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A NEW UPPER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH INFINITE VARIANCE

BY

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Abstract. Let $N_n(\omega)$ be the number of real roots of the algebraic equation $\sum_{v=0}^{n} \xi_v(\omega)x^v = 0$ where $\xi_v(\omega)$'s are random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$, C being a positive constant and $1 \le \alpha \le 2$. It is shown here that

$$Pr\left\{ \sup_{n>n_0} \frac{N_n(\omega)}{\Lambda_n} < 1 \right\} > 1 - \mu/n_0^{(5\alpha/2)-2-\gamma}$$

for all n_0 sufficiently large where

$$\Lambda_n = \frac{p}{\log 2}\left\{\frac{17}{2} + \frac{2}{\beta}\right\} \quad (\log n - \log\log n)\, \log n$$

and $0 < \gamma < 1/2$ with $\beta > 4$, μ being a positive constant. The result is also shown to be ture for the case when the coefficients are non-identically distributed in a prescribed manner.

1. Introduction. Let $N_n(\omega)$ be the number of real roots of the alge-

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braic equation

$$\sum_{v=0}^{n} \xi_v(\omega)x^v = 0$$

where $\xi_0(\omega), \xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$ are independent random variables assuming **real values** only.

Littlewood and Offord [1] have estimated that $N_n(\omega) < 25(\log n)^2$ outside an exceptional set E where $P_r(E) < (12\log n)/n$, n being sufficiently large. They suppose that the coefficients $\xi_v(\omega)$'s are either normally distributed or uniformly distributed in $(-1, +1)$, or assume only the value -1 and $+1$ with equal probabilities.

Samal [2] has considered the general case that the ξ-coefficients have identical distribution with expectation zero, the variance and the third absolute moment being finite and non-zero. He has shown (pp.436-437) that for a positive constant μ,

$$N_n(\omega) < \frac{(p\log n + 2)\log(\mu n^3)}{\log 2} + \frac{\log(\mu n^3)}{\log 2(1-\rho)},$$

where p is any number greater than $1/\log 2$ and $0 < \rho < 1/2$. As n tends to infinity, this bound is asymptotic to $3p(\log n)^2/\log 2$, so that Samal has proved that

$$N_n(\omega) < \frac{3p}{\log 2}(\log n)^2 \tag{1.1}$$

outside an exceptional set of small measure.

Evans [3] obtained for the case of normally distributed coefficients that for each n_0 sufficiently large and $n > n_0$,

$$N_n(\omega) \leq \alpha(\log\log n)^2 \log n; \qquad \alpha \text{ being a positive constant},$$

outside a set of measure at most $A/(\log n_0 \log\log\log n_0)$ where A is a positive constant.

Logan and Shepp [4] have considered the case when the coefficients have a common characteristic function $\exp(-|z|^\alpha)$, $0 < \alpha < 2$, and they have obtained that $EN_n(\omega)$, the expected value of $N_n(\omega)$, is asymptotically equal to $c_\alpha \log n$, c_α being constant. Using this we find that $N_n(\omega) < \mu(\log n)^2$ outside an exceptional set of measure $O(1/\log n)$ as n tends to infinity.

Samal and Mishra [5] have considered the case when $\xi_v(\omega)$'s are independent random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$ where C is a positive constant and $1 \leq \alpha \leq 2$. For $1 < \alpha < 2$, this represents a symmetric stable distribution with infinite variance, while for $\alpha = 1$ this gives Cauchy distribution whose variance does not exist. They have shown that, for each $n > n_0$, $N_n(\omega) < \mu(\log n)^2$ except for a set of measure at most $\mu'/n_0^{3\alpha-2-\beta}$, $0 < \beta < 1$, μ and μ' being positive constants.

Samal and Pratihari [7] have improved the upper bound (1.1) obtained by Samal. They have shown that

$$N_n(\omega) < \frac{p}{\log 2}\Big(\frac{5}{2} + \frac{2}{\beta}\Big)\ (\log n - \log\log n)\log n$$

where $p > (1/\log 2)$ and $\beta > 4$, outside an exceptional set of small measure. In another paper, Samal and Mishra [6] extend their results to the case of non-identically distributed coefficients.

Our aim is to improve the upper bound obtained by Samal and Mishra [5] and [6]. We show that, for n_0 sufficiently large,

$$Pr\left\{\sup_{n>n_0} \frac{N_n(\omega)}{\Lambda_n} < 1\right\} > 1 - \frac{\mu}{n_0^{(5\alpha/2)-2-\gamma}}; \qquad 0 < \gamma < 1/2$$

where

$$\Lambda_n = \frac{p}{\log 2}\Big(\frac{17}{2} + \frac{2}{\beta}\Big)(\log n - \log\log n)\log n$$

with $p > (1/\log 2)$ and $\beta > 4$.

Samal and Mishra [5] have shown that precisely

$$N_n(\omega) < \frac{(p\log n + 2)\log(e^2 n^9)}{\log 2} + \frac{\log(\mu n^9)}{\log 2}; \qquad \mu \quad \text{being constant.}$$

As n tends to infinity, this bound is asymptotic to $9p(\log n)^2/\log 2$ which is larger than Λ_n for $\beta > 4$. Thus our upper bound is an improved one. Also we have shown that $N_n(\omega)$ has the very same bound even if the coefficients are non-identically distributed as in case of Samal and Mishra [6].

Bharucha-Reid and Sambandham [8] have given an extensive survey of the field of study of random polynomials.

2. Theorem 1. *Let* $f_n(x) = \sum_{v=0}^{n} \xi_v(\omega) x^v$ *be a polynomial of degree* n *whose coefficients* $\xi_v(\omega)$*'s are independent real random variables identically distributed with a common characteristic function* $\exp(-C|t|^\alpha)$*, when* C *is a positive constant and* $1 \leq \alpha \leq 2$*. Then there exists an integer* n_0 *large enough such that for* $n > n_0$ *the number of real roots,* $N_n(\omega)$*, of most of the equations* $f_n(x) = 0$ *does not exceed.*

$$\Lambda_n = \frac{p}{\log 2}\left(\frac{17}{2} + \frac{2}{\beta}\right)(\log n - \log\log n)\log n$$

where $p > (1/\log 2)$ *and* $\beta > 4$*. The measure of the exceptional set is at most*

$$\frac{\mu}{n_0^{(5\alpha/2)-2-\gamma}}; \qquad 0 < \gamma < 1/2, \ \mu \text{ being a positive constant.}$$

The above theorem states that

$$Pr\left\{\sup_{n>n_0} \frac{N_n(\omega)}{\Lambda_n} < 1\right\} > 1 - n_0^{(5\alpha/2)-2-\gamma}.$$

Proof. In our proof positive constants are denoted by μ's. We shall suppose that our inequalities are satisfied when n is large. To avoid duplication, we shall indicate only the modifications necessary in the proof of the Theorem 1 of Samal and Mishra [5].

First of all we change the system of circles to cover the closed interval $[1/2, 1]$. As in Samal and Pratihari [7], we choose a fixed real number p greater than $(1/\log 2)$ and let k be given by formula $k = [p \log M]$ where $M = \beta n/\log n$, $\beta > 0$ and $[x]$ denotes the greatest integer not exceeding x.

We consider $k+1$ circles $C_m (m = 1, 2, \ldots, k, p \log M)$, C_m having its centre at $x_m = 1-(1/2)^m$ and radius $r_m = (1/2)(1-x_m) = (1/2)^{m+1}$. In addition, we also consider the circle C_0 with its centre at $x_0 = 1$ and radius $r_0 = 1/M$. It has been shown in [7] that the circles $C_0, C_1, C_2, \ldots, C_k, C_{p \log M}$ cover the closed segment $[1/2, 1]$.

Next, for every m, let Γ_m represent the circle concentric with C_m and with radius $2r_m$. All Γ-circles are interior to the circle $z = 1 + 2/M$.

In the Section 5 of [5], we have

$$Pr\{|\xi_v(\omega)| \geq (n+1)^{5/2} \log n\} \leq \mu\{(n+1)^{(5\alpha/2)} \cdot (\log n)^\alpha\}.$$

Hence

$$Pr\Big\{|\xi_v(\omega)| < (n+1)^{5/2} \log n;\ 0 \leq \nu \leq n\Big\} > 1 - \mu/\Big\{(n+1)^{(5\alpha/2)-1} \cdot (\log n)^\alpha\Big\}.$$

So, outside a set of measure at most

$$\mu/\Big\{(n+1)^{(5\alpha/2)-1} \cdot (\log n)^\alpha\Big\}, \tag{2.1}$$

$$\begin{aligned} \max_{|z| \leq 1+2/M} |f_n(z)| &\leq \max_{|z| \leq 1+2/M} \sum_{\nu=0}^{n} |\xi_\nu(\omega)||z|^\nu \\ &\leq (n+1)^{\frac{7}{2}} \log n \left(1 + \frac{2 \log n}{\beta n}\right)^n \\ &< 2^{\frac{7}{2}} n^{(7/2+2/\beta)} \log n. \end{aligned} \tag{2.2}$$

Since the characteristic function of $f_n(x_m)$ is $\exp\{-C|t|^\alpha \sum_{\nu=0}^{n} x_m^{\alpha\nu}\}$, we have

$$\text{(2.3)} \qquad Pr\{|f_n(x_m)| < 1/n^5\} \leq \frac{2\Gamma(1/\alpha)}{\pi\alpha C^{1/\alpha} n^5}\Big\{\sum_{\nu=0}^{n} x_m^{\alpha v}\Big\}^{-1/\alpha}$$

$$= \Big(\frac{\mu}{n^5}\Big)\left\{\sum_{v=0}^{n}\Big(1-\frac{1}{2^m}\Big)^{\alpha\nu}\right\}^{-1/\alpha} \qquad \text{for } m = 1, 2, \ldots, k, p \log M$$

and

$$\text{(2.4)} \qquad Pr\{|f_n(x_0)| < 1/n^5\} \leq \frac{2\Gamma(1/\alpha)}{\pi\alpha C^{1/\alpha} n^5}\Big\{\sum_{\nu=0}^{n} x_0^{\alpha\nu}\Big\}^{-1/\alpha} = \mu/n^{5+(1/\alpha)}.$$

It follows from (2.1)-(2.4) that, outside a set of measure at most

$$\frac{\mu}{(n+1)^{(5\alpha/2)-1}(\log n)^\alpha} + \frac{\mu}{n^5}\left\{\sum_{v=0}^{n}\Big(1-\frac{1}{2^m}\Big)^{\alpha\nu}\right\}^{-1/\alpha} \qquad \text{for } m > 0$$

and

$$\frac{\mu}{(n+1)^{(5\alpha/2)-1}(\log n)^\alpha} + \frac{\mu}{n^{5+(1/\alpha)}} \qquad \text{for } m = 0,$$

the number of zeros of $f_n(z)$ in C_m is at most

$$\text{(2.5)} \qquad \frac{1}{\log 2}\left\{\log\left(\frac{2^{7/2} n^{(7/2+2/\beta)} \log n}{1/n^5}\right)\right\}.$$

Considering the $k+2$ circles $C_0, C_1, C_2, \ldots, C_k, C_{p \log M}$, we obtain that the number of zeros inside all these circles does not exceed

$$\text{(2.6)} \qquad \frac{(k+2)}{(\log 2)}\left\{\log(2^{1/7} n^{(17/2+2/\beta)} logn)\right\}.$$

Finally, turning to the segment $[0, 1/2]$, we find as in [5] that the number of zeros inside the circle $|z| \leq 1/2$ dose not exceed

$$\text{(2.7)} \qquad \{\log(\mu n^9)\}/(\log 2),$$

outside an exceptional set of measure at most

$$\mu/n^5. \tag{2.8}$$

In view of (2.5)-(2.8), we have that

$$\begin{aligned} N_n(\omega) &< \frac{(k+2)}{\log 2}\Big\{\log(2^{1/7}n^{(17/2+2/\beta)}\log n)\Big\}+\frac{\log(\mu n^9)}{log2} \\ &\sim (p/\log 2)(17/2+2/\beta)(\log n-\log\log n)\log n, \end{aligned} \tag{2.9}$$

where β can be as large as we please, out side an exceptional set of measure at most

$$\frac{\mu}{(n+1)^{(5\alpha/2)-1}(\log n)^{\alpha}}+\frac{\mu}{n^{5+1/\alpha}}+\frac{\mu}{n^5}\sum_{m=1}^{p\log M}\left\{\sum_{\nu=0}^{n}\Big(1-\frac{1}{2^m}\Big)^{\alpha\nu}\right\}^{-1/\alpha}. \tag{2.10}$$

Now

$$\begin{aligned} \sum_{m=1}^{p\log M}\left\{\sum_{\nu=0}^{n}\Big(1-\frac{1}{2^m}\Big)^{\alpha\nu}\right\}^{-1/\alpha} &< \sum_{m=1}^{p\log M}\left\{\sum_{\nu=0}^{n}\Big(1-\frac{1}{2}\Big)^{\alpha\nu}\right\}^{-1/\alpha} \\ &= \sum_{m=0}^{p\log M}\left\{\frac{1-\frac{1}{2^{\alpha}}}{1-\frac{1}{2^{n}}}\right\}^{\frac{1}{\alpha}} < \sum_{m=0}^{p\log M}1=p\log M \\ &= p\{\log\beta+\log n-\log\log n\}<\mu\log n. \end{aligned} \tag{2.11}$$

Hence, the exceptional set (2.10) is of measure at most

$$\mu/n^{(5\alpha/2)-1-\gamma}$$

where $0<\gamma<1/2$.

Thus, for all $n>n_0$, we have

$$N_n(\omega)<(p/\log 2)\Big(\frac{17}{2}+\frac{2}{\beta}\Big)(\log n-\log\log n)\log n$$

outside an exceptional set of measure at most

$$\sum_{n=n_0+1}^{\infty} \mu/n^{(5\alpha/2)-1-\gamma} < \mu'/n_0^{(5\alpha/2)-2-\gamma}. \tag{2.12}$$

Hence the proof of the theorem.

3. Theorem 2. *Let $f_n(x) = \sum_{\nu=0}^{n} a_\nu \xi_\nu(\omega) x^\nu$ be a polynomial of degree n in which $\xi_v(\omega)$'s are independent random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$, C being a positive constant and $1 \le \alpha \le 2$, and $a_0, a_1, \ldots, a_n$ are non-zero real mumbers such that $k_n^\alpha = O(n^\delta/\log n)$, $0 < \delta < \gamma < 1/2$, where $k_n = \max_\nu |a_\nu|$, $t_n = \min_\nu |a_\nu|$. Then there exists an interger n_0 such that for each $n \ge n_0$, the number of real roots of the equation $f_n(x) = 0$ is at most $\Lambda_n = (p/\log 2)(17/2+2/\beta)(\log n - \log\log n)\log n$ where $p > 1/\log 2$ and $\beta > 4$. The measure of the exceptional set is at most $\mu/n_0^{(5\alpha/2)-2-\gamma}$; $0 < \gamma < 1/2$, μ being a constant.*

The above theorem means that

$$Pr\Big\{\sup_{n>n_0} (N_n(\omega)/\Lambda_n) < 1\Big\} > 1 - \mu/n_0^{(5\alpha/2)-2-\gamma}$$

for n_0 sufficiently large.

Proof. We have

$$Pr\{|a_\nu \xi_\nu(\omega)| \ge (n+1)^{5/2} \cdot \log n\} \le \mu |a_v|^\alpha / \{(n+1)^{5\alpha/2} \cdot (\log n)^\alpha\}.$$

Hence

$$\begin{aligned} & Pr\Big\{|a_\nu \xi_\nu(\omega)| < (n+1)^{5/2} \log n;\ 0 \le \nu \le n\Big\} \\ > \;& 1 - \mu k_n^\alpha / \Big\{(n+1)^{(5\alpha/2)-1} \cdot (\log n)^\alpha\Big\}. \end{aligned}$$

Since the characteristic function of $f_n(x_m)$ is $\exp(-C|t|^\alpha \sum_{\nu=0}^{n} |a_\nu|^\alpha x_m^{\alpha\nu})$, we have

$$Pr\Big\{|f_n(x_m)| < 1/n^5\Big\} < \frac{\mu}{t^n n^5}\Big(\sum_{\nu=0}^{n} x_m^{\alpha\nu}\Big)^{-1/\alpha}$$

and

$$Pr\Big\{|f_n(x_0)| < 1/n^5\Big\} < \mu/(n^{5+1/\alpha} t_n).$$

Thus, considering all the $(k+2)$ circles, we see that the number of zeros inside all these does not exceed a value, which is asymptotically equal to

$$\frac{p}{\log 2}\Big(\frac{17}{2} + \frac{2}{\beta}\Big)(\log n - \log\log n)\log n;\ \ (\beta > 4).$$

The measure of the exceptional set is at most

$$\frac{\mu' k_n^\alpha \log n}{(n+1)^{(5\alpha/2)-1}(\log n)^\alpha} + \frac{\mu''}{n^{5+(1/\alpha)}\cdot t_n} + \frac{\mu'''}{n^5\cdot t_n}\sum_{m=1}^{p\log M}\left\{\sum_{\nu=0}^{n}\Big(1-\frac{1}{2^m}\Big)^{\alpha\nu}\right\}^{-1/\alpha}$$

$$< \ \frac{\mu'}{(n+1)^{(5\alpha/2)-1-\delta}(\log n)^{\alpha-1}} + \frac{\mu''}{n^{5+(1/\alpha)}\cdot t_n} + \frac{\mu''' \log n}{n^5\cdot t_n}$$

$$< \ \mu'/n^{(5\alpha/2-1-\gamma)}.$$

where $0 < \delta < \gamma < 1/2$ considering $k_n \geq t_n > n^{(5\alpha/2)-6}$.

For all $n > n_0$, the measure of the exceptional set is at most.

$$\sum_{n=n_0+1}^{\infty} \frac{\mu}{n^{(5\alpha/2)-1-\gamma}} < \frac{\mu}{n_0^{(5\alpha/2)-2-\gamma}}$$

Hence the proof is completed.

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