm^{(k-1)}(b)$ exist. We shall denote by C the subset of $[a,b]$ where $g^{(k-1)}(x)$ exists and by D the set $[a,b]\setminus C$. With g given we shall denote by Ω a class of functions defined at each x in C, continuous over C at each point of C. Further F possesses either sided limits $F(x_0+)$, $F(x_0-)$ at each x_0 in D, over the points of C and also $F(x)=F(a_+)$ for $x<a$ and $F(x)=F(b_-)$ for $x>b$. F may or may not be defined in D. By Ω_0 we denote the subclass of Ω such that $F(x_0+)$ and $F(x_0-)$ exist finitely for $x_0\in D$.

The following notation will be required in the sequel.

Let $E\subset[a,b]$ and let $D: a\le x_0<x_1<\cdots<x_n\le b$ be any subdivision of $[a,b]$ with $x_i\in E$. For $F\in\Omega$, we define

$$W(F;E)=\sup\sum_{i=1}^{n}|F(x_i+)-F(x_{i-1}-)|$$

and call it the outer variation of F on E.

Definition 1.1. (Definition 2.1 of [3]) Let f be defined on $[a,b]$. A partition of $[a,b]$ is a set $P=\{x_0,x_1,\ldots,x_q;\xi_1,\xi_2,\ldots,\xi_q\}$ such that $a=x_0<x_1<\ldots<x_q=b$ and $x_{j-1}\le\xi_j\le x_j$, $j=1,2,\ldots,q$. For a given positive function δ on $[a,b]$, we say P is $\delta(gk)$-fine whenever $g_-^{(k-1)}(x_j)-g_+^{(k-1)}(x_{j-1})<\delta(\xi_j)$ for all $j=1,2,\ldots,q$. The points ξ_j are called the associated points and x_j the partition points.

Define

$$\begin{aligned} S(P,f,g) &= \sum_{j=1}^{q} f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! \\ &\quad + \sum_{j=1}^{q} f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \\ &= \sum_{j=1}^{q} T_j(P, x_{j-1}, x_j; \xi_j), \quad \text{say.} \end{aligned}$$

The HS_k-integral of f with respect to g, written as $(HS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$, is the real number I if for every $\varepsilon > 0$ there is a positive function δ on $[a,b]$ such that for every $\delta(gk)$-fine partition P of $[a,b]$, the inequality

$$|S(P,f,g) - I| < \varepsilon$$

is satisfied. If the HS_k-integral exists, we write $(f,g) \in HS_k[a,b]$ and

$$I = (HS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

For brevity we write $P = \{[u,v];\xi\}$ where $[u,v]$ denotes a typical interval in P tagged with the associated point $\xi \in [u,v]$. The approximating sum will be denoted by

$$S(P,f,g) = \sum T(P,u,v;\xi).$$

If $(f,g) \in HS_k[a,b]$, then $(f,g) \in HS_k[a,x]$ for every $x \in (a,b]$. We define the HS_k-primitive F of f on $[a,b]$ by

$$\begin{aligned} F(x) &= (HS_k)\int_a^x f(t)\frac{d^k g(t)}{dt^{k-1}} && \text{if } a < x \le b \\ &= 0 && \text{if } x = a. \end{aligned}$$

Theorem 1.2. (Theorem 2.11 of [3], Saks-Henstock Lemma.) *If $(f,g) \in HS_k[a,b]$, then there is a function F on $[a,b]$ such that for every $\varepsilon > 0$*

there is a positive function δ *on* $[a,b]$ *such that for every* $\delta(gk)$*-fine partition* $P=\{[u,v];\xi\}$ *of* $[a,b]$.

$$(P)\sum\Big|F(v)-F(u)-T(P,u,v;\xi)\Big|<\varepsilon.$$

Theorem 1.3. (Theorem 2.12 of [3], Cauchy Extension formula). *Let* $(f,g)\in HS_k[a,d]$ *for each* $d\in(a,b)$. *If* $\lim_{d\to b}(HS_k)\int_a^d f(x)\frac{d^k g(x)}{dx^{k-1}}$ *exists and equals* h, *then* $(f,g)\in HS_k[a,b]$ *and*

$$(HS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}=h+f(b)\Big[g_+^{(k-1)}(b)-g_-^{(k-1)}(b)\Big]/(k-1)!.$$

Theorem 1.4. (Theorem 2.14 of [3]). *The* HS_k*-integral is the gauge integral (Henstock-Kurzweil integral) induced by the* k*-convex function* g.

In view of Theorem 1.4 above, Das, Nath and Sahu [3] have the following remark.

Remark 1.5. (Remark 2.15 of [3]). The HS_k-integral of f with respect to g is the real number I if for every arbitrary $\varepsilon>0$ there is a positive function δ, called a gauge, on $[a,b]$ such that for every δ-fine partition

$$P=\{a=x_0<x_1<\cdots<x_q=b;\xi_1,\xi_2,\dots,\xi_q\},$$

$\xi_j\in[x_{j-1},x_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))$, of $[a,b]$ the inequality $|S(P,f,g)-I|<\varepsilon$ holds.

Finally Das, Nath and Sahu [3] obtain

Theorem 1.6. (Theorem 3.2 of [3]). *If* $(f,g)\in LS_k[a,b]$, *then* $(f,g)\in HS_k[a,b]$ *and the two integrals agree.*

This leads to the inclusive chain

$$(RS_k^*) \subset (\mathcal{R}S_k^*) \subset (LS_k) \subset (HS_k),$$

where (I) stands for the class of I-integrable functions on $[a, b]$.

2. The BV_{gk}^* and AC_{gk}^* Functions. Let $a \leq x_1 < x_1' \leq x_2 < x_2' \leq \cdots \leq x_n < x_n' \leq b$. The set of non-overlapping intervals $[x_i, x_i']$, $i = 1, 2, \ldots, n$ will be said to form an elementary system I in $[a, b]$ and we write

$$\sigma I = \sum_{i=1}^{n} [F(x_i'+) - F(x_i-)],$$
$$\sigma|I| = \sum_{i=1}^{n} |F(x_i'+) - F(x_i-)|,$$

and

$$|I|_{gk} = \sum_{i=1}^{n} [g_+^{(k-1)}(x_i') - g_-^{(k-1)}(x_i)].$$

Definition 2.1. The function F in calss Ω is said to be gk-absolutely continuous above, (respectively below) AC_{gk}-above (respectively AC_{gk}-be -low), if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sigma\ I < \varepsilon$ (respectively $\sigma I > -\varepsilon$) for any elementary system I in $[a, b]$ with $|I|_{gk} < \delta$.

The function F is said to be gk-absolutely continuous, AC_{gk}, if the relation $\sigma|I| < \varepsilon$ holds whenever $|I|_{gk} < \delta$ (cf. Definition 2.5 of [1]).

Definition 2.2. Let F be in class Ω. The least upper bound and the greatest lower bound of the aggregate $\{\sigma I\}$ of sums σI for all possible elementary systems I in $[a, b]$ are called respectively the positive and negative variations of F in $[a,b]$ and are denoted respectively by $V^+_{gk}[F;a,b]$ and $V^-_{gk}[F;a,b]$.

Clearly $V^+_{gk}[F;a,b] \geq 0$ and $V^-_{gk}[F;a,b] \leq 0$.

Definition 2.3. (Definition 2.5 of [2]). Let F be defined in $[a,b]$ and in class Ω and let

$$V_{gk}[F;a,b] = \sup_D \sum_{i=1}^{n} |F(x_i+) - F(x_{i-1}-)|,$$

where the supremum is taken for all subdivisions $D : a = x_0 < x_1 \ldots < x_n = b$ of $[a,b]$ with $g_+^{(k-1)}(x_{i-1}) < g_-^{(k-1)}(x_i)$, $i = 1, 2, \ldots, n$.

If $V_{gk}[F;a,b] < +\infty$, then F is said to be a function of bounded gk-variation, BV_{gk} on $[a,b]$ and we write $F \in BV_{gk}[a,b]$.

Definition 2.4. Let $X \subset [a,b]$. A function $F \in \Omega$ is said to be $AC_{gk}(BV_{gk})$ on X if the Definition 2.1 (Definition 2.3) holds with the end points belonging to X for all i.

It is evident that if F is AC_{gk} on X, then given any $\varepsilon > 0$, there exists a number $\delta > 0$ such that, for every sequence of non-overlapping intervals $\{I_n\}$ whose end points belong to E, the inequality $\sum |I_n|_{gk} < \delta$ implies $\sum_n W(F; X \cap I_n) < \varepsilon$.

Definition 2.5. A function $F \in \Omega_0$ is said to be AC^*_{gk} on a set $X \subset [a,b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any set of pairwise disjoint intervals $\{[c_r, d_r]\}$, having end points in X, the relations

$$\sum_r \overline{\text{bound}}_{c_r \leq x < d_r} |F(x+) - F(c_r^-)| < \varepsilon$$

and

$$\sum_r \overline{\text{bound}}_{c_r<x\le d_r}|F(d_r^+)-F(x-)|<\varepsilon$$

hold whenever $\sum_r\{g_+^{(k-1)}(d_r)-g_-^{(k-1)}(c_r)\}<\delta$.

Definition 2.6. A function $f\in\Omega_0$ is said to be BV_{gk}^* on a set $X\subset[a,b]$, if

$$\sup\sum_{i=1}^n \overline{bound}_{x_{i-1}\le x<x_i}|F(x+)-F(x_{i-1}-)|<+\infty$$

and

$$\sup\sum_{i=1}^n \overline{bound}_{x_{i-1}<x\le x_i}|F(x_i+)-F(x-)|<+\infty$$

for all possible subdivisions $D: a\le x_0<x_1<\cdots<x_n\le b$ of $[a,b]$ with each $x_i\in X$ and $g_+^{(k-1)}(x_{i-1})<g_-^{(k-1)}(x_i)$, $i=1,2,\dots,n$.

Note 2.7. Let $X\subset[a,b]$ and $a,b\in X$. If F is BV_{gk}^* on X then it is BV_{gk}^* on $\bar{X}$, where $\bar{X}$ is the closure of X.

Definition 2.8. A function F is said to be $AC_{gk}G^*$ (respectively $[AC_{gk}G^*]$) on X if X is the union of a sequence of sets (respectively closed sets) $\{X_i\}$ such that the function F is AC_{gk}^* on X_i for all i.

Definition 2.9. A function F is said to be $BV_{gk}G^*$ (respectively $[BV_{gk}G^*]$) on X if X is the union of a sequence of sets (respectively closed sets) $\{X_i\}$ such that the function F is BV_{gk}^* on X_i for all i.

The following lemma is immediate.

Lemma 2.10. $V_{gk}^+[F;a,b]$ *is finite if and only if* $V_{gk}^-[F;a,b]$ *is finite.*

Theorem 2.11. *If* F *in class* Ω_0, *then*

$$V_{gk}[F;a,b]\le V_{gk}^+[F;a,b]-V_{gk}^-[F;a,b].$$

Proof. If $V^+_{gk}[F;a,b]=+\infty$, then $V^-_{gk}[F;a,b]=-\infty$ and vice-versa. So the theorem is clear. Suppose that $V^+_{gk}[F;a,b]$ is finite. Then $V^-_{gk}[F;a,b]$ is also finite. Let $a\le x_0<x_1<\cdots<x_n\le b$ with $g^{(k-1)}_+(x_{i-1})<g^{(k-1)}_-(x_i)$ be any sub-division of $[a,b]$. Let M and N be two parts such that for $i\in M$, $F(x_i+)-F(x_{i-1}-)\ge 0$ and for $i\in N$, $F(x_i+)-F(x_{i-1}-)<0$, $i=1,2,\dots,n$.

Then the set of intervals $\{[x_{i-1},x_i]\}$ form two elementary systems I_1 and I_2 respectively for $i\in M$ and for $i\in N$. So

$$\sum_{i=1}^{n}|F(x_i+)-F(x_{i-1}-)|=\sigma I_1-\sigma I_2\le V^+_{gk}[F;a,b]-V^-_{gk}[F;a,b],$$

and the theorem is proved.

Following the process of the proof of Theorem 2.2 of Bhattacharyya and Das [2], we obtain

Theorem 2.12. *If F is AC_{gk}-above (below) on $[a,b]$, then it is BV_{gk} on $[a,b]$.*

Definition 2.13. (cf. Definition 2.6 of [1]). Let $F\in\Omega$ and for $x\in[a,b]$ and $x+h\in C\cap[a,b]$ the function $\psi(x,h)$ be defined by

$$\begin{aligned}\psi(x,h) &= \frac{F(x+h)-F(x-)}{|[x,x+h]|_{gk}} \quad \text{if} \quad h>0 \quad \text{and} \quad |[x,x+h]|_{gk}\ne 0\\ &= \frac{F(x+)-F(x+h)}{|[x,x+h]|_{gk}} \quad \text{if} \quad h<0 \quad \text{and} \quad |[x+h,x]|_{gk}\ne 0\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

If $\psi(x,h)$ tends to a limit as $h\to 0$, then this limit is called the gk-derivative of F at x and will be denoted by $F'_{gk}(x)$. The left, right, upper, lower etc, gk-derivatives are obtained in the usual way.

Theorem 2.14. *If a function F on class Ω_0 is AC_{gk}-below on $[a,b]$ and*

$F'_{gk}(x) \geq 0$ *except a set of gk-measure zero in* $[a,b]$, *then* F *is non-decreasing on* $[a,b] \cap C$.

Proof. In view of Theorem 2.12 above and Theorem 2.3 of Bhattacharyya and Das [2], F'_{gk} exists gk-almost everywhere on $[a,b]$. If α, β are any two points of $[a,b] \cap C$ and $E = \{x : x \in (\alpha,\beta) \cap C \text{ and } 0 \leq F'_{gk}(x) < \infty\}$, then $|E|_{gk} = |[\alpha,\beta]|_{gk}$.

Let $\varepsilon > 0$ be arbitrary. If $x \in E$, there is a sequence $\{h_i\}$, $(h_i > 0)$, such that $h_i \to 0$, $x + h_i \in C$ and

$$\frac{F(x+h_i) - F(x)}{|[x, x+h_i]|_{gk}} > F'_{gk}(x) - \varepsilon. \tag{1}$$

Let $\mathcal{F}$ denote the family of intervals $[x, x+h_i]$ thus associated to the points of E. Then by Lemma 1.1 of [2] for any $\delta > 0$, there exists a finite number of non-overlapping open intervals (x_1, x'_1), (x_2, x'_2),. . . ,(x_n, x'_n) from $\mathcal{F}$ with x_i, x'_i belonging to $(\alpha,\beta) \cap C$ for each i such that

$$\sum_{i=1}^{n} |E \cap (x_i, x'_i)|_{gk} > |[\alpha,\beta]|_{gk} - \delta.$$

Since F is AC_{gk}-below, then the number δ can be chosen such that

$$\sum_{i=1}^{n+1} [F(x_i) - F(x'_{i-1})] > -\varepsilon. \tag{2}$$

Combining (1) and (2), we obtain

$$F(\beta) - F(\alpha) > -\varepsilon + \sum_{i=1}^{n} [F'_{gk}(x) - \varepsilon] |[x_i, x'_i]|_{gk} > -\varepsilon - \varepsilon |[\alpha,\beta]|_{gk}.$$

Arbitrariness of $\varepsilon > 0$ gives $F(\beta) \geq F(\alpha)$. Since this is true for arbitrary α, $\beta (> \alpha)$ in $[a,b] \cap C$, the theorem is proved.

Lemma 2.15. *Let F be in $\Omega_0, F(x) = \frac{1}{2}[F(x+) + F(x-)]$ for $x \in D$ and let F be $[AC_{gk}G]$ below on $[a,b]$ and $F'_{gk} \geq 0$ in $[a,b]$ except a set of gk-measure zero. If P is a perfect subset of $[a,b]$ such that F is non-decreasing on the complementary intervals of P, then there is an interval $[1,m]$ containing points of P such that F is non-decreasing in $[1,m]$.*

Proof. Let $[a,b] = \cup_i E_i$, E_i be a closed set and F be AC_{gk}-below on E_i for each i. Then $P = \cup_i(P \cap E_i)$. So by Baire's theorem, there is an i_0 and an interval $[1,m]$ such that $P \cap [1,m] \subset P \cap E_{i_0}$. Hence F is AC_{gk}-below on $P \cap [1,m]$. Since F is non-decreasing in the complementary intervals of P, it follows that F is AC_{gk}-below on $[1,m]$. So, by Theorem 2.14, F is non-decreasing in $[1,m]$. This proves the lemma.

Lemma 2.16. *Let F be in Ω_0 and $[AC_{gk}G]$ below on $[a,b]$. If $F'_{gk}(x) \geq 0$ except a set of gk-measure zero in $[a,b]$, then F is non-decreasing on $[a,b] \cap C$.*

Proof. We define $s(x)$ in $[a,b]$ by

$$\begin{aligned} s(x) &= F(x) \quad \text{for} \quad x \in [a,b] \cap C \\ &= \frac{1}{2}[F(x+) + F(x-)] \quad \text{for} \quad x \in D. \end{aligned}$$

Clearly s is $[AC_{gk}G]$ below on $[a,b]$ and $s'_{gk}(x) \geq 0$ except a set of gk-measure zero in $[a,b]$. Let G be the set of all points x in $[a,b]$ such that there exists a neighbourhood of x in which s is non-decreasing. Then the set $P = [a,b] - G$ is perfect. If possible, suppose that P is non-empty. Let $\{(a_i, b_i)\}$ be the set of complementary intervals of P in $[a,b]$. Then s is non-decreasing in each interval (a_i, b_i). By Lemma 2.15, there is an interval $[1,m]$ containing points of P such that s is non-decreasing in $[1,m]$. This contradicts the definition of P. Thus P is empty and so s is non-decreasing in $[a,b] \cap C$. This proves the lemma.

Theorem 2.17. *If F is $[AC_{gk}G^*]$ on $[a,b]$ and if $F'_{gk}(x)$ vanishes except a set of gk-measure zero in $[a,b]$, then F is constant in $[a,b]$.*

Proof. Clearly F is $[AC_{gk}G]$ on $[a,b]$. By Lemma 2.16 and its analogue for $[AC_{gk}G]$ above function, we have F to be constant on $[a,b]\cap C$. For $x_0 \in D$

$$F(x_0+) = \lim_{\substack{x\to x_0+\\ x\in C}} F(x) = \lambda = \lim_{\substack{x\to x_0-\\ x\in C}} F(x) = F(x_0-) = F(x_0).$$

Hence F is constant in $[a,b]$ and the theorem is proved.

3. The DS_k^*-Integral.

Definition 3.1. Let f be defined on $[a,b]$. f is said to be DS_k^*-integrable on $[a,b]$ if there exists a function $F \in \Omega_0$ such that

(i) F is $[AC_{gk}G^*]$ on $[a,b]$,

(ii) $F'_{gk}(x) = f(x)$ except a set of gk-measure zero.

The DS_k^*-integral of f on $[a,b]$ is the number $F(b+) - F(a-)$ and we write

$$F(b+) - F(a-) = (DS_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

In view of Theorem 2.17, the DS_k^*-integral is uniquely defined. We further note that a DS_k^*-integrable function is gk-measurable and is finite gk-almost everywhere.

We shall show that DS_k^*-integral is the descriptive definition of the HS_k-integral.

We first show that $(DS_k^*) \subset (HS_k)$.

To this end we make the following observation.

Observation 3.2. Given an arbitrary function $\delta : [a,b] \to (0,\infty)$

(independent of the notion of integration), there always exists, in view of Lemma 2.2 and Theorem 2.14 of Das, Nath and Sahu [3], a $\delta(gk)$-fine (or a δ_1-fine) partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_q = b; \xi_1, \xi_2, \dots, x_q\}$ of $[a, b]$. Also observation 2.13 of Das, Nath and Sahu [3] shows that one can consider an associated point ξ_j to be a partition point. Here we further observe that the end points of two intervals $[x_{j-1}, \xi_j]$ and $[\xi_j, x_j]$ associated with ξ_j may be considered as points of C. Infact, if $x_j, a < x_j < b$, belongs to D, we consider two points x'_j, x''_j of C so that $x'_j \in (\xi_j, \xi_j + \delta_1(\xi_j)) \cap (x_j - \delta_1(x_j), x_j)$ and $x''_j \in (x_j, x_j + \delta_1(x_j)) \cap (\xi_{j+1} - \delta_1(\xi_{j+1}), \xi_{j+1})$.

We consider x_j as an associated point having $[x'_j, x_j]$ and $[x_j, x''_j]$ as two intervals associated with x_j. If $x_j = a$ we consider only $[x_j, x''_j]$ and if $x_j = b$, the associated interval is only $[x'_j, x_j]$. This is consistent as because we ultimately confine ourselves to all $\delta(gk)$-fine (or δ_1-fine) partitions that invite additional associated points.

Thus if $P = \{[u, \xi] \cup [\xi, v]; \xi\}$ is a $\delta(gk)$-fine (or equivalently a δ_1-fine, see Theorem 2.14 and Remark 2.15 of Das, Nath and Sahu [3]) partition of $[a, b]$ with $u, v \in C$, then the approximating sum for the HS_k-integral will be given by

$$\begin{aligned} S^1(P, f, g) &= \sum_{\xi} f(\xi)\Big[\{g_-^{(k-1)}(\xi) - g^{(k-1)}(u)\} + \{g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi)\} \\ &\quad + \{g^{(k-1)}(v) - g_+^{(k-1)}(\xi)\}\Big]/(k-1)! \\ &= \sum_{\xi} f(\xi)\{g^{(k-1)}(v) - g^{(k-1)}(u)\}/(k-1)! \end{aligned}$$

where the concerned term for $\xi = a$ is only $f(\xi)\{g^{(k-1)}(v) - g_+^{(k-1)}(\xi)\}/(k-1)!$ and the associated term for $\xi = b$ is $f(\xi)[g_-^{(k-1)}(\xi) - g^{(k-1)}(u)\} + \{g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi)\}]/(k-1)!$.

We note that the approximating sum $S(P,f,g)$ of Das, Nath and Sahu [3], namely

$$\begin{aligned} S(P,f,g) &= \sum f(\xi)\{g_-^{(k-1)}(v) - g_+^{(k+1)}(u)\}/(k-1)! \\ &\quad + \sum f(v)\{g_+^{(k-1)}(v) - g_-^{(k-1)}(v)\}/(k-1)! \end{aligned}$$

actually accomodates the k-saltus of g in $D=[a,b]-C$. Thus $S^1(P,f,g)$ can equivalently be considered for $S(P,f,g)$.

Theorem 3.3. *If* $(f,g)\in DS_k^*[a,b]$, *then* $(f,g)\in HS_k[a,b]$ *and the two integrals agree.*

Proof. Let F be the DS_k^*-primitive of f and let $F'_{gk}(x)=f(x)$ for x in $[a,b]-H$ where H is of gk-measure zero.

Clearly $H\subset C$.

For $\xi\notin H$, given $\varepsilon>0$ there is a $\delta(\xi)>0$ such that whenever $\xi-\delta(\xi)<u<\xi$, $u\in C$ and $\xi<v<\xi+\delta(\xi)$, $v\in C$ we have respectively

$$|F(\xi+)-F(u)-f(\xi)|[u,\xi]|_{gk}|<\varepsilon|[u,\xi]|_{gk} \tag{1}$$

and

$$|F(v)-F(\xi-)-f(\xi)|[\xi,v]|_{gk}|<\varepsilon|[\xi,v]|_{gk}. \tag{2}$$

Since F is $[AC_{gk}G^*]$ on $[a,b]$ there is a sequence of closed sets $\{E_i\}$ such that $[a,b]=\cup_i E_i$ and F is AC_{gk}^* on each E_i. Let $X_1=E_1$, $X_i=E_i-\cup_{j=1}^{i-1}E_j$ for $i=2,3,\ldots$, and let $H_{ij}=\{x : x\in H\cap X_i \text{ and } j-1<|f(x)|<j\}$. Clearly $H_{ij}, i,j=1,2\ldots$ are pairwise disjoint and $\cup_{i,j}H_{ij}=H$. Since F is also AC_{gk}^* on H_{ij}, there is a positive $\delta_{ij}<\varepsilon/(j\cdot 2^{i+j+1})$ such that for any sequence of non-overlapping intervals $\{[a_r,b_r]\}$ with end points a_r,b_r

belonging to H_{ij} and satisfying

(3) $$\sum_r |[a_r, b_r]|_{gk} < \delta_{ij}$$

we have

(4) $$\sup_{a_r<x\le b_r} \sum_r |F(b_r+) - F(x-)| < \varepsilon/2^{i+j+1}$$

and

(5) $$\sup_{a_r\le x<b_r} \sum_r |F(x+) - F(a_r-)| < \varepsilon/2^{i+j+1}.$$

Choose G_{ij} to be the union of a sequence of open intervals such that $G_{ij} \supset H_{ij}$ and $|G_{ij}|_{gk} < \delta_{ij}$. Now for $\xi \in H_{ij}, i, j = 1, 2, \ldots$, define $\delta(\xi)$ such that $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_{ij}$ (infact contained in a component interval of G_{ij}). Hence we have defined a function $\delta : [a, b] \to (0, \infty)$.

Consider any δ-fine partition $P = \{[u, v]; \xi\}$ of $[a, b]$. Without any loss of generality (see observation 2.13 of Das, Nath and Sahu [3] and Observation 3.2 of this article) we can assume $u = \xi$, $v \in C$ or $v = \xi$, $u \in C$. Then

(6) $$\begin{aligned} & |S^1(P, f, g) - (DS_k^*)\int_a^b f| \\ = \; & |\sum f(\xi)[\{g_-^{(k-1)}(\xi) - g^{(k-1)}(u)\} + \{g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi)\} \\ & +\{g^{(k-1)}(v) - g_+^{(k-1)}(\xi)\}]/(k-1)! - \sum\{F(v) - F(u)\}| \\ \le \; & \sum[|f(\xi)\{g_+^{(k-1)}(\xi) - g^{(k-1)}(u)\}/(k-1)! - \{F(\xi+) - F(u)\}| \\ & +|f(\xi)\{g^{(k-1)}(v) - g_-^{(k-1)}(\xi)\}/(k-1)! - \{F(v) - F(\xi-)\}| \\ & +|f(\xi)\{g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi)\}/(k-1)! - \{F(\xi+) - F(\xi-)\}|] \\ = \; & \sum_{\xi\notin H} + \sum_{\xi\in H}, \quad \text{say} \end{aligned}$$

Using (1) and (2) and noting that

$$F(\xi+) = \lim_{v\to\xi+} F(v) \quad \text{and} \quad g_+^{(k-1)}(\xi) = \lim_{v\to\xi+} g^{(k-1)}(v),\ v \in C,$$

we have

$$(7) \qquad \sum_{\xi\notin H} < \varepsilon \sum\{|[u,\xi]|_{gk} + |[\xi,v]|_{gk} + |\{\xi\}|_{gk}\} < 3\varepsilon|[a,b]|_{gk}.$$

Again, using (3), (4), (5) and noting that the last member of each term of $\sum_{\xi\in H}$ is zero (since $\xi \in H \subset C$), we have

$$(8) \qquad \sum_{\xi\in H} < \sum_{i,j} 2\varepsilon/2^{i+j+1} + \sum_{i,j} 2\cdot j\cdot \delta_{ij} < \varepsilon + \varepsilon = 2\varepsilon.$$

In view of (7) and (8) it follows from (6) that

$$|S^1(P,f,g) - (DS_k^*)\int_a^b f| < \varepsilon[3|[a,b]|_{gk} + 2].$$

Hence by virtue of Theorem 2.14 and Remark 2.15 of Das, Nath and Sahu [3], it follows that $(f,g) \in HS_k[a,b]$ and that

$$(HS_k)\int_a^b f = (DS_k^*)\int_a^b f.$$

This proves the theorem.

4. The Equivalence of the DS_k^*-Integral and the HS_k-Integral. Theorem 3.3 shows that $(DS_k^*) \subset (HS_k)$. In this section we proceed to show that $(HS_k) \subset (DS_k^*)$. To this end we first show that if $(f,g) \in HS_k[a,b]$, then f is $[AC_{gk}G^*]$ on $[a,b]$ and if F is the HS_k-primitive then $F'_{gk}(x) = f(x)$ gk-almost everywhere in $[a,b]$. Theorem 4.6 is our desired result and Remark 4.7 is the equivalence statement.

Theorem 4.1. *If $(f,g) \in HS_k[a,b]$ and F is the HS_k-primitive then $F'_{gk}(x) = f(x)$ in $[a,b]$ except for a set of gk-measure zero.*

Proof. Since $(f,g) \in HS_k[a,b]$, in view of Theorem 1.4, Remark 1.5 and Theorem 1.2, for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine partition $P = \{[u, v]; \xi\}$ we have

$$(1) \qquad \sum |F(v) - F(u) - f(\xi)|(u, v)|_{gk} - f(v)|\{v\}|_{gk}| < \varepsilon.$$

Let $X = \{x : a \leq x \leq b$, either $F'_{gk}(x)$ does not exist or $F'_{gk}(x) \neq f(x)\}$. We prove $|X|_{gk} = 0$.

For every $x \in X$, there is $\eta(x) > 0$ such that for every $\delta_1(x) > 0$, $0 < \delta_1(x) \leq \delta(x)$, either there is a point $u \in C$, $x - \delta_1(x) < u < x$ and

$$(2) \qquad |F(x+) - F(u) - f(x)|[u, x]|_{gk}| > \eta(x)|[u, x]|_{gk}$$

or there is a point $v \in C$, $x < v < x + \delta_1(x)$ and

$$(3) \qquad |F(v) - F(x-) - f(x)|[x, v]|_{gk}| > \eta(x)|[x, v]|_{gk}.$$

By Theorem 1.3, we have for $a < x \leq b$

$$F(x-) = F(x) - f(x)|\{x\}|_{gk}.$$

Since the approximating sum for the HS_k-integral accomodates the gk-saltus at the right end point of each partition interval $[u, v]$, it is evident that $F(x+) = F(x)$ for $a \leq x < b$. Thus (2) and (3) respectively reduce to

$$|[F(x) - F(u) - f(x)|(u, x)|_{gk} - f(x)|\{x\}|_{gk}]| > \eta(x)|[u, x]|_{gk},$$

and

$$|[F(v) - F(x) - f(x)|(x, v)|_{gk}]| > \eta(x)|[x, v]|_{gk}.$$

For a fixed n, let $X_n \subset X$ for which $\eta(x) \geq \frac{1}{n}$. Let $X_n = X_{n_1} \cup X_{n_2}$ such that the intervals $\{[u, x]\}$ covers the set X_{n_1} with right end points $x \in X_{n_1}$, and the intervals $\{[x, v]\}$ covers the set X_{n_2} with left end points $x \in X_{n_2}$. Using Lemma 1.1 of Bhattacharyya and Das [2] and the subsequent note there, it follows that we can find $[u_i, x_i]$, $i = 1, 2 \ldots, p$ with $x_i \in X_{n_1}$ and $[x_i, v_i]$, $i = 1, 2, \ldots, q$ with $x_i \in X_{n_2}$ such that

$$|X_{n_1}|^0_{gk} < \sum_{i=1}^{p} |[u_i, x_i]|_{gk} + \varepsilon$$

and

$$|X_{n_2}|^0_{gk} < \sum_{i=1}^{q} |[x_i, v_i]|_{gk} + \varepsilon.$$

Clearly the intervals $\{[u_i, x_i]\}$ and also $\{[x_i, v_i]\}$ form δ_1-fine partial partitions of $[a, b]$. So, using (1)

$$\sum_{i=1}^{p} |[F(x_i) - F(u_i) - f(x_i)|(u_i, x_i)|_{gk} - f(x_i)|\{x_i\}|_{gk}]| < \varepsilon$$

and

$$\sum_{i=1}^{q} |[F(v_i) - F(x_i) - f(x_i)|(x_i, v_i)|_{gk}]| < \varepsilon.$$

Hence

$$\begin{aligned} |X_n|^0_{gk} &< \varepsilon + \sum_{i=1}^{p} |[F(x_i) - F(u_i) - f(x_i)|(u_i, x_i)|_{gk} \\ &\quad - f(x_i)|\{x_i\}|_{gk}]|(\eta(x_i))^{-1} \\ &\quad + \sum_{i=1}^{q} |[F(v_i) - F(x_i) - f(x_i)|(x_i, v_i)|_{gk}]|(\eta(x_i))^{-1} \\ &< \varepsilon + n\varepsilon + n\varepsilon \\ &= (2n+1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $|X_n|^0_{gk} = 0$, and so $|X|^0_{gk} = 0$. This proves the theorem.

Theorem 4.2. *If $(f, g) \in HS_k[a, b]$ and F is the HS_k-primitive, then F is $[BV_{gk}G^*]$ on $[a, b]$.*

Proof. For every $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that for any δ-fine partition $P = \{[u, v]; \xi\}$ we have

$$\sum |F(v) - F(u) - f(\xi)|(u, v)|_{gk} - f(v)|\{v\}|_{gk}| < \varepsilon. \tag{1}$$

For convenience, we take $\varepsilon = 1$ and $\delta(\xi) \leq 1$.

Let X_{n_i} denote the set of all points $x \in [a + \frac{i-1}{n}, a + \frac{i}{n}] \cap [a, b]$ such that $|f(x)| \leq n$ and $\frac{1}{n} < \delta(\xi) < \frac{1}{n-1}$. Obviously, the union of $X_{n_i}, i = 1, 2, \ldots; n = 2, 3, \ldots$ is $[a, b]$.

Take any finite sequence of non-overlapping intervals $\{[a_j, b_j]\}$ with $a_j, b_j \in X_{n_i}$ such that $g_+^{(k-1)}(a_j) < g_-^{(k-1)}(b_j)$ for all j. Then these intervals form a δ-fine partial partition of $[a, b]$. Hence for any $x \in (a_j, b_j)$, using (1) and Theorem 1.3, we have

$$\begin{aligned} |F(x+) - F(a_j-)| &= |F(x) - F(a_j) + f(a_j)|\{a_j\}|_{gk}| \\ &\leq |F(b_j) - F(a_j)| + |F(b_j) - F(x)| + |f(a_j)||\{a_j\}|_{gk} \\ &< 2\varepsilon + 2|f(b_j)||\{b_j\}|_{gk} + 2|f(b_j)||(a_j, b_j)|_{gk} \\ &\quad +|f(a_j)||\{a_j\}|_{gk} \\ &< 2 + 5n|[a_j, b_j]|_{gk}. \end{aligned}$$

Also for every $y \in C$ and $a_j < y < x < b_j$,

$$\begin{aligned} |F(b_j+) - F(y)| &< \varepsilon + |f(b_j)||(y, b_j)|_{gk} + |f(b_j)||\{b_j\}|_{gk} \\ &< 1 + 2n|[a_j, b_j]|_{gk}. \end{aligned}$$

Passing to the limit as $y \to x-$, we get

$$|F(b_j+) - F(x-)| \leq 1 + 2n|[a_j, b_j]|_{gk}.$$

It, therefore, follows that

$$\sum_j \overline{bound}_{a_j \le x < b_j} |F(x+) - F(a_j-)| < +\infty$$

and

$$\sum_j \overline{bound}_{a_j < x \le b_j} |F(b_j+) - F(x-)| < +\infty.$$

Hence F is BV^*_{gk} on X_{n_i} and so by Note 2.7, F is BV^*_{gk} on $\bar{X}_{n_i}$, the closure of X_{n_i}. Hence F is $[BV_{gk}G^*]$ on $[a,b]$ and the theorem is proved.

Lemma 4.3. *If F is in Ω_0 and AC_{gk} on $[1,m] \subset [a,b]$, then F is AC^*_{gk} on $[1,m]$.*

Proof. Since F is AC_{gk} on $[1,m]$, to every $\varepsilon > 0$ there is $\eta > 0$ such that for every sequence of intervals $\{[c_j,d_j]\}$ in $[1,m]$ with $|[c_j,d_j]|_{gk} < \eta/2^j$, we have for $c_j < x < d_j$ and $j = 1,2,3\ldots.$

$$|F(x+) - F(c_j-)| < \varepsilon/2^j \quad \text{and} \quad |F(d_j+) - F(x-)| < \varepsilon/2^j.$$

We note that by Lemma 1.3 of Bhattacharyya and Das [1] there is only a finite set

$$P_j = \{x : x \in D, [g_+^{(k-1)}(x) - g_-^{(k-1)}(x)]/(k-1)! = |\{x\}|_{gk} > \eta/2^j\}.$$

It, therefore, follows that none of the end points $c_j, d_j \in P_j$ and that $[c_j,d_j] \subset [1,m] - P_j$. Consequently,

$$\sum_j \overline{bound}_{c_j \le x < d_j} |F(x+) - F(c_j-)| \le \varepsilon$$

and

$$\sum_j \overline{bound}_{c_j < x \le d_j} |F(d_j+) - F(x-)| \le \varepsilon,$$

and so F is AC^*_{gk} on $[1,m]$. This proves the lemma.

Lemma 4.4. *Let $E \subset [a,b]$ be a closed set, $\{[c_k,d_k]\}$ the sequence of the intervals contiguous to E, and $I_0 = [u_0,v_0] \subset [a,b]$ the smallest interval containing E. Then for any $F \in \Omega_0$, we have*

$$\begin{aligned} &\overline{bound}_{u_0 \le x < v_0} |F(x+) - F(u_0-)| \\ \le\ & W(F; E \cap I_0) + 2\sum_k \overline{bound}_{c_k \le x < d_k} |F(x+) - F(c_k-)| \end{aligned}$$

and

$$\begin{aligned} &\overline{bound}_{u_0 < x \le v_0} |F(v_0+) - F(x-)| \\ \le\ & W(F; E \cap I_0) + 2\sum_k \overline{bound}_{c_k < x \le d_k} |F(d_k+) - F(x-)|. \end{aligned}$$

Proof. We prove the first inequality; the latter is similar.

Let M_1 be any number less than $\overline{bound}_{u_0 \le x < v_0} |F(x+) - F(u_0-)|$ and let $x_0 \in I_0$ be such that $M_1 \le |F(x_0+) - F(u_0-)|$. If $x_0 \in E$, then $M_1 \le W(F; E \cap I_0)$. If $x_0 \in [c_k, d_k]$ for some k, then

$$M_1 \le W(F; E \cap I_0) + 2\overline{bound}_{c_k \le x < d_k} |F(x+) - F(c_k-)|.$$

Hence

$$\begin{aligned} &\overline{bound}_{u_0 \le x < v_0} |F(x+) - F(u_0-)| \\ \le\ & W(F; E \cap I_0) + 2\overline{bound}_{c_k \le x < d_k} |F(x+) - F(c_k-)|. \end{aligned}$$

This proves the lemma.

Theorem 4.5. *Let $Q \subset [a,b]$ be a closed set having complementary intervals $\{(c_n,d_n)\}$. If F is in Ω_0, then F is AC^*_{gk} on Q if and only if*

(i) *F is AC_{gk} on Q,*

(ii) $\sum_n \overline{bound}_{c_n \le x < d_n} |F(x+) - F(c_n-)| < \infty$ and $\sum_n \overline{bound}_{c_n < x \le d_n} |F(d_n+) - F(x-)| < \infty$,

(iii) *F is constant in $[\alpha, \beta]$ if $g_+^{(k-1)}(\beta) = g_-^{(k-1)}(\alpha)$, where α, $\beta \in Q$.*

Proof. The necessary part is obvious. We prove the sufficient part. If $[1, m] \subset Q$, then by Lemma 4.3 F is AC^*_{gk} on $[1, m]$. We, therefore, assume that Q has no portion $[1, m]$. Since F is AC_{gk} on Q, to every $\varepsilon > 0$ there exists a number $\delta > 0$ such that, for every sequence of non-overlapping intervals $\{[a_r, b_r]\}$, $a_r, b_r \in Q$, the inequality

$$\sum_r |[a_r, b_r]|_{gk} < \delta \quad \text{implies} \quad \sum_r W(F; Q \cap [a_r, b_r]) < \varepsilon/2.$$

In view of (ii) there exists a positive integer r_0 such that

$$\sum_{r=r_0+1}^{\infty} \overline{bound}_{c_r \le x < d_r} |F(x+) - F(c_r-)| < \varepsilon/4,$$
$$\sum_{r=r_0+1}^{\infty} \overline{bound}_{c_r < x \le d_r} |F(d_r+) - F(x-)| < \varepsilon/4.$$

Denote $[c_r, d_r]$ by $Jr, r = 1, 2, \ldots$ and let $\delta_0 = \min\{\delta, |J_1|_{gk}, \ldots, |J_{r_0}|_{gk}\}$. Consider any sequence of non-overlapping intervals $\{I_r\}$, $I_r = [u_r, v_r]$ with end points belonging to Q such that $\sum_r |I_r|_{gk} < \delta_0$.

By Lemma 4.4,

$$\begin{aligned} & \sum_r \overline{bound}_{u_r \le x < v_r} |F(x+) - F(u_r-)| \\ \le \quad & \sum_r W(F; Q \cap I_r) + 2 \sum_r \overline{bound}_{c_r \le x < d_r} |F(x+) - F(c_r-)| \\ < \quad & \varepsilon/2 + \varepsilon/2 = \varepsilon; \\ & \sum_r \overline{bound}_{u_r < x \le v_r} |F(v_r+) - F(x-)| \\ \le \quad & \sum_r W(F; Q \cap I_r) + 2 \sum_r \overline{bound}_{c_r < x \le d_r} |F(d_r+) - F(x-)| \end{aligned}$$

$$< \quad \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, the function F is AC^*_{gk} on Q, and the theorem is proved.

Theorem 4.6. *If* $(f, g) \in HS_k[a, b]$ *and* F *is the* HS_k*-primitive, then* F *is* $[AC_{gk}G^*]$ *on* $[a, b]$.

Proof. By Theorem 4.2, F is $[BV_{gk}G^*]$ on $[a, b]$ and so $[a, b] = UX_i$, where each X_i is closed and F is BV^*_{gk} on each X_i. In view of Theorem 4.5, it is sufficient to show that F is AC_{gk} on each X_i.

Let F_{X_i} be the HS_k-primitive of f_{X_i}, where $f_{X_i}(x) = f(x)$ for $x \in X_i$ and $f_{X_i}(x) = 0$ for $x \notin X_i$. If $(a, b) - X_i = \cup_j (c_j, d_j)$, define $F(x) = F_{X_i}(x) + G(x)$ and $G(x) = \sum_j F([a, x] \cap [c_j, d_j])$ for $x \in [a, b]$.

Clearly F_{X_i} is BV_{gk} on $[a, b]$ and the gk-derivative of F_{X_i} equals f_{X_i} gk-almost everywhere in $[a, b]$. By Theorem 1.1 of Bhattacharyya and Das [2], $f_{X_i} \in LS_k[a, b]$ and so $F_{X_i} \in AC_{gk}[a, b]$. We show that G is AC_{gk} on X_i.

Choose a sufficiently large N such that

$$\sum_{j=N+1}^{\infty} \overline{bound}_{c_j \le x < d_j} |F(x+) - F(c_j-)| < \varepsilon,$$
$$\sum_{j=N+1}^{\infty} \overline{bound}_{c_j < x \le d_j} |F(d_j+) - F(x-)| < \varepsilon.$$

and let $0 < \eta < \min\{|[c_j, d_j]|_{gk}, j = 1, 2 \ldots, N\}$.

Then for any sequence of non-overlapping intervals $\{[a_j,\ b_j]\}$ with a_j, $b_j \in X_i$ and satisfying $\sum_j |[a_j,\ b_j]|_{gk} < \eta$ we have

$$\sum_j \overline{bound}_{a_j \le x < b_j} |G(x+) - G(a_j-)| < \varepsilon,$$
$$\sum_j \overline{bound}_{a_j < x \le b_j} |G(b_j+) - G(x-)| < \varepsilon.$$

(We note that no $[c_j, d_j]$, $j = 1, 2, \ldots, N$ is a member of the sequence $\{[a_j, b_j]\}$). This shows that G is AC_{gk} on X_i. Consequently, F is AC_{gk} on X_i, and the theorem is proved.

Theorem 4.7. *If* $(f, g) \in HS_k[a, b]$, *then* $(f, g) \in DS_k^*[a, b]$.

Proof. Let F be the HS_k-primitive of f. Then F is $[AC_{gk}G^*]$ on $[a, b]$ and by Theorem 4.1. $F'_{gk}(x) = f(x)$, gk-almost everywhere in $[a, b]$.

Hence $(f, g) \in DS_k^*[a, b]$, and the theorem is proved.

Remark 4.8. In view of Theorem 3.3 and 4.7 we conclude that the DS_k^*-integral is the equivalent descriptive definition of the HS_k-integral, and that these integrals are countable extension of the LS_k-integral of Bhattacharyya and Das [1].

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