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PROPERTIES FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS*

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Abstract. Let H be the class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{+\infty} a_k z^k$, which are analytic in the unit disk $U = \{z; |z| < 1\}$. In this paper, we introduce two subclasses $B(\lambda, \alpha, \rho)$ and $C(\lambda, \alpha, \sigma)$ of H and study their some properties. The results obtained extend the related results of some authors. We also get some new univalent criterions.

1. Introduction. Let H be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k,$$

which are analytic in the unit disk $U = \{z; |z| < 1\}$. Let S denote the class of all functions in H which are univalent in the disk U.

Assume $\alpha > 0$, $\lambda \geq 0$, $\rho < 1$, $\sigma > 1$. A function $p(z) = 1+p_1 z+p_2 z^2+\cdots$ is said to be in the class P_ρ if and only if $p(z)$ is analytic in the unit disk U and satisfy $\mathrm{Re}p(z) > \rho$, $z \in U$; A function $f(z) \in H$ is said to be in the

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class $B(\lambda, \alpha, \rho)$ if and only if

$$\text{Re}\Big[(1-\lambda)(\frac{f(z)}{z})^{\alpha} + \lambda \frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}\Big] > \rho, \quad z \in U, \tag{1.1}$$

where the power are unstood as principle values. Below we apply this agreement. A function $f(z) \in H$ is said to be in the class $C(\lambda, \alpha, \sigma)$ if and only if

$$\text{Re}[(1-\lambda)(\frac{f(z)}{z})^{\alpha} + \lambda \frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] < \sigma, \quad z \in U. \tag{1.2}$$

It is obvious that the subclass $B(1, \alpha, 0)$ is the subclass of Bazilevič functions, which is the subclass of univalent functions S. The subclass $B(1, \alpha, \rho)(0 \leq \rho < 1)$ has been studied by Bazilevič [1], Singh [9], Owa [8], respectively. The subclass $B(\lambda, 1, \rho)(0 \leq \rho < 1)$ has been studied by Chichra [3], Ding, Ling and Bao [5], respectively. The subclass $B(0, 1, \rho)$ has been studied by Chen [4]. The subclass $B(0, \alpha, \rho)$ has been studied by Liu [6].

In this paper, we shall study the properties of $B(\lambda, \alpha, \rho)$ and $C(\lambda, \alpha, \sigma)$. The results obtained generalize the related works of some authors. We obtain some new univalent criterions too. In order to derive our main results, we need recall the following linear operator.

For the function $f(z) = \sum_{k=1}^{+\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{+\infty} b_k z^k$ which are analytic in U, let $(f * g)(z)$ denote the Hadamard product or convolution of $f(z)$ and $g(z)$, defined by

$$(f * g)(z) = \sum_{k=1}^{+\infty} a_k b_k z^k.$$

Now define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad (c \neq 0, -1, -2, \ldots, z \in U)$$

where

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & k=0, \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & k \in N \end{cases}$$

Carlson and Shaffer [2] defined a linear opertor $L(a,c)$ on H by using the Hadamard product.

$$L(a,c)f = \phi(a,c;z) * f(z), \quad f \in H.$$

It is known in [2] that $L(a,c)$ maps H into itself. If $a \neq 0, -1, -2, \ldots$, then $L(a,c)$ has a continuous inverse $L(c,a)$. Clearly, $L(a,a)$ is the unit operator. Moreover, if $c > a > 0$, then $L(a,c)$ has the integral representation

$$L(a,c)f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-2}(1-u)^{c-a-1} f(uz)du.$$

2. Some lemmas.

Lemma 1[4]. $z[\phi(c,c+1)]' = c\phi(c+1,c+1) - (c-1)\phi(c,c+1)$.

Lemma 2[4]. *Let $\rho < 1, 0 < u < 1, p(z) \in P_\rho$, then for $|z| = r < 1$, we have*

$$Re[p(z) - p(uz)] \geq \frac{2(1-\rho)(u-1)r}{(1+r)(1+ur)},$$

and the inequality is sharp.

Lemma 3. *Let $c > 0, \lambda > 0, \rho < 1$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in U and*

$$Re[p(z) + c\lambda z p'(z)] > \rho, \quad z \in U, \tag{2.1}$$

then for $|z| = r < 1$,*we have*

$$(2.2) \quad Re[p(z)+czp^{'}(z)] \geq 2\rho-1+\frac{2(1-\rho)}{\lambda(1+r)}+2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+ur}du,$$

and

$$(2.3) \quad Re[p(z)+czp^{'}(z)] \geq 2\rho-1+\frac{1-\rho}{\lambda}+2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+u}du,$$

and these results are sharp.

Proof. Set $F(z) = p(z)+c\lambda zp^{'}(z)$, then it follows from (2.1) that $F(z) \in P_\rho$ and

$$zF(z) = (1-c\lambda)[zp(z)] + c\lambda z[zp(z)]^{'} = L(\frac{1}{c\lambda}+1, \frac{1}{c\lambda})[zp(z)],$$

that is,

$$(2.4) \qquad zp(z) = L(\frac{1}{c\lambda}, \frac{1}{c\lambda}+1)[zF(z)] = \frac{1}{c\lambda}\int_0^1 u^{\frac{1}{c\lambda}-1}zF(uz)du.$$

Let $b = \frac{1}{c\lambda}$, then

$$p(z) = b\int_0^1 u^{b-1}F(uz)du,$$

According to Lemma 1 and (2.4), we get

$$\begin{aligned} z[zp(z)]' &= [z(\phi(b,b+1;z))^{'}] * [zF(z)] \\ &= bL(b+1,b+1)[zF(z)] - (b-1)L(b,b+1)[zF(z)] \\ &= bzF(z) - b(b-1)z\int_0^1 u^{b-1}F(uz)du \end{aligned}$$

On the other hand, we have

$$[zp(z)]^{'} = p(z) + zp^{'}(z),$$

Thus

$$(2.5)\quad p(z)+czp'(z)=(1-c)p(z)+c[zp(z)]'=bcF(z)+b(1-bc)\int_0^1 u^{b-1}F(uz)du,$$

Now we distinguish two cases.

(A) If $\lambda > 1$, then $0 < bc = \frac{1}{\lambda} < 1$ and

$$\begin{aligned}\mathrm{Re}[p(z)+czp'(z)] &= bc\mathrm{Re}[F(z)] + b(1-bc)\int_0^1 u^{b-1}\mathrm{Re}[F(uz)]du \\ &\geq bc\cdot\frac{1-(1-2\rho)r}{1+r} + b(1-bc)\int_0^1 u^{b-1}\frac{1-(1-2\rho)ur}{1+ur}du \\ &= 2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+ur}du.\end{aligned}$$

(B) If $0 < \lambda \leq 1$, then $bc = \frac{1}{\lambda} \geq 1$ and it follows from Lemma 2 and (2.5) that

$$\begin{aligned}\mathrm{Re}[p(z)+czp'(z)] &= \mathrm{Re}[bcF(z) - b(bc-1)\int_0^1 u^{b-1}F(uz)du] \\ &= \mathrm{Re}F(z) + b(bc-1)\int_0^1 u^{b-1}\mathrm{Re}[F(z)-F(uz)]du \\ &\geq \frac{1-(1-2\rho)r}{1+r} + b(bc-1)\int_0^1 u^{b-1}\frac{2(1-\rho)(u-1)r}{(1+r)(1+ur)}du \\ &= 2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+ur}du.\end{aligned}$$

Since the function

$$2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+ur}du,$$

is decreasing with respect to r, therefore

$$\mathrm{Re}[p(z)+czp'(z)] \geq 2\rho - 1 + \frac{1-\rho}{\lambda} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+u}du.$$

Note that

$$p_{\lambda,c,\rho}(z)=\frac{1}{c\lambda}\int_0^1 u^{\frac{1}{c\lambda}-1}\frac{1+(1-2\rho)uz}{1-uz}du, \tag{2.6}$$

satisfies (2.1), we obtain that the inequalities (2.2) and (2.3) are sharp.

Lemma 4. *Let* $\sigma>1, 0<u<1$. *If* $q(z)=1+q_1z+q_2z^2+\cdots$ *is analytic in* U *and*

$$Re q(z)<\sigma, \quad z\in U, \tag{2.7}$$

then for $|z|=r<1$, *we have*

$$Re[q(z)-q(uz)]\geq\frac{2(1-\sigma)(1-u)r}{(1-r)(1-ur)}, \tag{2.8}$$

and the inequality is sharp.

Proof. Let $z=re^{i\theta},\theta\in[0,2\pi)$, then

$$\begin{aligned}
\mathrm{Re}\left(\frac{1}{1-z}-\frac{1}{1-uz}\right) &= \mathrm{Re}\left(\frac{1}{1-r\cos\theta-ir\sin\theta}-\frac{1}{1-ur\cos\theta-iur\sin\theta}\right)\\
&= \frac{(1-u)[r\cos\theta-r^2(1+u)+ur^3\cos\theta]}{(1-2r\cos\theta+r^2)(1-2ur\cos\theta+u^2r^2)}\\
&\leq \frac{(1-u)[r\cos 0-r^2(1+u)+ur^3\cos 0]}{(1-2r\cos 0+r^2)(1-2ur\cos 0+u^2r^2)}\\
&= \frac{(1-u)r}{(1-r)(1-ur)},
\end{aligned}$$

According to (2.7), there exits a function $p(z)\in P_0$ such that

$$q(z)=(1-\sigma)p(z)+\sigma,$$

Since $\sigma > 1$, by the Herglotz formula

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x),$$

where $\mu(x)$ is a probability measure on $|x| = 1$. Thus we have

$$\begin{aligned} \mathrm{Re}[q(z) - q(uz)] &= (1-\sigma)\mathrm{Re}[p(z) - p(uz)] \\ &= 2(1-\sigma)\int_{|x|=1} \mathrm{Re}(\frac{1}{1-xz} - \frac{1}{1-xuz})d\mu(x) \\ &\geq \frac{2(1-\sigma)(1-u)r}{(1-r)(1-ur)}. \end{aligned}$$

Finally, the inequality (2.8) is sharp for the function $q_0(z) = \frac{1+(1-2\sigma)z}{1-z}$.

Lemma 5. *Let $c > 0, \lambda > 0, \sigma > 1$. If $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ be analytic in U and*

$$Re[q(z) + c\lambda z q'(z)] < \sigma, \quad z \in U, \tag{2.9}$$

then

$$Re[q(z)+czq'(z)] \geq 2\sigma - 1 + \frac{2(1-\sigma)}{\lambda(1+r)} + 2(1-\sigma)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1-ur}du, \tag{2.10}$$

where $r = |z| < 1$. The result is sharp.

Proof. Set $F(z) = q(z) + c\lambda z q'(z)$, then

$$zF(z) = (1-c\lambda)[zq(z)] + c\lambda z[zq(z)]' = L(\frac{1}{c\lambda}+1, \frac{1}{c\lambda})[zq(z)]$$

that is,

$$zq(z) = L(\frac{1}{c\lambda}, \frac{1}{c\lambda}+1)[zF(z)] = \frac{1}{c\lambda}\int_0^1 u^{\frac{1}{c\lambda}-1} zF(uz)du, \tag{2.11}$$

Let $b=\frac{1}{c\lambda}$, then

$$q(z)=b\int_0^1 u^{b-1}F(uz)du,$$

According to Lemma 1 and (2.11), we get

$$\begin{aligned} z[zq(z)]' &= [z((b,b+1;z))'] * [zF(z)] \\ &= bL(b+1,b+1)[zF(z)]-(b-1)L(b,b+1)[zF(z)] \\ &= bzF(z)-b(b-1)z\int_0^1 u^{b-1}F(uz)du \end{aligned}$$

On the other hand, we have

$$[zq(z)]' = q(z)+zq'(z),$$

Thus

$$q(z)+czq'(z)=(1-c)q(z)+c[zq(z)]'=bcF(z)+b(1-bc)\int_0^1 u^{b-1}F(uz)du, \tag{2.12}$$

From (2.9), there exits a function $F_1(z)\in P_0$ such that

$$F(z)=(1-\sigma)F_1(z)+\sigma,$$

Thus

$$\mathrm{Re}F(z)=(1-\sigma)\mathrm{Re}F_1(z)+\sigma\geq(1-\sigma)\frac{1+r}{1-r}+\sigma=\frac{1+(1-2\sigma)r}{1-r}.$$

Now we distinguish two cases.

(A) If $\lambda>1$, then $0<bc=\frac{1}{\lambda}<1$ and

$$\mathrm{Re}[q(z)+czq'(z)] = bc\mathrm{Re}[F(z)]+b(1-bc)\int_0^1 u^{b-1}\mathrm{Re}[F(uz)]du$$

$$\geq bc\cdot\frac{1+(1-2\sigma)r}{1-r}+b(1-bc)\int_0^1 u^{b-1}\frac{1+(1-2\sigma)ur}{1-ur}du$$

$$=2\sigma-1+\frac{2(1-\sigma)}{\lambda(1-r)}+2(1-\sigma)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1\frac{u^{\frac{1}{c\lambda}-1}}{1-ur}du.$$

(B) If $0<\lambda\leq 1$, then $bc=\frac{1}{\lambda}\geq 1$ and it follows from Lemma 4 and (2.12) that

$$\begin{aligned}\text{Re}[q(z)+czq^{'}(z)] &= \text{Re}[bcF(z)-b(bc-1)\int_0^1 u^{b-1}F(uz)du]\\ &= \text{Re}F(z)+b(bc-1)\int_0^1 u^{b-1}\text{Re}[F(z)-F(uz)]du\\ &\geq \frac{1+(1-2\sigma)r}{1-r}+b(bc-1)\int_0^1 u^{b-1}\frac{2(1-\sigma)(1-u)r}{(1-r)(1-ur)}du\\ &= 2\sigma-1+\frac{2(1-\sigma)}{\lambda(1-r)}+2(1-\sigma)(1-\frac{1}{\lambda})\frac{1}{c\lambda}\int_0^1\frac{u^{\frac{1}{c\lambda}-1}}{1-ur}du.\end{aligned}$$

Note that

$$q_{\lambda,c,\sigma}(z)=\frac{1}{c\lambda}\int_0^1 u^{\frac{1}{c\lambda}-1}\frac{1+(1-2\sigma)uz}{1-uz}du, \tag{2.13}$$

satisfies (2.9), we obtain the inequality (2.10) is sharp.

Lemma 6[1]. *Let $\alpha>0, f(z)\in H$ and for $|z|<R\leq 1$,*

$$Re[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}]>0,$$

then $f(z)$ is univalent in $|z|<R$.

3. Main results.

Theorem 1[7]. *Let $\lambda_2>\lambda_1\geq 0, 1>\rho_2\geq\rho_1$, then*

$$B(\lambda_2,\alpha,\rho_2)\subset B(\lambda_1,\alpha,\rho_1).$$

Corollary 1. *Let* $\lambda \geq 1, \rho \geq 0$, *then*

$$B(\lambda, \alpha, \rho) \subset B(1, \alpha, 0) \subset S.$$

Theorem 2. *Let* $\alpha > 0, \lambda > 0, \rho < 1$. *If* $f \in B(\lambda, \alpha, \rho)$, *then*

$$\text{(3.1)} \qquad \operatorname{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] \geq 2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du,$$

and

$$\text{(3.2)} \qquad \operatorname{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] \geq 2\rho - 1 + \frac{1-\rho}{\lambda} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du,$$

where for $r = |z| < 1$. *These results are sharp.*

Proof. Let $p(z) = [\frac{f(z)}{z}]^{\alpha}$ for $f \in B(\lambda, \alpha, \rho)$, then $p(z) = 1 + \alpha a_2 z + \cdots$ is analytic in U and

$$[f(z)]^{\alpha} = z^{\alpha} p(z),$$

By taking the derivatives in the both sides of the above equation, we obtain

$$(1-\lambda)(\frac{f(z)}{z})^{\alpha} + \lambda \frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha} = p(z) + \frac{\lambda}{\alpha} zp'(z),$$

Since $f \in B(\lambda, \alpha, \rho)$, we have

$$\operatorname{Re}[p(z) + \frac{\lambda}{\alpha} zp'(z)] > \rho, \quad z \in U,$$

According to Lemma 3, we obtain that

$$\begin{aligned} \operatorname{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] &= \operatorname{Re}[p(z) + \frac{1}{\alpha} zp'(z)] \\ &\geq 2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du. \end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] &= \mathrm{Re}[p(z)+\frac{1}{\alpha}zp^{'}(z)]\\ &\geq 2\rho-1+\frac{1-\rho}{\lambda}+2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du.\end{aligned}$$

Note that

$$f_{\lambda,\alpha,\rho}(z)=z[\frac{\alpha}{\lambda}\int_0^1 u^{\frac{\alpha}{\lambda}-1}\frac{1+(1-2\rho)uz}{1-uz}du]^{\frac{1}{\alpha}}\in B(\lambda,\alpha,\rho), \tag{3.3}$$

we obtain the inequalities (3.1) and (3.2) are sharp.

Remark 1. Setting $\alpha=1$ in Theorem 2, we get Theorem 1(ii) of [5].

From [6] we know that there exists $f(z)\in B(0,\alpha,\rho)$ such that $f(z)$ is not univalent in U. But in the following theorem, we shall prove that $f(z)\in B(\lambda,\alpha,\rho)$ is univalent in U for $0<\lambda<1$ and $\rho_0\leq\rho<1$. And we shall also prove that $f(z)\in B(\lambda,\alpha,\rho)$ is univalent in U for $\lambda>1$ and ρ is larger than some negative number.

Theorem 3. *Let $\alpha>0,\lambda>0,\rho_0\leq\rho<1$, then $B(\lambda,\alpha,\rho)\subset S$,where*

$$\rho_0=1-\frac{1}{2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du}, \tag{3.4}$$

and the constant ρ_0 can not replaced by any smaller one.

Proof. Let $f(z)\in B(\lambda,\alpha,\rho)$, then it follows from Theorem 2 that

$$\begin{aligned}\mathrm{Re}[\frac{zf^{'}(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] &> 2\rho-1+\frac{1-\rho}{\lambda}+2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du,\\ &= 1-(1-\rho)[2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du].\end{aligned}$$

Since

$$\frac{1}{2} < \frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du = \frac{1}{2} + \int_0^1 \frac{u^{\frac{\alpha}{\lambda}}}{(1+u)^2}du < 1,$$

so

$$2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du > 0,$$

Thus from $\rho_0 \le \rho < 1$, we have

$$\mathrm{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] > 1-(1-\rho_0)[2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du] = 0,$$

Therefore it follows from Lemma 6 that $f(z)$ is univalent in U or $f(z) \in S$. Hence

$$B(\lambda,\alpha,\rho) \subset S.$$

In order to show that the result is sharp. If $\rho < \rho_0$, we consider the function $f_{\lambda,\alpha,\rho}(z)$, which is defined by (3.3), we have

$$f'_{\lambda,\alpha,\rho}(z) = [\tfrac{\alpha}{\lambda}\textstyle\int_0^1 u^{\frac{\alpha}{\lambda}-1}\frac{1+(1-2\rho)uz}{1-uz}du]^{\frac{1}{\alpha}} +$$

$$\frac{z}{\lambda}[\frac{\alpha}{\lambda}\int_0^1 u^{\frac{\alpha}{\lambda}-1}\frac{1+(1-2\rho)uz}{1-uz}du]^{\frac{1}{\alpha}-1}\cdot\int_0^1 \frac{2(1-\rho)zu}{(1-zu)^2}u^{\frac{\alpha}{\lambda}-1}du,$$

$$f'_{\lambda,\alpha,\rho}(-1) = [\frac{\alpha}{\lambda}\int_0^1 u^{\frac{\alpha}{\lambda}-1}\frac{1-(1-2\rho)u}{1+u}du]^{\frac{1}{\alpha}-1}.$$

$$\cdot\{1-(1-\rho)[2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du]\} < 0,$$

and $f'_{\lambda,\alpha,\rho}(0) = 1 > 0$, therefore there exists a points $z = -r_0 \in (-1,0)$ such that $f'_{\lambda,\alpha,\rho}(-r_0) = 0$. Hence $f_{\lambda,\alpha,\rho}(z)$ is not univalent in U when $\rho < \rho_0$ as required.

Remark 2. It is easy to see that $\rho_0 < 0$ for $\lambda > 1$. Setting $\alpha = 1$ in Theorem 3, we get Theorem 2 of [5]; setting $\lambda = 1$ in Theorem 6, we get the result of [1].

Setting $\lambda = \alpha = \frac{1}{2}$ in Theorem 3, we have

Corollary 2. *If $f(z) \in H$ and*

$$Re[\sqrt{\frac{f(z)}{z}} + z(\sqrt{\frac{f(z)}{z}})'] > 1 - \frac{1}{2\ln 2} \approx 0.275, \quad z \in U,$$

then $f(z)$ is univalent in U. The result is sharp.

Setting $\alpha = 2, \lambda = \frac{1}{2}$ in Theorem 3, we have

Corollary 3. *If $f(z) \in H$ and*

$$Re[(\frac{f(z)}{z})^2 + \frac{f(z)f'(z)}{z}] > \frac{17 + 24\ln 2}{10 + 12\ln 2} \approx 1.836, \quad z \in U,$$

then $f(z)$ is univalent in U. The result is sharp.

Setting $\lambda = \alpha = 2$ in Theorem 3, we have

Corollary 4. *If $f(z) \in H$ and*

$$Re[\frac{2f(z)f'(z)}{z} - (\frac{f(z)}{z})^2] > \frac{1 - 2\ln 2}{3 - 2\ln 2} \approx -0.234, \quad z \in U,$$

then $f(z)$ is univalent in U. The result is sharp.

Theorem 4. *Let $\alpha > 0, \lambda > 0, \rho < \rho_0$. If $f(z) \in B(\lambda, \alpha, \rho)$, then $f(z)$ is univalent in $|z| < r_0$, where ρ_0 is defined by (3.4) and r_0 is the smallest positive root of the equation*

$$2\rho - 1 + \frac{2(1-\rho)}{\lambda(1+r)} + 2(1-\rho)(1 - \frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du = 0, \tag{3.5}$$

and the constant r_0 can not replaced by any larger one.

Proof: Since $f(z) \in B(\lambda, \alpha, \rho)$, it follows from Theorem 5 that for $|z| =$

$r<1$, we have

$$\mathrm{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}]\geq 2\rho-1+\frac{2(1-\rho)}{\lambda(1+r)}+2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du.$$

Let $h(r)=2\rho-1+\frac{2(1-\rho)}{\lambda(1+r)}+2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du$, then $h(r)$ is continuous in [0,1] and $h(0)=1>0$,

$$\begin{aligned} h(1) &= 2\rho-1+\frac{(1-\rho)}{\lambda}+2(1-\rho)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du \\ &< 1-(1-\rho_0)[2-\frac{1}{\lambda}-2(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1+ur}du]=0, \end{aligned}$$

therefore the equation (3.5) has at least one positive root in $(0,1)$, let r_0 denotes the smallest positive root of (3.5), then

$$\mathrm{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}]>0,$$

for $|z|<r_0$. Hence it follows from Lemma 6 that $f(z)$ is univalent in $|z|<r_0$ and the result is sharp.

Remark 3. Setting $\alpha=1$ in Theorem 4, we get Theorem 4 of [5].

Setting $\lambda=1$ in Theorem 4, we have

Corollary 5. *Let $\alpha>0,\rho<0$. If $f(z)\in B(1,\alpha,\rho)$, then $f(z)$ is univalent in $|z|<\frac{1}{1-2\rho}$ and the result is sharp.*

By applying the same method as in Theorem 2 of [7], we have

Theorem 5. *Let $\lambda_2>\lambda_1\geq 0,\sigma_1\geq\sigma_2>1$, then*

$$C(\lambda_2,\alpha,\sigma_2)\subset C(\lambda_1,\alpha,\sigma_1).$$

Theorem 6. *Let $\alpha>0,\lambda>0,\sigma>1$. If $f\in C(\lambda,\alpha,\sigma)$, then for*

$|z| = r < 1$, *we have*

$$(3.6)\quad Re[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] \geq 2\sigma-1+\frac{2(1-\sigma)}{\lambda(1-r)}+2(1-\sigma)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1-ur}du,$$

and the result is sharp.

Proof. Let $p(z) = [\frac{f(z)}{z}]^{\alpha}$ for $f \in C(\lambda, \alpha, \sigma)$, then $p(z) = 1 + \alpha a_2 z + \cdots$ is analytic in U and

$$[f(z)]^{\alpha} = z^{\alpha} p(z),$$

By taking the derivatives in the both sides of the above equation, we obtain

$$(1-\lambda)(\frac{f(z)}{z})^{\alpha} + \lambda \frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha} = p(z) + \frac{\lambda}{\alpha} zp'(z),$$

Since $f \in C(\lambda, \alpha, \sigma)$, we have

$$\text{Re}[p(z) + \frac{\lambda}{\alpha} zp'(z)] < \sigma, \quad z \in U,$$

According to Lemma 5, we obtain that

$$\begin{aligned} \text{Re}[\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\alpha}] &= \text{Re}[p(z) + \frac{1}{\alpha} zp'(z)] \\ &\geq 2\sigma-1+\frac{2(1-\sigma)}{\lambda(1-r)}+2(1-\sigma)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1-ur}du. \end{aligned}$$

Note that

$$f_{\lambda,\alpha,\sigma}(z) = z[\frac{\alpha}{\lambda}\int_0^1 u^{\frac{\alpha}{\lambda}-1}\frac{1+(1-2\sigma)uz}{1-uz}du]^{\frac{1}{\alpha}} \in C(\lambda, \alpha, \sigma),$$

we obtain that the inequality (3.6) is sharp.

By using Theorem 6 and Lemma 6 and the similar method as in Theorem 4, we have

Theorem 7. *Let $\alpha>0,\lambda>0,\sigma>1$. If $f(z)\in C(\lambda,\alpha,\sigma)$, then $f(z)$ is univalent in $|z|<r_1$,where r_1 is the smallest positive root of the equation*

$$2\sigma-1+\frac{2(1-\sigma)}{\lambda(1-r)}+2(1-\sigma)(1-\frac{1}{\lambda})\frac{\alpha}{\lambda}\int_0^1\frac{u^{\frac{\alpha}{\lambda}-1}}{1-ur}du=0,$$

and the constant r_1 can not replaced by any larger one.

Setting $\lambda=1$ in Theorem 9, we have

Corollary 6. *Let $\alpha>0,\sigma>1$. If $f(z)\in C(1,\alpha,\sigma)$, then $f(z)$ is univalent in $|z|<\frac{1}{2\sigma-1}$ and the result is sharp.*

Theorem 8[7]. *Let $f(z)=z+\sum_{k=2}^{+\infty}a_kz^k\in B(\lambda,\alpha,\rho)$, then*

$$|a_2|\le\frac{2-2\rho}{\lambda+\alpha}, \tag{3.7}$$

and the inequality is sharp, with the extremal function defined $f_{\lambda,\alpha,\rho}(z)$ by (3.3).

Theorem 9.(Covering Theorem) *Let $\alpha>0,\lambda>0,\rho_0\le\rho<1$, $f(z)\in B(\lambda,\alpha,\rho)$, then the unit disk U is mapped on a domain that contain the disk $|w|<r_1$, where ρ_0 defined by* (3.4) *and*

$$r_1=\frac{\alpha+\lambda}{2-2\rho+2(\alpha+\lambda)}. \tag{3.8}$$

Proof. Let w_0 be any complex number such that $f(z)\ne w_0$ for $z\in U$, then $w_0\ne0$ and

$$\frac{w_0f(z)}{w_0-f(z)}=z+(a_2+\frac{1}{w_0})z^2+\cdots,$$

is univalent in U by Theorem 3, so

$$|a_2+\frac{1}{w_0}|\le2,$$

Therefore according to Theorem 8, we obtain

$$|w_0| \geq \frac{\alpha+\lambda}{2-2\rho+2(\alpha+\lambda)} = r_1,$$

Hence we have completed the proof of the theorem.

Setting $\alpha = 1$ in Theorem 9, we have

Corollary 7.(Covering Theorem) *Let* $f(z) \in B(\lambda, 1, \rho)$ *with* $\lambda > 0, \rho_1 \leq \rho < 1$, *then the unit disk* U *is mapped on a domain that contain the disk* $|w| < \frac{1+\lambda}{4-2\rho+2\lambda}$, *where*

$$\rho_1 = 1 - \frac{1}{2 - \frac{1}{\lambda} - 2(1-\frac{1}{\lambda})\frac{1}{\lambda}\int_0^1 \frac{u^{\frac{1}{\lambda}-1}}{1+u}\mathrm{du}}.$$

Setting $\lambda = 1$ in Theorem 9, we have

Corollary 8.(Covering Theorem) *Let* $f(z) \in B(1, \alpha, \rho)$ *with* $0 \leq \rho < 1$, *then the unit disk* U *is mapped on a domain that contain the disk* $|w| < \frac{1+\alpha}{4-2\rho+2\alpha}$.

References

1. I. E. Bazilevic, *On a case of integrability in quadratures of the Loewner-Kuarev equation*, Mat. Sb., **37**(1955), 471-476.

2. B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15**(1984),737-745.

3. P. N. Chichra, *New subclass of the class of close-to-convex functions*, Proc. Amer. Math. Soc., **62**(1977), 37-43.

4. M. P. Chen, *On the regular functions satisfying* $Re[\frac{f(z)}{z}] > \alpha$, Bull. Inst. Math. Acad. Sinica, **3**(1975), 65-70.

5. S. S. Ding, Y. Ling and G. J.Bao, *Some properties of a class of analytic functions*, J. of Math. Anal. Appl., **195**(1995), 71-81.

6. M. S. Liu, *The Radius of univalence for certain class of analytic functions*, In:Lu Jianke and Wen Guochun eds, Boundary Value Problems, Integral Equations and Related Problems, Singapore: World Scientific Publishing, 2000, 122-128.

7. M. S. Liu, *On certain subclass of p-valent functions*, Soochow J. Math., **26**(2000), 163-171.

8. S. Owa and M. Obradovic, *Certain subclasses of Bazilevic functions of type* α, Internat. J. Math. and Math. Sci., **9**(1986), 97-105.

9. R. Singh, *On Bazilevic functions*, Proc. Amer. Math. Sci., **38**(1973), 261-267.

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