BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 30, Number 1, March 2002

DIPOLARIZATIONS AND COMPLETE SOLVABLE LIE ALGEBRAS*

BY

SHAOQIANG DENG(鄧少強) AND DAOJI MENG(孟道驥)

Abstract. In this paper, we find a method to construct dipolarizations in complex complete solvable Lie algebras. This sheds some light on the theory of representations of complete Lie algebras and those Lie groups with complete Lie algebras. In general, the dipolarizations we find are nonsymmetric.

Introduction. Let $\mathfrak{g}$ be a Lie algebra over field $\mathbb{F}$, [1] defines a *dipolarization* of $\mathfrak{g}$ as a triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$, where $\mathfrak{g}^\pm$ are subalgebras of $\mathfrak{g}$, f is a linear function on $\mathfrak{g}$, and the following conditions are satisfied:

(D1) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$,

(D2) $f([X, \mathfrak{g}]) = 0$ if and only if $X \in \mathfrak{g}^+ \cap \mathfrak{g}^-$,

(D3) $f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$.

A dipolarization is called *symmetric* if the two subalgebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are Lie-isomorphic to each other. Otherwise it is called *nonsymmetric*.

The notion of dipolarizations in Lie algebras is closely related to that of polarizations, which plays an important role in the theory of unitary representations of Lie groups(cf. [3]). In fact, let $\mathfrak{g}^\pm$ be two subalegbras of

Received by the editors September 22, 2000 and in revised form January 19, 2001.

AMS 1991 Subject Classification: 17B20, 17B05, 22E46, 15, 30.

Key words and phrases: Dipolarization, nonsymmetric, complete Lie algebra.

*Project 19901015, 19971044 supported by NSFC and the Research Fund for the Doctoral Program of Higher Education of China No. 97005511.

the Lie algebra $\mathfrak{g}$, and $f \in \mathfrak{g}^*$, then the triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization in $\mathfrak{g}$, if and only if $\mathfrak{g}^\pm$ are two polarizations in $\mathfrak{g}$ at f and $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ (cf. [7]). Thus dipolarizations offer a method to construct polarizations in Lie algebras. For fundamental properties of polarizations, one can refer to [3].

The background of the definition of dipolarizations is the geometry of homogeneous parakähler manifolds, i.e., homogeneous symplectic manifolds with two invariant transversal Lagrangian foliations (cf. ([1]). In [4] Kaneyuki obtained a remarkable class of symmetric dipolarizations in real semisimple Lie algebras by using gradations. In [5] the authors constructed an example of nonsymmetric dipolarization in the Lie algebra of upper triangular matrices, which is the first known nonsymmetric dipolarization. In [6], the authors constructed a large number of nonsymmetric dipolarizations in subalgebras of some real forms of complex semisimple Lie algebras, which can be viewed as a generalization of the example of [5]. Recently we proved that any dipolarization in a real or complex semisimple Lie algebra is symmetric ([7]). The most important applications of dipolarizations are to prove the fact that a connected compact homogeneous parakähler manifold is a torus ([2]) and that a homogeneous parakähler manifold of a semisimple Lie group is a covering space of a hyperbolic semisimple adjoint orbit ([7]).

In this paper we find a method to construct dipolarizations in complex complete solvable Lie algebras. In general, it is very difficult to construct polarizations in a Lie algebra. But we can find many new example of polarizations using our method. This sheds some light on the research of representations of complete Lie algebras and those Lie groups with complete Lie algebras, which should be the main subject of the research on complete Lie algebras after D. Meng, S. P. Wang's deep research on the construction of this kind of Lie agebras. On the other hand, this also gives some geometric meaning to those coset spaces of Lie groups with solvable complete Lie algebras, since using Kaneyuki's result, we can contruct a great deal of invariant parakähler structures on them. (cf. [1])

In general, the dipolarizations we find are nonsymmetric. Therefore, our method can be viewed as a promotion of the results of [5] and [6]. As an interesting problem, the necessary and sufficient conditions for a Lie algebra to have nonsymmetric dipolarizations is still unknown.

In the following, if $\mathfrak{g}$ is a Lie algebra, we use $C(\mathfrak{g})$ and $\mathrm{Der}\mathfrak{g}$ to denote the center and the set of derivations of $\mathfrak{g}$, respectively. The set of inner derivations of $\mathfrak{g}$ will be denoted by $\mathrm{ad}\mathfrak{g}$.

1. Root systems of complete solvable Lie alegbras. Let $\mathbb{F}$ be an algebraically closed field of charateristic 0 and $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. According to [8], $\mathfrak{g}$ is called complete if $C(\mathfrak{g}) = 0$ and $\mathrm{Der}\mathfrak{g}= \mathrm{ad}\mathfrak{g}$. For fundamental properties of complete Lie algebras, one should refer to [9].

Now suppose that $\mathfrak{g}$ is complete and solvable. Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{t} + \mathfrak{n}$, where $\mathfrak{n}$ is the nil radical of $\mathfrak{g}$ (cf. [10]. Theorem 1). Therefore $\mathfrak{n}$ can be decomposed into the direct sum of the root spaces with respect to $\mathrm{ad}\mathfrak{t}$:

$$\mathfrak{n} = \mathfrak{n}_0 + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{n}_0 = \{X \in \mathfrak{n}|[X, \mathfrak{t}] = 0\}$, $\mathfrak{g}_\alpha = \{X \in \mathfrak{n}|[T, X] = \alpha(T)X, \forall T \in \mathfrak{t}\}$ and $\Delta = \{\alpha \in \mathfrak{t}^* - \{0\}|\mathfrak{g}_\alpha \neq 0\}$. We call Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Denote

$$\mathfrak{n}_1 = \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

define

$$\prod = \{\alpha \in \Delta|\alpha - \beta \notin \Delta, \qquad \forall\beta \in \Delta\},$$

i.e., $\prod$ is the subset of Δ consisting of the elements in Δ which can not be decomposed into the sum of two elements of Δ.

Lemma 1.1. $\Delta \cap -\Delta = \emptyset$, $\prod \neq \emptyset$. *Furthermore, every root in* Δ *can be written as the sum of some roots in* $\prod$.

Proof. (cf. [10]) For the first assertion we use the Lie's Theorem. Since the adjoint representation of $\mathfrak{g}$ is faithful, we can select a basis $X_1, X_2, \ldots$, X_m of $\mathfrak{g}$ such that $\forall X \in \mathfrak{g}$, the matrix of ad$X$ under this basis is upper triangular. Suppose $\mathrm{ad}\, X = (\mathrm{ad}\, X)_s + (\mathrm{ad}\, X)_n$ is the Jordan decomposition of ad$X$. Then it is well known $(\mathrm{ad}\, X)_s \in Der\mathfrak{g}$ and $(\mathrm{ad}\, X)_n \in Der\mathfrak{g}$. By completeness of $\mathfrak{g}$ there exist unique X_s,$X_n \in \mathfrak{g}$ such that

$$\mathrm{ad}\, X_g = (\mathrm{ad}\, X)_g; \qquad \mathrm{ad}\, X_n = (\mathrm{ad}\, X)_n; \qquad X = X_g + X_n,$$

and

$$\mathfrak{g} = \{X_g | X \in \mathfrak{g}\} + \{X_n | X \in \mathfrak{g}\}.$$

Set $\mathfrak{t}_1 = \{X_s | X \in \mathfrak{g}\}$. Then $\mathfrak{t}_1$ is a maximal toral subalgebra in $\mathfrak{g}$ and $\forall X \in \mathfrak{t}_1$, $\mathrm{ad}\, X$ is diagonal under the basis $X_1, X_2, \ldots, X_m$, $\forall Y \in \{X_n | X \in \mathfrak{g}\}$, adY is strictly upper triangular. Now the first assertion follows, because the maximal toral subalgebras in $\mathfrak{g}$ are isomorphic to each other under the group of inner automorphisms of $\mathfrak{g}$. Other assertions are the direct corollaries of the first assertion.

Since $C(\mathfrak{g}) = 0$, we easily see that Δ generates $\mathfrak{t}^*$. By Lemma 1.1, we can select a subset $\prod_0$ of $\prod$ such that the elements of $\prod_0$ are linearly independent and that $\prod_0$ generates $\mathfrak{t}^*$. Then the number of the elements of $\prod_0$ equals $N = \dim \mathfrak{t}$.

Now we suppose further that $\mathfrak{g}$ is simply complete and dim$\mathfrak{g} \geq 3$. Since $\mathfrak{n}_0$ is nilpotent, we have $[\mathfrak{n}_0, \mathfrak{g}_\alpha] \neq \mathfrak{g}_\alpha$, $\forall \alpha \in \Delta$. Thus for $\beta \in \prod_0$, we can select a basis of $\mathfrak{g}_\beta, Y_\beta^1, Y_\beta^2, \ldots, Y_\beta^{N_\beta}$ such that $[\mathfrak{n}_0, \mathfrak{g}_\beta]$ is generated by $Y_\beta^{m_\beta}, Y_\beta^{m_\beta+1}, \ldots, Y_\beta^{N_\beta}$, where $m_\beta \geq 2$ and $N_\beta = \dim \mathfrak{g}_\beta$.

Lemma 1.2. $\mathfrak{n}_1 \neq \sum_{\beta \in \Pi_0} \mathbb{F} Y_\beta^1$.

Proof. Suppose that $\mathfrak{n}_1 = \sum_{\beta\in\Pi_0} \mathbb{F}Y_\beta^1$. Then we have $\mathfrak{g}_\beta = \mathbb{F}Y_\beta^1$, $\forall\beta \in \prod_0$. Thus $[\mathfrak{n}_0, \mathfrak{g}_\beta] = 0$, $\forall\beta \in \prod_0$, i.e.,$[\mathfrak{n}_0, \mathfrak{n}_1] = 0$. There are only two cases:

Case 1: $\mathfrak{n}_0 \neq 0$. In this case by $[\mathfrak{t}, \mathfrak{n}_0] = 0$ and $[\mathfrak{n}_0, \mathfrak{n}_1] = 0$, we have $C(\mathfrak{n}_0) \subset C(\mathfrak{g})$. Since $\mathfrak{n}_0$ is nilpotent, $C(\mathfrak{n}_0) \neq 0$. This contradicts the fact $C(\mathfrak{g}) = 0$.

Case 2: $\mathfrak{n}_0 = 0$. Then $\mathfrak{g} = \mathfrak{t} + \sum_{\beta\in\prod_0} \mathbb{F}Y_\beta^1$, and $[Y_\gamma^1, Y_\eta^1] = 0$, for $\gamma \neq \eta$. It is easily seen that $\mathfrak{g}$ can be decomposed into the sum of $N = \dim \mathfrak{t}$ ideals such that each of them is isomorphic to the Borel subalgebra of $\mathfrak{sl}\ (2, \mathbb{F})$. This is also a contradiction.

2. Nonsymmetric dipolarizations in complete solvable Lie algebra. In this section we suppose that $\mathfrak{g}$ is a complex simply complete solvable Lie algebra. As in §1, we select a maximal torus $\mathfrak{t}$ of $\mathfrak{g}$. Let $\Delta, \prod, \prod_0$, and $\{Y_\beta^1, Y_\beta^2, \ldots, Y_\beta^{N_\beta}\}$, $\beta \in \prod_0$ be as in §1. Let

$$\mathfrak{n}_2 = \sum_{\beta\in\Pi_0} (\sum_{i=2}^{N_\beta} \mathbb{C}Y_\beta^i) + \sum_{\alpha\in\Delta-\Pi_0} \mathfrak{g}_\alpha.$$

We define

$$\mathfrak{g}^+ = \mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_1,$$

$$\mathfrak{g}^- = \mathfrak{t} + \mathfrak{n}_0 + \mathfrak{n}_2.$$

Lemma 2.1. $\mathfrak{g}^\pm$ *are subalgebras of* $\mathfrak{g}$.

Proof. It is obvious that $\mathfrak{g}^+$ is a subalgebra of $\mathfrak{g}$. By the definitions of $\prod_0$ and $Y_\beta^j, \beta \in \prod_0$, $1 \leq j \leq N_\beta$, we have

$$[\mathfrak{n}_0, \mathfrak{n}_2] \subset \mathfrak{n}_2, \qquad [\mathfrak{n}_2, \mathfrak{n}_2] \subset \mathfrak{n}_2.$$

Thus $\mathfrak{g}^-$ is a subalgebra of $\mathfrak{g}$.

Now choose a Hermitian inner product $<,>$ in $\mathfrak{g}$ such that the sum

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{n}_0 + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

is orthogonal and for $\beta \in \prod_0$, $Y_\beta^1, y_\beta^2, \ldots, Y_\beta^{N_\beta}$ form an orthogonal basis of $\mathfrak{g}_\beta$. Choose a basis of $\mathfrak{t}$, $\{T_\beta|\beta \in \prod_0\}$ such that $\gamma(T_\beta) = \delta_{\beta\gamma}$, $\forall\beta$, $\gamma \in \prod_0$. Define a linear function f in $\mathfrak{g}$ by

$$f(Y) = \sum_{\beta\in\Pi_0} (< Y, T_\beta > + < Y, Y_\beta^1 >), Y \in \mathfrak{g}.$$

Lemma 2.2. *$f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$.*

Proof. Since

$$[\mathfrak{n}_0, \mathfrak{n}_1] \subset \mathfrak{n}_2,$$

$$[\mathfrak{n}_0, \mathfrak{n}_2] \subset \mathfrak{n}_2.$$

we have

$$[\mathfrak{g}^+, \mathfrak{g}^+] \subset \mathfrak{n}_0 + \mathfrak{n}_2,$$

$$[\mathfrak{g}^-, \mathfrak{g}^-] \subset \mathfrak{n}_0 + \mathfrak{n}_2.$$

Since $f(\mathfrak{n}_0 + \mathfrak{n}_2) = 0$, the lemma follows.

Lemma 2.3. *For $X \in \mathfrak{g}$, $f([X, \mathfrak{g}]) = 0$ if and only if $X \in \mathfrak{g}^+ \cap \mathfrak{g}^-$.*

Proof. Suppoe first that $X \in \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $X \in \mathfrak{n}_0 + \mathfrak{n}_2$, thus $[X, \mathfrak{g}] \subset \mathfrak{n}_0 + \mathfrak{n}_2$. Therefore $f([X, \mathfrak{g}]) = 0$. On the other hand, let $f([X, \mathfrak{g}]) = 0$. Suppose

$$X = X_t + X_0 + \sum_{\beta\in\Pi_0} c_\beta Y_\beta^1 + X_2,$$

where $X_{\mathfrak{t}} \in \mathfrak{t}, X_0 \in \mathfrak{n}_0, X_2 \in \mathfrak{n}_2$ and $c_\beta \in \mathbb{C}$. We first assert that $X_{\mathfrak{t}} = 0$. Suppose this is not true. Then we can choose $\eta \in \prod_0$ such that $\eta(X_{\mathfrak{t}}) \neq 0$. Then

$$[X, Y_\eta] = \eta(X_{\mathfrak{t}})Y_\eta + Z$$

where $Z \in \mathfrak{n}_0 + \mathfrak{n}_2$. Thus

$$f([X, Y_\eta]) = \eta(X_{\mathfrak{t}}) < Y_\eta, Y_\eta > \neq 0.$$

This is a contradiction. Now for $\beta \in \prod_0$, we have

$$[T_\beta, X] = c_\beta Y_\beta^1 + [T_\beta, X_2].$$

Since $[T_\beta, X_2] \in \mathfrak{n}_2$, we have $f([T_\beta, X_2]) = 0$. Thus $0 = f([T_\beta, X]) = c_\beta$. Thus $X \in \mathfrak{n}_0 + \mathfrak{n}_2$. Now the lemma follows.

Now we can prove the main result of the paper.

Theorem 1. *Let $\mathfrak{g}^+, \mathfrak{g}^-$ and f be as above, then $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolariation in $\mathfrak{g}$. Futhermore, if $\dim\mathfrak{g} \geq 3$, then the dipolarization is nonsymmetric.*

Proof. By Lemma 2.1, 2.2, 2.3 and the obvious fact $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ we see that $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization in $\mathfrak{g}$. Now suppose $\dim\mathfrak{g} \geq 3$. Then Lemma 1.2 shows that $\mathfrak{n}_2 \neq 0$. Thus $\mathfrak{g}^-$ is not nilpotent. Thereforce it is not isomorphic to $\mathfrak{g}^+$, which is a nilpotent Lie algebra.

Corollary 1. *Let $\mathfrak{g}^+, \mathfrak{g}^-$, and f be as above, then $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are two polarizations in $\mathfrak{g}$ at f.*

Theorem 2. *Let $\mathfrak{u}$ be a complex complete solvable Lie algebra. Suppose $\mathfrak{u}$ contains a simply complete ideal which is not isomophic to the Borel subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. Then there exists nonsymmetric dipolarization in $\mathfrak{u}$.*

Proof. Suppose $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \ldots \oplus \mathfrak{u}_s$ is the decomposition of $\mathfrak{u}$ into the sum of its simply complete ideals ([9]). For convienence we suppose $\mathfrak{u}_1$ is not isomorphic to the Borel subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. Then Theorem 1 shows that there exists nonsymmetric dipolarization $\{\mathfrak{u}_1^+, \mathfrak{u}_1^-, g\}$ in $\mathfrak{u}_1$. Let $\mathfrak{u}^\pm = \mathfrak{u}_1^\pm \oplus \mathfrak{u}_2 \oplus \ldots \oplus \mathfrak{u}_s$ and extend g to $\mathfrak{u}$ by $g(Y_1 + Y_2 + \ldots + Y_s) = g(Y_1)$, where $Y_j \in \mathfrak{u}_j (j = 1, 2, \ldots, s)$. Then it is obvious that $\{\mathfrak{u}^+, \mathfrak{u}^-, g\}$ is a nonsymmetric dipolarization in $\mathfrak{u}$.

References

1. S. Kaneyuki, *Homogeneous symplectic manifolds and dipolarizations in Lie algebra*, Tokyo J. Math., **15**(1992), 313-325.

2. Z. Hou, S. Deng and S. Kaneyuki, *Dipolarizations in compact Lie algebras and homogeneous parakähler manifolds*, Tokyo J. Math., **20**(1997), 381-388.

3. J. Dixmier, Algébres Enveloppants, Gauthier-Villars, Paris, 1974.

4. S. Kaneyuki, *On a remarkable class of homogeneous symplectic manifolds.* Proc. Japan Acad., **67, A**(1991), 128-131.

5. S. Deng and S. Kaneyuki, *An example of nonsymmetric dipolarizations in a Lie algebra*, Tokyo J. Math., **16**(1993).

6. D. Meng, S. Deng and S. Kanyuki, *A remarkable class of nonsymmetric dipolarization in Lie algebras*, Yokohama Math. J., **43**(1995), 117-123.

7. Z. Hou, S. Deng, S. Kaneyuki and K. Nishiyama, *Dipolarizations in semisimple Lie algebras and homegeneous parakähler manifolds*, Journal of Lie Theory, **9**(1999), 215-232.

8. N. Jacobson, Lie Algebras, Wiley (Interscience), New York, 1962.

9. D. J. Meng, *Some results on complete Lie algebras*, communications in algebra, **22**(1994), 5457-5507.

10. D. J. Meng, and L. S. Zhu, *Solvable Complete Lie algebras I*, communications in algebra, **24**(1996), 4181-4197.

Department of Mathematics, Nankai Univesity, Tianjin 300071, P. R. China.