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ON SOME CLASSES OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

BY

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Abstract. The object of this paper is to investigate some inclusion properties of certain analytic functions defined by using the Ruscheweyh derivative. The results obtained here provide a sharpening of earlier results proved by several authors.

1. Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}\qquad (p\in N=\{1,2,3,\ldots\}),$$

that are analytic in the unit disk $E=\{z\in C:|z|<1\}$. Let $M(p,n,\alpha,\beta)$ be the subclass of $A(p)$ whose members satisfy the condition

$$\text{(1.1)}\quad \operatorname{Re}\left\{(1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}+\alpha\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}\right\}>\beta\ (\beta<1;\ \alpha\in R;\ z\in E),$$

where

$$D^{n+p-1}f(z)=\left(\frac{z^p}{(1-z)^{n+p}}\right)*f(z)\qquad (n>-p).$$

Here "$*$" stands for the Hadamard product (convolution of power series, i.e.,

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if

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{p+k} \in A(p) \quad \text{for} \quad j = 1, 2,$$

then

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{p+k}).$$

Since

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!} \qquad (f \in A(p); n \in N_0 = \{0, 1, 2, \ldots\}),$$

$D^{n+p-1} f(z)$ is usually called the Ruscheweyh derivative of $f(z)$ of order $n + p - 1$.

The class $M(p, n, \alpha, \beta)$, for special values of the parameters p, n, α, and β, generalizes several known classes (cf.[4], [11], [12], [15]), and has aroused considerable interest. For example, Goel and Sohi [6] generalized a result of Ruscheweyh [13] and proved that

$$M(p, n, 1, \frac{1}{2}) \subset M(p, n, 0, \frac{1}{2}) \tag{1.2}$$

where n is any integer greater than $-p$. Let $f \in M(p, n, \alpha, \beta)$, Chen and Lan [4] obtained lower bound of $\operatorname{Re}\{D^{n+p} f(z) / D^{n+p-1} f(z)\}$ in the cases when $\alpha/(n+p-1) \leq \beta < 1$, and when $1/2 \leq \beta < 1$ and $\alpha = n+p+1$ (see also [8]). Owa et al. [10] derived lower bound of $\operatorname{Re}\left\{(D^{n+p} f(z) / D^{n+p-1} f(z))^{1/2}\right\}$ in the case when $-p < \alpha \leq n + p + 1$, and considered its numerous consequences. On the other hand, Chen and Owa [5] obtained lower bound of $\operatorname{Re}\left\{(D^{n+p-1} f(z)/z^p)^{\lambda}\right\}$ for $f \in M(p, n, 0, \beta)$, $\beta \in [0, 1)$ and $0 < \lambda < 1/(2(n+p)(1-\beta))$.

One of the main objectives of this note is to give a sharp form of these results in a simple manner.

Let $T_{n+p-1}(\alpha)$ be the subclass of $A(p)$ consisting of functions which satisfy

$$\text{Re}\left[\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right] > \alpha \qquad (z \in E), \tag{1.3}$$

for some $\alpha < 1$ and $n > -p$. The class $T_{n+p-1}(\alpha)$ introduced by Goel and Sohi [7], also extends some known classes. In [7], the authors showed that (i) $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$ for $n \in N_0$ and $\alpha \in [0,1)$; (ii) for $c > -p$, the integral operator

$$F_c(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{1.4}$$

preserves the class $T_{n+p-1}(\alpha)$, where $n \in N_0$ and $\alpha \in [0,1)$.

More recently, Aouf and Darwish [3] proved that if $f \in T_{n+p+1}(\alpha)$ with $n \in N_0$ and $\alpha \in [0,1)$ then

$$\text{Re}\left\{\sqrt{\frac{(D^{n+p}f(z))'}{pz^{p-1}}}\right\} > \frac{1+\sqrt{4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)} \qquad (z \in E). \tag{1.5}$$

In the present paper we shall extend and sharpen these results and some related results.

2. Preliminaries. We need the following lemmas to prove our results.

Lemma 1. *Let $s > 0$ and $\delta \in [0,1)$. If $p(z) = 1 + p_1 z + \cdots$ is analytic in E and satisfies*

$$\text{Re}\left\{p(z) + \frac{zp'(z)}{sp(z)}\right\} > \delta \qquad (z \in E), \tag{2.1}$$

then

$$\text{Re}\{p(z)\} > \omega(s,\delta) \equiv \inf\{\text{Re } H(z) : z \in E\}, \tag{2.2}$$

where

$$H(z)=\frac{(1-z)^{2s(\delta-1)}}{S\int_0^1 t^{s-1}(1-tz)^{2s(\delta-1)}dt}.$$

This result is sharp and $\omega(s,\delta)\geq\delta$.

Remark 1. More general forms of this lemma may be found in the work of Mocanu et al. [9]. Furthermore, in the case when

$$\max\left\{\frac{s-1}{2s},0\right\}=\delta_0\leq\delta<1, \tag{2.3}$$

the value of $\omega(s,\delta)$ given by (2.2) can be replaced by

$$\omega(s,\delta)=H(-1)=\frac{2^{-2s(1-\delta)}}{{}_2F_1(2s(1-\delta),s;s+1;-1)}, \tag{2.4}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function.

The next Lemma 2 is of elementary, so we omit its proof.

Lemma 2. *Let* $w\in C$ *with* $\mathrm{Re}\{w\}>0$. *Then for any* $\rho\in[0,1]$

$$\mathrm{Re}\{w^\rho\}\geq\{\mathrm{Re}\ w\}^\rho. \tag{2.5}$$

Let F and G be analytic in E. The function F is subordinate to G, written $F\prec G$ or $F(z)\prec G(z)$, if G is univalent, $F(0)=G(0)$ and $F(E)\subset G(E)$.

Lemma 3. *Let* $\beta<1$ *and* $m\geq 2(1-\beta)$. *If* $p(z)=1+p_1z+\cdots$ *is analytic in* E *and satisfies*

$$\mathrm{Re}\left\{1+m\frac{zp'(z)}{p(z)}\right\}>\beta \qquad (z\in E), \tag{2.6}$$

then for any $\lambda \in (0,1]$,

$$\text{(2.7)} \qquad \mathrm{Re}\left\{p^{\lambda}(z)\right\} > 2^{2\lambda(\beta-1)/m} \qquad (z \in E).$$

This result is sharp.

Proof. It follows from (2.6) that the subordination

$$\frac{zp'(z)}{p(z)} \prec \frac{2(1-\beta)}{m}\frac{z}{1-z}$$

hods in E, Using a result of Suffridge [16], we have

$$\text{(2.8)} \qquad p(z) \prec (1-z)^{2(\beta-1)/m}.$$

Hence,

$$\text{(2.9)} \qquad \mathrm{Re}\left\{p^{\lambda}(z)\right\} \geq \inf_{z\in E} \mathrm{Re}(1-z)^{2\lambda(\beta-1)/m}.$$

From Lemma 2, we obtain

$$\mathrm{Re}\left\{(1-z)^{2\lambda(\beta-1)/m}\right\} \geq \left\{\mathrm{Re}\frac{1}{1-z}\right\}^{2\lambda(1-\beta)/m} > (\frac{1}{2})^{2\lambda(1-\beta)/m} \qquad (z \in E),$$

and (2.7) is proved.

For the function $p_0(z) = (1-z)^{2(\beta-1)/m}$, we observe that

$$\text{(2.10)} \qquad \begin{aligned} &\inf_{z\in E} \mathrm{Re}\left\{1 + m\frac{z{p'}_0(z)}{p_0(z)}\right\} = \inf_{z\in E}\left\{1 + 2(1-\beta)\mathrm{Re}\frac{z}{1-z}\right\} = \beta, \\ &\inf_{z\in E} \mathrm{Re}\left\{p_0^{\lambda}(z)\right\} = \inf_{z\in E} \mathrm{Re}(\frac{1}{1-z})^{2\lambda(1-\beta)/m} = 2^{2\lambda(\beta-1)/m}. \end{aligned}$$

This shows that the result is sharp.

Remark 2. Using the same technique as we have detailed above, one can show that if $p(z) = 1 + p_1 z + \cdots$ is analytic in E and satisfies (2.6) for

$\beta < 1$ and $m > 0$ then for any $0 < \lambda \le m/(2(1-\beta))$ the sharp result (2.7) still holds.

3. The class $M(p, n, \alpha, \beta)$.

Theorem 1. *If $f(z) \in M(p, n, \alpha, \beta)$ with $0 < \alpha \le \beta(n+p+1)$, or $\alpha = n+p+1$ and $\beta \ge 1/2$, then for any $\lambda \in (0, 1]$,*

$$\text{Re}\left\{\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right)^{\lambda}\right\} > \gamma(p, n, \alpha, \beta, \lambda) \qquad (z \in E), \tag{3.1}$$

where

$$\gamma(p, n, \alpha, \beta, \lambda) = \begin{cases} \omega^{\lambda}\left(\frac{n+p+1-\alpha}{\alpha}, \frac{\beta(n+p+1)-\alpha}{n+p+1-\alpha}\right), & \text{if } 0 < \alpha \le \beta(n+p+1) \\ 2^{2\lambda(\beta-1)}, & \text{if } \alpha = n+p+1,\ \beta \ge \frac{1}{2} \end{cases} \tag{3.2}$$

and ω is defined by (2.2). This result is sharp.

Proof. It is easily seen that, for every $n > -p$,

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z), \tag{3.3}$$

which yields

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = (n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - n. \tag{3.4}$$

Let $p(z) = \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = 1 + p_1 z + \cdots$. Using (3.4), we have

$$\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} = \frac{1}{n+p+1}\left\{\frac{zp'(z)}{p(z)} + (n+p)p(z) + 1\right\}.$$

This shows that $f(z) \in M(p,n,\alpha,\beta)$ if and only if

$$\text{(3.5)}\quad \operatorname{Re}\left\{\alpha\frac{zp'(z)}{p(z)}+(n+p+1-\alpha)p(z)\right\}>\beta(n+p+1)-\alpha \qquad (z\in E).$$

(a) If $0<\alpha\le\beta(n+p+1)$, then (3.5) gives

$$\operatorname{Re}\left\{p(z)+\frac{zp'(z)}{sp(z)}\right\}>\delta \qquad (z\in E)$$

where

$$s=\frac{n+p+1-\alpha}{\alpha}>0$$
$$\delta=\frac{\beta(n+p+1)-\alpha}{n+p+1-\alpha}\in[0,1).$$

Hence, Theorem 1 follows from Lemma 1 and Lemma 2.

(b) If $\alpha=n+p+1$ and $\beta\ge\frac{1}{2}$, then (3.5) yields

$$\operatorname{Re}\left\{1+\frac{zp'(z)}{p(z)}\right\}>\beta \qquad (z\in E).$$

Thus, Theorem 1 follows from Lemma 3. This completes the proof of Theorem 1.

Remark 3. By assigning appropriate special values to the various parameters involved in Theorem 1, we can derive several consequences of Theorem 1. Only a few of them will be listed here

If we let $\alpha=1$, $\beta=1/2$, and $\lambda=1$ in Theorem 1, we obtain

$$\text{(3.6)}\qquad M(p,n,1,\frac{1}{2})\subset M(p,n,0,\omega(s,\delta))$$

where $n+p\ge1$, $\omega(s,\delta)=H(-1)$ is defined by (2.2) with $s=n+p$ and $\delta=(n+p-1)/(2(n+p))$. This sharpens the above-mentioned result of Goel and Sohi [6].

If we let $\alpha = 1$, $\beta = (n+\xi p+1)/(n+p+1)$, and $\lambda = 1$ in Theorem 1, we obtain

$$M(p,n,1,\frac{n+\xi p+1}{n+p+1}) \subset M(p,n,0,\omega(s,\delta)) \tag{3.7}$$

where $n+\xi p \geq 0$, $0 \leq \xi < 1$, $\omega(s,\delta)$ is defined by (2.2) with $s = n+p$ and $\delta = (n+\xi p)/(n+p)$. This sharpens the result of Ahuja [1]. As a special case ($\xi = (p-1)/p$), (3.7) also sharpens the result of Soni [14].

If we let $p = 1$, $\beta = 1/2$, and $\lambda = 1$ in Theorem 1, we obtain

$$M(1,n,\alpha,\frac{1}{2}) \subset M(1,n,0,\omega(s,\delta)) \tag{3.8}$$

where $n > -1$, $0 < \alpha \leq (n+2)/2$, and $\omega(s,\delta) = H(-1)$ is defined by (2.2) with $s = (n+2-\alpha)/\alpha$ and $\delta = (n+2-2\alpha)/(2(n+2-\alpha))$. This sharpens the result of A1-Amiri [2] in the case when $0 < \alpha \leq (n+2)/2$.

Theorem 1 also sharpens the result of Chen and Lan [4] in the case when $\lambda = 1$, and the result of Owa et al. [10] in the case when $\lambda = 1/2$.

Theorem 2. *If $f(z) \in M(p,n,0,\beta)$ for $n > -p$ and $\beta < 1$, then for any $0 < \lambda \leq 1/(2(n+p)(1-\beta))$, we have*

$$\operatorname{Re}\left\{\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\lambda}\right\} > 2^{2\lambda(n+p)(\beta-1)} \qquad (z \in E). \tag{3.9}$$

This result is sharp.

Proof. Let $p(z) = \frac{D^{n+p-1}f(z)}{z^p} = 1 + p_1 z + \cdots$. Then from (3.3), we have

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = 1 + \frac{1}{n+p}\frac{zp'(z)}{p(z)}.$$

Thus,

$$f(z) \in M(p,n,0,\beta) \Leftrightarrow \operatorname{Re}\left\{1 + \frac{1}{n+p}\frac{zp'(z)}{p(z)}\right\} > \beta, \qquad (z \in E) \tag{3.10}$$

and the desired result follows from Lemma 3. This completes the proof.

Remark 4. It is easy to see that Theorem 2 extends and sharpens the above-mentioned result of Chen and Owa [5].

4. The class $T_{n+p-1}(\alpha)$.

Theorem 3. *If $n > -p$ and $\alpha < 1$, then $T_{n+p}(\alpha) \subset T_{n+p-1}(\mu)$, where*

$$\mu = 1 + 2(1-\alpha)(p+n)\sum_{k=1}^{\infty}\frac{(-1)^k}{n+p+k} \geq \alpha.$$

This result is sharp.

Proof. It follows from (3.3) that

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{p+n}{z^n}\int_0^z t^{n-1}D^{n+p}f(t)dt \\ &= (p+n)\int_0^1 t^{n-1}D^{n+p}f(tz)dt, \end{aligned} \tag{4.1}$$

which yields

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = (p+n)\int_0^1 t^{n+p-1}\frac{(D^{n+p}f(tz))'}{p(tz)^{p-1}}dt \qquad (z \in E). \tag{4.2}$$

Since $f(z) \in T_{n+p}(\alpha)$, we have

$$\mathrm{Re}\left\{\frac{(D^{n+p}f(tz))'}{p(tz)^{p-1}}\right\} \geq (1-\alpha)\frac{1-t}{1+t} + \alpha \qquad (z \in E,\ t \in [0,1]). \tag{4.3}$$

Therefore, form (4.2) and (4.3), we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} &> (p+n)\int_0^1 t^{n+p-1}\frac{1-(1-2\alpha)t}{1+t}dt \\ &= 1 + 2(1-\alpha)(p+n)\sum_{k=1}^{\infty}\frac{(-1)^k}{n+p+k} \end{aligned} \tag{4.4}$$

$$= \mu \geq \alpha.$$

To show that this result is sharp, we consider the function

$$f_0(z) = z^p + 2(1-\alpha)p \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+n+1)}{\Gamma(k+p+n+1)(p+k)} z^{p+k} \tag{4.5}$$

which is an element of $T_{n+p}(\alpha)$. Since

$$\left\{ \frac{(D^{n+p-1} f_0(z))'}{pz^{p-1}} \right\} = (p+n) \int_0^1 t^{n+p-1} \frac{1+(1-2\alpha)tz}{1-tz} dt, \tag{4.6}$$

one can easily show that $f_0(z) \in T_{n+p-1}(\mu)$, but $f_0(z) \notin T_{n+p-1}(\mu')$ if $\mu' > \mu$. This completes the proof of Theorem 3.

Remark 5. Theorem 3 extends and sharpens the above-mentioned result of Goel and Sohi [7]. From Theorem 3 and Lemma 2, we obtain the following Corollary which extends and sharpens the inequality (1.5).

Corollary. *Let $n > -p$ and $\alpha \in [0, 1)$. If $f \in T_{n+p+1}(\alpha)$, then*

$$\begin{aligned} & \operatorname{Re}\left\{ \sqrt{\frac{(D^{n+p} f(z))'}{pz^{p-1}}} \right\} \\ & > \left\{ 1 + 2(1-\alpha)(1+p+n) \sum_{k=1}^{\infty} \frac{(-1)^k}{1+p+n+k} \right\}^{1/2} \qquad (z \in E). \end{aligned} \tag{4.7}$$

This estimate is sharp.

Theorem 4. *Let $n > -p$, $c > -p$, and $\alpha < 1$. If $f(z) \in T_{n+p-1}(\alpha)$, then the function $F_c(z)$ defined by (1.4) belongs to $T_{n+p-1}(v)$, where*

$$v = 1 + 2(1-\alpha)(c+p) \sum_{k=1}^{\infty} \frac{(-1)^k}{c+p+k}.$$

This result is sharp.

Proof. It is easily seen that

$$D^{n+p-1}F_c(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}D^{n+p-1}f(t)dt = (c+p)\int_0^1 t^{c-1}D^{n+p-1}f(tz)dt. \tag{4.8}$$

Since $f(z)\in T_{n+p-1}(\alpha)$, we deduce from (4.3) and (4.8) that

$$\begin{aligned} \mathrm{Re}\left\{\frac{(D^{n+p-1}F_c(z))'}{pz^{p-1}}\right\} &= (c+p)\int_0^1 t^{c+p-1}\mathrm{Re}\left\{\frac{(D^{n+p-1}f(tz))'}{p(tz)^{p-1}}\right\}dt \\ &> (c+p)\int_0^1 t^{c+p-1}\frac{1-(1-2\alpha)t}{1+t}dt \\ &= 1+2(1-\alpha)(c+p)\sum_{k=1}^{\infty}\frac{(-1)^k}{c+p+k} \\ &= v\geq\alpha. \end{aligned} \tag{4.9}$$

The estimate (4.9) is best possible for the function (4.5) with n replaced by $n-1$. This completes the proof of Theorem 4.

Remark 6. Theorem 4 also extends and sharpens the above-mentioned result of Goel and Sohi [7].

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